MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume 14, Number 5&6, October & December 2017

doi:10.3934/mbe.2017080

pp. 1535–1563

THRESHOLD DYNAMICS OF A TIME PERIODIC AND TWO–GROUP EPIDEMIC MODEL WITH DISTRIBUTED DELAY

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Abstract. In this paper, a time periodic and two–group reaction–diffusion epidemic model with distributed delay is proposed and investigated. We firstly introduce the basic reproduction number R_0 for the model via the next generation operator method. We then establish the threshold dynamics of the model in terms of R_0, that is, the disease is uniformly persistent if $R_0 > 1$, while the disease goes to extinction if $R_0 < 1$. Finally, we study the global dynamics for the model in a special case when all the coefficients are independent of spatio–temporal variables.

1. **Introduction.** Mathematical modeling is a basic but efficient tool to study the spread mechanism of diseases, by which the future course of an outbreak can be predicted and then be controlled. In order to establish a theoretical framework for mathematical analysis of transmission of malaria, Ross [44] firstly proposed a system of ordinary differential equations which is the origin of the modern susceptible–infected–recovered (SIR) compartmental model. Since then the SIR compartmental model and many of its extensions, which are independent of the spatial variables, have been well investigated by many scholars [2, 8, 20, 40, 36]. At the same time, the heterogeneity of living environment and mobility of the host individuals play a crucial role in the geographic spread of infectious disease. In fact, there have been many articles which have analyzed mathematically the spatial dynamics of epidemic models, see [3, 16, 42, 45, 43, 46, 61, 59, 64, 65, 66] and the references therein. As reported by [29], multi–group epidemic models have been proposed to describe the spread of various infectious diseases in heterogeneous populations, such as measles, mumps, gonorrhea, and HIV/AIDS. In such models, a heterogeneous host population can be divided into several homogeneous groups according to the modes of transmission, contact patterns, or geographic distributions, so that within–group and inter–group interactions could be modeled separately. The works involved with multi-group models with or without spread diffusion can be found in [7, 13, 14, 15, 17, 21, 23, 32, 34, 50, 60, 70, 68].

Many infectious diseases, such as measles, chicken pox, cholera, influenza, HIV, SARS, etc., exhibit a latent period, namely, the infected individuals do not infect

2010 *Mathematics Subject Classification.* Primary: 35K57; Secondary: 35B10, 35B35, 34B40, 92D30.

Key words and phrases. Two–group epidemic model, time periodic, distributed delay, threshold dynamics, Lyapunov functional.

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other susceptible individuals until some time later. Meanwhile, the infected individuals may move from one spatial location to another spatial location with time, which give rise to spatial nonlocal effect. Generally, such nonlocal infection may effect the outbreak and transmission of the diseases (see, e.g. Li and Zou [27], Lou and Zhao [32], Wang and Zhao [58]). Li and Zou [28] proposed a time–delayed SIR epidemic model with nonlocal terms among n–patches in a fixed latent period, where a demographic structure is incorporated by adding recruitment (including births) and natural deaths. They found that nonlocal terms can enhance the basic reproduction number R_0, and thus, may leads to an otherwise dying–out disease to persist. When the habitat is a continuous domian, Guo et al. [18] derived a reaction–diffusion epidemic model with time–delay and non–locality in a fixed latent period and investigated the threshold dynamics of the epidemic model by means of the basic reproduction number R_0. In addition, there have been other papers studying diffusion–reaction epidemic models with fixed latent period, see [27, 32, 58, 63, 67] and the references therein.

However, it is common that the length of the latent period differs from disease to disease; even for the same disease, the length of the latent period is also different from individuals to individuals. Based on this point, instead of using the discrete (fixed) delay, we employ distributed delay to characterize the variable latency (see, e.g., van den Driessche et al. [53]). The distributed delay allows infectivity to be a function of the duration since infection, up to some maximum duration (see [38]). To characterize the distributed delay, a distribution function $p(u) : [0, \infty) \to [0, \infty)$ which accounts for the variance that the infected individuals become infectious and is assumed to have compact support, $p(u) \geq 0$ and $\int_0^\infty p(u)du = 1$ can be used. Epidemic models with distributed delay independent of the spatial variables have been studied, see [6, 10, 22, 29, 49, 56] and the references therein.

It is well known that seasonality can impact host–pathogen interactions, including seasonal changes in host social behaviour and contact rates, variation in encounters with infective stages in the environment, annual pulses of host births and deaths and changes in host immune defences (see [1]). For an infectious disease, it is crucial and more realistic to take into account temporal heterogeneity, which gives rise to non–autonomous evolution equations. Bacaër and Guernaoui [5] defined the basic reproduction number R_0 in a periodic environment. For further developments, we refer to Bacaër et al. [4] and Inaba [24] and the references therein. Wang and Zhao [57] developed the basic reproduction number R_0 of a large class of compartmental epidemic models in periodic environments and studied the impact of periodic contacts or periodic migrations on the disease transmission by analyzing the global dynamics of a periodic epidemic model with patch structure. Peng and Zhao [41] studied the threshold dynamics of a time–periodic reaction–diffusion SIS model and showed that the persistence of the infectious disease can be enhanced by incorporating the spatial heterogeneity and temporal periodicity into the model. Recently, the theory of the basic reproduction number on the periodic and time–delayed compartmental models is established by Zhao [71] and can be applied to periodic SEIR models with incubation period. Zhang et al. [67] proposed a time–periodic reaction–diffusion epidemic model which incorporates simple demographic structure and a fixed latent period of the infectious disease, introduced the basic reproduction number R_0 via a next generation operator, and investigated the threshold dynamics of the epidemic model in terms of R_0. Some other studies on the dynamics of time heterogeneous epidemic models can be found in [33, 54, 55, 65, 68] and the references therein. However, for such non-autonomous (even autonomous) diffusion–reaction epidemic models with distributed delays, much less is done. The purpose of this paper is to incorporate spatial diffusion, distributed latency of the disease and temporal heterogeneity into a multi–group SIR disease model and to investigate the threshold dynamics of the derived model.

The rest of this paper is organized as follows. In the next section, we derive a two–group reaction–diffusion epidemic model with seasonality and distributed delay. In section 3, we introduce the basic reproduction number R_0 for the system via the next generation operator method and then establish the threshold dynamics for the system in term of R_0, namely, the disease is uniformly persistent if $R_0 > 1$, while the disease goes to extinction if $R_0 < 1$. Section 4 is devoted to the global dynamics for the model in a special case where all the coefficients are independent of spatio–temporal variables.

2. **Model formulation.** Assume that an infectious disease spreads in two populations or sub–populations living in a bounded domain $\Omega \in \mathbb{R}^n$ with smooth boundary $\partial\Omega$. We always define two populations or sub–populations by the subscript 1 and 2. Without loss of generality, we divide each population/sub–population into four compartments: the susceptible compartment, the latent compartment, the infectious compartment and the removed compartment. Then we denote the densities of four compartments at time t and location x by $S_i(t,x)$, $L_i(t,x)$, $I_i(t,x)$ and $R_i(t,x)$, respectively, where $i = 1,2$ and $(t,x) \in \mathbb{R}^+ \times \bar{\Omega}$.

Let $E_1(t,a,x)$ and $E_2(t,a,x)$ be the densities of two exposed populations or sub–populations at time $t \geq 0$, infection age variable $a \geq 0$ and location $x \in \bar{\Omega}$, repectively. Then $E_i (i = 1,2)$ satisfy the following model

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) E_i(t,a,x) &= D_i \Delta_x E_i(t,a,x) \\ &\quad -\Big(\bar{D}_i(t,a,x) + M_i(t,a,x) + d_i(t,x)\Big) E_i(t,a,x), \quad (1) \\ &\quad t, a > 0, \ x \in \Omega \end{aligned}$$

with Neumann boundary condition

$$\frac{\partial E_i(t,a,x)}{\partial n} = 0, \ t, a > 0, \ x \in \partial\Omega, \ i = 1,2,$$

where n is the outward normal to $\partial\Omega$, D_i represents the diffusion rate of the i–th population, $\bar{D}_i(t,a,x)$ and $M_i(t,a,x)$ mean the disease–induced mortality rate and the recovery rate of the i–th population which are dependent upon the infection age a, time t and location x, respectively and $d_i(t,x)$ denotes the natural death rate of the i–th population at time t and location x for $i = 1,2$.

Suppose that an infectious disease has a period of latency which is not fixed. Namely, for each population, we assume that infectious individuals must have capable of infecting others after the infection age $\tau_i \in [0,\infty)$. But, between the infection age 0 and τ_i, the infected individual may or may not have an infection ability. Assume that $f_i(r)dr$ denotes the probability of becoming into the individuals who are capable of infecting others between the infection ages r and $r + dr$ and $F_i(a) := \int_0^a f_i(r)dr$ represents the probability of turning into the individuals with infecting others before the infection age a for $i = 1,2$. Then we have

$$L_i(t,x) = \int_0^{\tau_i} (1 - F_i(a)) E_i(t,a,x) da$$

and

$$I_i(t,x)=\int_0^{\tau_i}F_i(a)E_i(t,a,x)da+\int_{\tau_i}^{+\infty}E_i(t,a,x)da,\ i=1,2.$$

It is clear that $f_i(a)\geq 0$ for $a\in(0,\tau_i)$ and $F_i(a)\equiv 1$ for $a\in[\tau_i,\infty)$ for $i=1,2$. For the sake of simplicity, we assume that the functions $\bar{D}_i(t,a,x)$ and $M_i(t,a,x)$ are independent of the infection age a, namely,

$$\bar{D}_i(t,a,x)=\bar{D}_i(t,x),\ M_i(t,a,x)=M_i(t,x),\ \forall t\geq 0,\ a\in[0,\infty),\ x\in\Omega,\ i=1,2.$$

For a convenience, we assume

$$I_{i,1}:=\int_0^{\tau_i}F_i(a)E_i(t,a,x)da \quad \text{and} \quad I_{i,2}:=\int_{\tau_i}^{+\infty}E_i(t,a,x)da.$$

We now aim to find partial differential equations satisfied by $L_i(t,x)$ and $I_i(t,x)$. Integrating (1) with respect to a and using the expressions of $L_i(t,x)$ and $I_i(t,x)$, one has

$$\begin{aligned}\frac{\partial L_i}{\partial t}&=D_i\Delta L_i(t,x)-\Big(\bar{D}_i(t,x)+M_i(t,x)+d_i(t,x)\Big)L_i(t,x)\\&\quad-\int_0^{\tau_i}f_i(a)E_i(t,a,x)da+E_i(t,0,x),\\\frac{\partial I_{i,1}}{\partial t}&=D_i\Delta I_{i,1}(t,x)-\Big(\bar{D}_i(t,x)+M_i(t,x)+d_i(t,x)\Big)I_{i,1}(t,x)\\&\quad+\int_0^{\tau_i}f_i(a)E_i(t,a,x)da-E_i(t,\tau_i,x)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial I_{i,2}(t,x)}{\partial t}&=D_i\Delta I_{i,2}(t,x)-\Big(\bar{D}_i(t,x)+M_i(t,x)+d_i(t,x)\Big)I_{i,2}(t,x)\\&\quad+E_i(t,\tau_i,x)-E_i(t,\infty,x),\end{aligned}$$

where $i=1,2$. Let $E_i(t,\infty,x)=0(i=1,2)$, then we can obtain

$$\begin{aligned}\frac{\partial I_i(t,x)}{\partial t}&=D_i\Delta I_i(t,x)-\Big(\bar{D}_i(t,x)+M_i(t,x)+d_i(t,x)\Big)I_i(t,x)\\&\quad+\int_0^{\tau_i}f_i(a)E_i(t,a,x)da,\ i=1,2.\end{aligned}$$

As the new infection individuals come from the contact of the infectious and susceptible individuals, we adopt the following form:

$$E_i(t,0,x)=\beta_{i1}(t,x)g_{i1}(S_i(t,x),I_1(t,x))+\beta_{i2}(t,x)g_{i2}(S_i(t,x),I_2(t,x)),\ i=1,2,$$

where $\beta_{ij}(t,x)\geq 0$ is called the infection rate for $i,j=1,2$. In this paper, we assume that the contacts between susceptible individuals and infectious individuals are defined by incidence functions $g_{ij}(u,v)(i,j=1,2)$, which satisfy the following conditions:

(H1): (i) $g_{ij}(u,v):\mathbb{R}_+^2\to\mathbb{R}_+(i,j=1,2)$ are continuously differentiable for all $u,v\geq 0$;

(ii) $g_{ij}(u,0)=0$ and $g_{ij}(0,v)=0$ for all $u,\ v\geq 0$ and $i,j=1,2$;

(iii) $\frac{\partial}{\partial u}g_{ij}(u,v)\geq 0$ and $\frac{\partial}{\partial v}g_{ij}(u,v)\geq 0$ for all $u,v\geq 0$ and $i,j=1,2$. In particular, $\partial_u g_{ij}(u,0)=0$ and $\partial_v g_{ij}(u,0)>0$ for all $u>0$;

(iv) there exist $\eta_i>0(i=1,2)$ such that $g_{ij}(u,v)\leq\eta_i u$ for all $u,v\geq 0$;

(v) $\mathcal{N}_{ij}(u,v) := \frac{g_{ij}(u,v)}{v}$, $\mathcal{N}_{ij}(u,v) > 0$, $\frac{\partial}{\partial u}\mathcal{N}_{ij}(u,v) \geq 0$ and $\frac{\partial}{\partial v}\mathcal{N}_{ij}(u,v) \leq 0$ for all $u, v > 0$ and $i, j = 1, 2$.

Note that the class of $g_{ij}(u,v)(i,j = 1,2)$ satisfying (H1) include many common incidence functions such as

$$g_{ij}(u,v) = \frac{uv}{u+v},\ \ g_{ij}(u,v) = \frac{uv}{1 + a_{ij}u + b_{ij}v + c_{ij}uv}$$

and

$$g_{ij}(u,v) = \frac{uv}{1 + a_{ij}u},$$

where $a_{ij}, b_{ij}, c_{ij} > 0$ for $i, j = 1, 2$, see [39].

We use the following simple demographic equation for a population $\mathcal{Q}(t,x)$ that admits a dynamics of global convergence to a positive periodic solution

$$\frac{\partial \mathcal{Q}(t,x)}{\partial t} = D_{\mathcal{Q}}\Delta\mathcal{Q}(t,x) + \mu(t,x) - d(t,x)\mathcal{Q}(t,x),$$

where $\mu(t,x)$ is the recruiting rate, $D_{\mathcal{Q}}$ is the diffusion rate and $d(t,x)$ is the natural death rate. We also assume that the disease under consideration does not transmit vertically. On the basis of the above assumptions, the disease dynamics is expressed by the following system

$$\begin{cases} \frac{\partial S_i(t,x)}{\partial t} = D_{S_i}\Delta S_i(t,x) + \mu_i(t,x) - d_i(t,x)S_i(t,x) \\ \qquad\qquad -\beta_{i1}(t,x)g_{i1}(S_i(t,x), I_1(t,x)) - \beta_{i2}(t,x)g_{i2}(S_i(t,x), I_2(t,x)), \\ \frac{\partial L_i(t,x)}{\partial t} = D_i\Delta L_i(t,x) - \Big(\bar{D}_i(t,x) + M_i(t,x) + d_i(t,x)\Big)L_i(t,x) \\ \qquad\qquad +\beta_{i1}(t,x)g_{i1}(S_i(t,x), I_1(t,x)) + \beta_{i2}(t,x)g_{i2}(S_i(t,x), I_2(t,x)) \\ \qquad\qquad -\int_0^{\tau_i} f_i(a)E_i(t,a,x)da, \\ \frac{\partial I_i(t,x)}{\partial t} = D_i\Delta I_i(t,x) - \Big(\bar{D}_i(t,x) + M_i(t,x) + d_i(t,x)\Big)I_i(t,x) \\ \qquad\qquad +\int_0^{\tau_i} f_i(a)E_i(t,a,x)da, \\ \frac{\partial R_i(t,x)}{\partial t} = D_{R_i}\Delta R_i(t,x) + M_i(t,x)L_i(t,x) + M_i(t,x)I_i(t,x) - d_i(t,x)R_i(t,x). \end{cases} \tag{2}$$

We make the following basic assumptions:

(H2): D_{S_i} and D_i are positive constants for $i = 1, 2$; $\mu_i(t,x)$ and $\bar{D}_i(t,x)$ are Hölder continuous and nonnegative nontrivial functions on $\mathbb{R}\times\bar{\Omega}$, and periodic in time t with the same period $T > 0$; $d_i(t,x)(i = 1,2)$ are Hölder continuous and positive functions on $\mathbb{R}\times\bar{\Omega}$, and periodic in time t with the same period $T > 0$; $\beta_{ij}(t,x)(i,j = 1,2)$ are Hölder continuous and nonnegative nontrivial functions on $\mathbb{R}\times\bar{\Omega}$, and periodic in time t with the same period $T > 0$.

The reminder is to derive functions $E_i(t,a,x)(i = 1,2)$ by integration along characteristics. For a convenience, let $r_i(t,\cdot) = \bar{D}_i(t,\cdot) + M_i(t,\cdot) + d_i(t,\cdot)$. For any $\xi \geq 0$, we consider the solutions of (1) along the characteristic line $t = a + \xi$ by letting $v_i(\xi,a,x) = E_i(a+\xi,a,x)(i = 1,2)$. Then for $a \in (0,\tau_i]$, we have

$$\begin{cases} \frac{\partial v_i(\xi,a,x)}{\partial a} = D_i\Delta v_i(\xi,a,x) - r_i(a+\xi,x)v_i(\xi,a,x), \\ v_i(\xi,0,x) = \beta_{i1}(\xi,x)g_{i1}(S_i(\xi,x), I_1(\xi,x)) + \beta_{i2}(\xi,x)g_{i2}(S_i(\xi,x), I_2(\xi,x)), \end{cases}$$

where $i = 1, 2$. For the above system, we can regard ξ as a parameter. Then we have

$$v_i(\xi, a, x) = \int_\Omega \Gamma_i(\xi + a, \xi, x, y)\Big(\beta_{i1}(\xi, y)g_{i1}(S_i(\xi, y), I_1(\xi, y)) + \beta_{i2}(\xi, y)g_{i2}(S_i(\xi, y), I_2(\xi, y))\Big)dy,$$

where $\Gamma_i(t, s, x, y)$ with $t > s$ and $x, y \in \Omega$ is the fundamental solution associated with the partial differential operator $\partial_t - D_i\Delta - r_i(t, \cdot)$ and Neumann boundary condition for $i = 1, 2$. Note that $\Gamma_i(t, s, x, y) = \Gamma_i(t+T, s+T, x, y)$ for all $t > s \geq 0$ and $x, y \in \Omega$ because of $r_i(t + T, x) = r_i(t, x)$ for any $t \geq 0$. It then follows from $E_i(t, a, x) = v_i(t - a, a, x)$ that

$$E_i(t, a, x) = \int_\Omega \Gamma_i(t, t - a, x, y)\Big(\beta_{i1}(t - a, y)g_{i1}(S_i(t - a, y), I_1(t - a, y)) + \beta_{i2}(t - a, y)g_{i2}(S_i(t - a, y), I_2(t - a, y))\Big)dy, \ i = 1, 2. \tag{3}$$

Substituting (3) into the second equation and the third equation of (2) respectively, and ignoring the $L_i(t, x)$ and $R_i(t, x)$ equations from (2) because they are decoupled from the $S_i(t, x)$ and $I_i(t, x)$ equations, we obtain the following system:

$$\begin{cases} \frac{\partial S_i(t,x)}{\partial t} = D_{S_i}\Delta S_i(t, x) + \mu_i(t, x) - d_i(t, x)S_i(t, x) \\ \qquad -\beta_{i1}(t, x)g_{i1}(S_i(t, x), I_1(t, x)) - \beta_{i2}(t, x)g_{i2}(S_i(t, x), I_2(t, x)), \\ \qquad t > 0, \ x \in \Omega, \\ \frac{\partial I_i(t,x)}{\partial t} = D_i\Delta I_i(t, x) - r_i(t, x)I_i(t, x) + \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t, t - a, x, y) \\ \qquad \times\Big(\beta_{i1}(t - a, y)g_{i1}(S_i(t - a, y), I_1(t - a, y)) \\ \qquad +\beta_{i2}(t - a, y)g_{i2}(S_i(t - a, y), I_2(t - a, y))\Big)dyda, \ t > 0, \ x \in \Omega, \\ \frac{\partial}{\partial n}S_i(t, x) = \frac{\partial}{\partial n}I_i(t, x) = 0, \ t > 0, \ x \in \partial\Omega \end{cases} \tag{4}$$

for $i = 1, 2$. We assume

$$\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t, t - a, x, y)\beta_{ij}(t - a, y)dyda > 0, \ \forall(t, x) \in (0, +\infty) \times \bar{\Omega} \tag{5}$$

for $i, j = 1, 2$. For simplicity, letting $(u_{S_1}, u_{S_2}, u_1, u_2) = (S_1, S_2, I_1, I_2)$, we focus on the following reaction–diffusion system with Neumann boundary condition:

$$\begin{cases} \frac{\partial u_{S_i}(t,x)}{\partial t} = D_{S_i}\Delta u_{S_i}(t, x) + \mu_i(t, x) - d_i(t, x)u_{S_i}(t, x) \\ \qquad -\beta_{i1}(t, x)g_{i1}(u_{S_i}(t, x), u_1(t, x)) - \beta_{i2}(t, x)g_{i2}(u_{S_i}(t, x), u_2(t, x)), \\ \qquad t > 0, \ x \in \Omega, \\ \frac{\partial u_i(t,x)}{\partial t} = D_i\Delta u_i(t, x) - r_i(t, x)u_i(t, x) + \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t, t - a, x, y) \\ \qquad \times\Big(\beta_{i1}(t - a, y)g_{i1}(u_{S_i}(t - a, y), u_1(t - a, y)) \\ \qquad +\beta_{i2}(t - a, y)g_{i2}(u_{S_i}(t - a, y), u_2(t - a, y))\Big)dyda, \ t > 0, \ x \in \Omega, \\ \frac{\partial}{\partial n}u_{S_i}(t, x) = \frac{\partial}{\partial n}u_i(t, x) = 0, \ t > 0, \ x \in \partial\Omega \end{cases} \tag{6}$$

for $i = 1, 2$.

3. **Threshold dynamics.**

In this section, we explore the threshold dynamics of system (6).

3.1. **Global existence of solution.** In this subsection, we investigate the existence and uniqueness of time–global solutions of system (6). Set $\tau = \max\{\tau_1, \tau_2\}$. Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^4)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Let $\mathbb{C}_\tau := C([-\tau, 0], \mathbb{X})$ be the Banach space with the norm $\|\phi\| = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{\mathbb{X}}$, $\forall \phi \in \mathbb{C}_\tau$. Define $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^4_+)$ and $\mathbb{C}^+_\tau := C([-\tau, 0], \mathbb{X}^+)$, then $(\mathbb{X}, \mathbb{X}^+)$ and $(\mathbb{C}_\tau, \mathbb{C}^+_\tau)$ are strongly ordered spaces. For $\sigma > 0$ and a given function $u(t) : [-\tau, \sigma] \to \mathbb{X}$, we denote $u_t \in \mathbb{C}_\tau$ by

$$u_t(\theta) = u(t + \theta),\ \theta \in [-\tau, 0].$$

Set $\mathbb{Y} := C(\bar{\Omega}, \mathbb{R})$ and $\mathbb{Y}^+ := C(\bar{\Omega}, \mathbb{R}^+)$. Furthermore, we consider the following system:

$$\begin{cases} \frac{\partial w_i}{\partial t} = D_{S_i}\Delta w_i(t, x) - d_i(t, x)w_i(t, x),\ t > 0,\ x \in \Omega,\ i = 1, 2, \\ \frac{\partial}{\partial n} w_i(t, x) = 0,\ t > 0,\ x \in \partial\Omega,\ i = 1, 2, \\ w_i(0, x) = \phi_{S_i}(x),\ x \in \Omega,\ \phi_{S_i} \in \mathbb{Y}^+,\ i = 1, 2, \end{cases} \tag{7}$$

where $D_{S_i} > 0 (i = 1, 2)$ and $d_i(t, x)(i = 1, 2)$ are Hölder continuous and nonnegative nontrivial functions on $\mathbb{R} \times \bar{\Omega}$ and T–periodic in t. It follows from [19, Chapter II] that (7) admits an evolution operator $V_{S_i}(t, s) : \mathbb{Y}^+ \to \mathbb{Y}^+$ for $s \le t$ satisfying $V_{S_i}(t, t) = I$, $V_{S_i}(t, s)V_{S_i}(s, \rho) = V_{S_i}(t, \rho)$ for $0 \le \rho \le s \le t$ and $V_{S_i}(t, 0)(\phi_{S_i})(x) = w_i(t, x; \phi_{S_i})$ for $t \ge 0$, $x \in \Omega$ and $\phi_{S_i} \in \mathbb{Y}^+$, where $w_i(t, x; \phi_{S_i})$ is a solution of (7) for $i = 1, 2$. Similarly, we take into account the following system:

$$\begin{cases} \frac{\partial \bar{w}_i}{\partial t} = D_i\Delta \bar{w}_i(t, x) - r_i(t, x)\bar{w}_i(t, x),\ t > 0,\ x \in \Omega,\ i = 1, 2, \\ \frac{\partial}{\partial n} \bar{w}_i(t, x) = 0,\ t > 0,\ x \in \partial\Omega,\ i = 1, 2, \\ \bar{w}_i(0, x) = \phi_i(x),\ x \in \Omega,\ \phi_i \in \mathbb{Y}^+,\ i = 1, 2, \end{cases}$$

where $D_i > 0 (i = 1, 2)$ and $r_i(t, x)(i = 1, 2)$ are Hölder continuous and nonnegative nontrivial functions on $\mathbb{R} \times \bar{\Omega}$ and T–periodic in t. Let $V_i(t, s)(i = 1, 2)$ be the evolution operators determined by the above system and have the similar properties as $V_{S_i}(t, s)$. Due to the periodicity of coefficients, it follows from [11, Lemma 6.1] that $V_{S_i}(t, s) = V_{S_i}(t + T, s + T)$ and $V_i(t, s) = V_i(t + T, s + T)$ hold for $(t, s) \in \mathbb{R}^2$, $t \ge s$ and $i = 1, 2$. In addition, for any $t, s \in \mathbb{R}$ and $s < t$, $V_{S_i}(t, s)$ and $V_i(t, s)$ are compact, analytic and strongly positive operators on $\mathbb{Y}^+$ for $i = 1, 2$. Together [11, Theorem 6.6] with $\alpha = 0$, we get that there exist constants $\mathcal{Q} \ge 1$ and $c_0 \in \mathbb{R}$ such that

$$\|V_{S_i}(t, s)\|, \|V_i(t, s)\| \le \mathcal{Q}e^{-c_0(t-s)},\ \forall t \ge s,\ t, s \in \mathbb{R},\ i = 1, 2.$$

Define $F = (F_{S_1}, F_{S_2}, F_1, F_2)^T : [0, \infty) \times \mathbb{C}^+_\tau \to \mathbb{X}$ by

$$\begin{aligned} F_{S_i}(t, \phi) =& \mu_i(t, \cdot) - \beta_{i1}(t, \cdot)g_{i1}(\phi_{S_i}(0, \cdot), \phi_1(0, \cdot)) - \beta_{i2}(t, \cdot)g_{i2}(\phi_{S_i}(0, \cdot), \phi_2(0, \cdot)), \\ F_i(t, \phi) =& \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(t, t - a, \cdot, y)\Big(\beta_{i1}(t - a, y)g_{i1}(\phi_{S_i}(-a, y), \phi_1(-a, y)) \\ &+ \beta_{i2}(t - a, y)g_{i2}(\phi_{S_i}(-a, y), \phi_2(-a, y))\Big)dyda \end{aligned}$$

for $t \ge 0$, $x \in \bar{\Omega}$, $\phi = (\phi_{S_1}, \phi_{S_2}, \phi_1, \phi_2) \in \mathbb{C}^+_\tau$ and $i = 1, 2$. Let

$$U(t, s) := \begin{pmatrix} V_{S_1}(t, s) & 0 & 0 & 0 \\ 0 & V_{S_2}(t, s) & 0 & 0 \\ 0 & 0 & V_1(t, s) & 0 \\ 0 & 0 & 0 & V_2(t, s) \end{pmatrix}$$

and $U(t,s):\mathbb{X}\to\mathbb{X}$ be an evolution operator for $(t,s)\in\mathbb{R}^2$ with $t\geq s$. Let

$$A(t):=\begin{pmatrix} A_{S_1}(t) & 0 & 0 & 0\\ 0 & A_{S_2}(t) & 0 & 0\\ 0 & 0 & A_1(t) & 0\\ 0 & 0 & 0 & A_2(t)\end{pmatrix},$$

where $A_{S_i}(t)$ and $A_i(t)(i=1,2)$ are defined by

$$\begin{aligned}&D(A_{S_i}(t))=\{\phi\in C^2(\bar{\Omega})\mid \partial_n\phi=0 \text{ on } \partial\Omega\},\\ &A_{S_i}(t)\phi(x)=D_{S_i}\Delta\phi(x)-d_i(t,x)\phi(x),\ \forall\phi\in D(A_{S_i}(t))\end{aligned}$$

and

$$\begin{aligned}&D(A_i(t))=\{\phi\in C^2(\bar{\Omega})\mid \partial_n\phi=0 \text{ on } \partial\Omega\},\\ &A_i(t)\phi(x)=D_i\Delta\phi(x)-r_i(t,x)\phi(x),\ \forall\phi\in D(A_i(t)),\end{aligned}$$

respectively. Then (6) can be written as the following Cauchy problem:

$$\begin{cases}\frac{\partial u(t,x)}{\partial t}=A(t)u(t,x)+F(t,u_t),\ t>0,\ x\in\Omega,\\ u(\zeta,x)=\phi(\zeta,x),\ \zeta\in[-\tau,0],\ x\in\Omega,\end{cases}\tag{8}$$

where $u(t,x):=(u_{S_1}(t,x),u_{S_2}(t,x),u_1(t,x),u_2(t,x))$. Moreover, it can be rewritten as the following integral equation

$$u(t,\phi)=U(t,0)\phi(0)+\int_0^t U(t,s)F(s,u_s)ds,\ t\geq 0,\ \phi\in\mathbb{C}_\tau^+.\tag{9}$$

A solution of (9) is called a mild solution of (8).

Lemma 3.1. *For every initial value function $\phi\in\mathbb{C}_\tau^+$, system (6) has a unique mild solution $u(t,\phi)$ on $[0,+\infty)$ with $u_0=\phi$. Furthermore, system (6) generates a T–periodic semiflow $\Phi_t(\cdot):=u_t(\cdot):\mathbb{C}_\tau^+\to\mathbb{C}_\tau^+$, namely, $\Phi_t(\phi)(s,x)=u_t(\phi)(s,x)=u(t+s,x;\phi)$ for each $\phi\in\mathbb{C}_\tau^+$, $t\geq 0$, $s\in[-\tau,0)$, $x\in\Omega$ and $\Phi_T:\mathbb{C}_\tau^+\to\mathbb{C}_\tau^+$ has a global compact attractor in $\mathbb{C}_\tau^+$.*

Proof. We firstly show the local existence of the unique mild solution. It is obvious that $F(t,\phi)$ is locally Lipschitz continuous. By Martin and Smith [37, Corollary 3] and Smith [47, Theorem 7.3.1], it is necessary to prove

$$\lim_{k\to 0^+}\operatorname{dist}(\phi(0)+kF(t,\phi),\ \mathbb{X}^+)=0,\ \forall(t,\phi)\in[0,\infty)\times\mathbb{C}_\tau^+.\tag{10}$$

For any $t\geq 0$, $x\in\bar{\Omega}$, $\phi\in\mathbb{C}_\tau^+$ and $k\geq 0$, we have
$\phi(0,x)+kF(t,\phi)(x)$

$$\begin{aligned}&=\begin{pmatrix}\phi_{S_1}(0,x)+k\left(\mu_1(t,x)-(\sum_{i=1}^2\beta_{1i}(t,x)g_{1i}(\phi_{S_1},\phi_i)(0,x))\right)\\ \phi_{S_2}(0,x)+k\left(\mu_2(t,x)-(\sum_{i=1}^2\beta_{2i}(t,x)g_{2i}(\phi_{S_2},\phi_i)(0,x))\right)\\ \phi_1(0,x)+k\Big(f_1\circ\big(\Gamma_1*\big(\beta_{11}g_{11}+\beta_{12}g_{12}\big)\big)\Big)(t,x)\\ \phi_2(0,x)+k\Big(f_2\circ\big(\Gamma_2*\big(\beta_{21}g_{21}+\beta_{22}g_{22}\big)\big)\Big)(t,x)\end{pmatrix}\\ &\geq\begin{pmatrix}\phi_{S1}(0,x)\left(1-k\left(\sum_{i=1}^2\beta_{1i}(t,x)\frac{g_{1i}(\phi_{S_1},\phi_i)(0,x)}{\phi_{S_1}(0,x)}\right)\right)\\ \phi_{S2}(0,x)\left(1-k\left(\sum_{i=1}^2\beta_{2i}(t,x)\frac{g_{2i}(\phi_{S_2},\phi_i)(0,x)}{\phi_{S_2}(0,x)}\right)\right)\\ \phi_1(0,x)\\ \phi_2(0,x)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}&\Big(f_i \circ \big(\Gamma_i * (\beta_{i1}g_{i1} + \beta_{i2}g_{i2})\big)\Big)(t,x)\\ =&\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\Big(\beta_{i1}(t-a,y)g_{i1}(\phi_{S_i}(-a,y),\phi_1(-a,y))\\ &+\beta_{i2}(t-a,y)g_{i2}(\phi_{S_i}(-a,y),\phi_2(-a,y))\Big)dyda,\ i=1,2\end{aligned}$$

and $g_{ij}(\phi_{S_i},\phi_j)(0,x) = g_{ij}(\phi_{S_i}(0,x),\phi_j(0,x))$. The above inequality implies that (10) holds when k is small enough. Consequently, by [37, Corollary 4] with $K = \mathbb{X}^+$ and $S(t,s) = U(t,s)$, system (6) admits a unique mild solution $u(t,x;\phi)$ with $u_0(\cdot,\cdot;\phi) = \phi$ on its maximal interval of existence $t \in [0,\tilde{t}_\phi)$, where $\tilde{t}_\phi \le \infty$ and $u(t,\cdot;\phi) \in \mathbb{X}^+$, $\forall t \in [0,\tilde{t}_\phi)$. Furthermore, $u(t,x;\phi)$ is a classic solution for $t > \tau$ by using the analytic of $U(t,s)$ for any $s,t \in \mathbb{R}$ with $s < t$.

Consider the following time–periodic reaction–diffusion equation:

$$\begin{cases}\frac{\partial \omega_i(t,x)}{\partial t} = D_{Si}\Delta\omega_i(t,x) + \mu_i(t,x) - d_i(t,x)\omega_i(t,x),\ t>0,\ x\in\Omega,\\ \frac{\partial}{\partial n}\omega_i(t,x) = 0,\ t>0,\ x\in\partial\Omega,\end{cases} \tag{11}$$

where $i = 1,2$. It follows from [67, Lemma 2.1] that system (11) admits a unique positive T–periodic solution $\omega_i^*(t,x)$ which is globally asymptotically stable in $\mathbb{Y}^+$ for $i = 1,2$. Since the $u_{S_i}(i = 1,2)$ equations of system (6) are dominated by (11), respectively, there exists a positive constant B_s such that for any $\phi \in \mathbb{C}_\tau^+$, there is a positive integer $l_s = l_s(\phi) > 0$ such that $u_{S_i}(t,x;\phi) \le B_s$ for any $t \ge l_sT$, $x \in \bar{\Omega}$ and $i = 1,2$.

In view of (iv) of (H1), we have for $t > 0$ and $x \in \Omega$,

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} \le\ & D_i\Delta u_i(t,x) - r_i(t,x)u_i(t,x) + \eta_i\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\\ & u_{Si}(t-a,y)\left(\beta_{i1}(t-a,y)+\beta_{i2}(t-a,y)\right)dyda\end{aligned}$$

with Neumann boundary condition

$$\frac{\partial}{\partial n}u_i(t,x) = 0,\ \forall x\in\partial\Omega.$$

It follows from the comparison principle that there exists a constant $\tilde{B} > 0$ such that for any $\phi \in \mathbb{C}_\tau^+$, there is a positive integer $l_i > l_s$ such that $u_i(t,x;\phi) \le \tilde{B}$ for any $t \ge l_iT + \tau$, $x \in \bar{\Omega}$ and $i = 1,2$.

Define $\Phi_t : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ by $\Phi_t(\phi)(s,x) = u_t(\phi)(s,x) = u(t+s,x;\phi)$ for $t > 0$, $s \in [-\tau,0]$, $x \in \bar{\Omega}$ and $\phi \in \mathbb{C}_\tau^+$. Similar to the proof of [67, Lemma 2.1], we get that $\{\Phi_t\}_{t\ge0}$ is a T–periodic semiflow on $\mathbb{C}_\tau^+$. From the above discussion, we have that Φ_t is point dissipative. Let $n_0 := \min\{n \in \mathbb{N} : nT > 2\tau\}$. Then by the standard parabolic estimates, we conclude that $\Phi_T^{n_0} = u_{n_0T}$ is compact. Following from [35, Theorem 2.9], one has that $\Phi_T : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ has a global compact attractor. The proof is completed. □

3.2. **Basic reproduction number.** Let $C_T(\mathbb{R}\times\bar{\Omega},\mathbb{R})$ be the ordered Banach space consisting of all T–periodic and continuous functions from $\mathbb{R}\times\bar{\Omega}$ to $\mathbb{R}$, where $\|\phi\|_{C_T} = \max_{t\in[0,T],x\in\bar{\Omega}}|\phi(t,x)|$ for any $\phi \in C_T$. Denote C_T^+ as the positive cone of C_T, that is,

$$C_T^+ := \{\phi \in C_T : \phi(t,x) \ge 0, \forall t\in\mathbb{R}, x\in\bar{\Omega}\}.$$

Let $\mathbf{C}_T(\mathbb{R}\times\bar{\Omega},\mathbb{R}\times\mathbb{R}) = C_T(\mathbb{R}\times\bar{\Omega},\mathbb{R})\times C_T(\mathbb{R}\times\bar{\Omega},\mathbb{R})$ with the norm $\|\phi\|_{\mathbf{C}_T} = \sum_{i=1}^2 \|\phi_i\|_{C_T}$ for any $\phi\in\mathbf{C}_T$. Similarly, we define $\mathbf{C}_T^+$ as the positive cone of $\mathbf{C}_T$, namely,

$$\mathbf{C}_T^+ := \{\phi = (\phi_1,\phi_2)\in\mathbf{C}_T : \phi_i(t,x)\geq 0, \forall t\in\mathbb{R}, x\in\bar{\Omega}, i=1,2\}.$$

For $\tau\geq 0$, define $\mathbb{D} = C([-\tau,0],\mathbb{Y}\times\mathbb{Y})$ with the norm $\|\phi\|_{\mathbb{D}} = \max_{\theta\in[-\tau,0]}\|\phi(\theta)\|_{\mathbb{Y}\times\mathbb{Y}}$ and $\mathbb{D}^+ := C([-\tau,0],\mathbb{Y}^+\times\mathbb{Y}^+)$, then $(\mathbb{D},\mathbb{D}^+)$ is a strongly ordered Banach space.

Setting $u_1\equiv 0$ and $u_2\equiv 0$, we have the following equations for the densities of the susceptible population $u_{S_i}(t,x)(i=1,2)$

$$\begin{cases} \frac{\partial u_{S_i}(t,x)}{\partial t} = D_{S_i}\Delta u_{S_i}(t,x) + \mu_i(t,x) - d_i(t,x)u_{S_i}(t,x),\ t>0,\ x\in\Omega,\ i=1,2, \\ \frac{\partial}{\partial n}u_{S_i}(t,x) = 0,\ t>0,\ x\in\partial\Omega,\ i=1,2, \end{cases} \tag{12}$$

respectively. It follows from [67, Lemma 2.1] that (12) admit positive solutions $u_{S_i}^*(i=1,2)$ which are unique, globally asymptotically stable and T–periodic in $t\in\mathbb{R}$, respectively. As a consequence, the function $(u_{S_1}^*, u_{S_2}^*, 0, 0)$ is called the disease–free periodic solution of (6). Linearizing the third and the forth equations of system (6) at $(u_{S_1}^*, u_{S_2}^*, 0, 0)$ and according to (iii) of (H1), we have the following system:

$$\begin{cases} \frac{\partial\omega_1(t,x)}{\partial t} = D_1\Delta\omega_1(t,x) - r_1(t,x)\omega_1(t,x) + \int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad\times\Big(\beta_{11}(t-a,y)\partial_v g_{11}(u_{S_1}^*(t-a,y),0)\omega_1(t-a,y) + \beta_{12}(t-a,y) \\ \qquad\partial_v g_{12}(u_{S_1}^*(t-a,y),0)\omega_2(t-a,y)\Big)dyda,\ t>0,\ x\in\Omega, \\ \frac{\partial\omega_2(t,x)}{\partial t} = D_2\Delta\omega_2(t,x) - r_2(t,x)\omega_2(t,x) + \int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad\times\Big(\beta_{21}(t-a,y)\partial_v g_{21}(u_{S_2}^*(t-a,y),0)\omega_1(t-a,y) + \beta_{22}(t-a,y) \\ \qquad\partial_v g_{22}(u_{S_2}^*(t-a,y),0)\omega_2(t-a,y)\Big)dyda,\ t>0,\ x\in\Omega, \\ \omega_i(s,x) = \phi_i(s,x),\ i=1,2,\ \phi=(\phi_1,\phi_2)\in\mathbb{D},\ s\in[-\tau,0],\ x\in\Omega, \\ \frac{\partial\omega_i(t,x)}{\partial n} = 0,\ t>0,\ x\in\partial\Omega,\ i=1,2. \end{cases} \tag{13}$$

Define operators $C_{ij} : C_T(\mathbb{R}\times\bar{\Omega},\mathbb{R})\to C_T(\mathbb{R}\times\bar{\Omega},\mathbb{R})(i,j=1,2)$ by

$$\begin{aligned} &\big(C_{ij}\psi_j\big)(t,x) \\ = &\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\beta_{ij}(t-a,y) \\ &\times\partial_v g_{ij}(u_{S_i}^*(t-a,y),0)\psi_j(t-a,y)dyda. \end{aligned}$$

Suppose that $\phi(s,x) := (\phi_1(s,x),\phi_2(s,x))$ is the initial distribution of infectious individuals at time $s\in\mathbb{R}$ and the spatial location $x\in\bar{\Omega}$. Given $t\in\mathbb{R}$. Due to the synthetical influence of mobility, mortality and recovery, $\left(V_i(t-a,s)\phi_i(s)\right)(x)$, where $s<t-a$ represents the density distribution at location x of those infective individuals who were infected at time s and remained infective at time $t-a$ when time evolved from s to $t-a$ for $a\in[0,\tau]$. Furthermore, $\int_{-\infty}^{t-a}\left(V_i(t-a,s)\phi_i(s)\right)(x)ds$ denotes the density distribution of the accumulative infective individuals of the i–th group at locations x and time $t-a$ for all previous time $s<t-a$ when time evolved from the previous time s to $t-a$. After that, the term

$$\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\Big\{\beta_{i1}(t-a,y)\partial_v g_{i1}(u_{S_i}^*(t-a,y),0)$$

$$
\begin{aligned}
&\times \int_{-\infty}^{t-a} \Big(V_1(t-a,s)\phi_1(s)\Big)(y)ds + \beta_{i2}(t-a,y)\partial_v g_{i2}(u^*_{S_i}(t-a,y),0) \\
&\times \int_{-\infty}^{t-a} \Big(V_2(t-a,s)\phi_2(s)\Big)(y)ds\Big\}dyda \\
= \quad & \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\Big\{\sum_{j=1}^{2}\beta_{ij}(t-a,y)\partial_v g_{ij}(u^*_{S_i}(t-a,y),0) \\
&\times \int_0^{+\infty}\Big(V_j(t-a,t-a-s)\phi_j(t-a-s)\Big)(y)ds\Big\}dyda
\end{aligned}
$$

represents the distribution of new infected individuals of the i–th group at location x and time t for $i=1,2$. As a consequence, we can define the next generation infection operator $\mathcal{L}$ as

$$\mathcal{L}(\phi)(t,x) := (\mathcal{L}_1(\phi)(t,x), \mathcal{L}_2(\phi)(t,x)),$$

where

$$
\begin{aligned}
&\mathcal{L}_i(\phi)(t,x) \\
=&\int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\Big\{\sum_{j=1}^{2}\beta_{ij}(t-a,y)\partial_v g_{ij}(u^*_{S_i}(t-a,y),0) \\
&\times \int_0^{+\infty}\Big(V_j(t-a,t-a-s)\phi_j(t-a-s)\Big)(y)ds\Big\}dyda
\end{aligned}
$$

for $i=1,2$. It is obvious that $\mathcal{L}$ is a positive and bounded linear operator on $\mathbf{C}_T$. Let $r(\mathcal{L})$ be the spectral radius of $\mathcal{L}$. Similar to [5, 12, 57, 71, 67], denote the spectral radius of $\mathcal{L}$ as the basic reproduction number R_0 of model (6), that is,

$$R_0 := r(\mathcal{L}).$$

Next, we define an operator $\hat{\mathcal{L}}(\phi)(t,x) : \mathbf{C}_T \to \mathbf{C}_T$ by

$$\hat{\mathcal{L}}(\phi)(t,x) := (\hat{\mathcal{L}}_1(\phi)(t,x), \hat{\mathcal{L}}_2(\phi)(t,x)),$$

where

$$\hat{\mathcal{L}}_i(\phi)(t,x) = \int_0^\infty \sum_{j=1}^{2}\Big(V_i(t,t-s)\left(C_{ij}\phi_j\right)(t-s)\Big)(x)ds, \ t\in\mathbb{R}, \ s\geq 0, \ i=1,2.$$

Clearly, $\hat{\mathcal{L}}$ is a compact, positive and bounded linear operator on $\mathbf{C}_T$. Let

$$\mathcal{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix}, \ \tilde{V}(t,s) = \begin{pmatrix} V_1(t,s) & 0 \\ 0 & V_2(t,s)\end{pmatrix}$$

and

$$A(\phi)(t,x) = \mathcal{C}(\phi)(t,x), \ B(\phi)(t,x) = \int_0^\infty (\tilde{V}(t,t-s)\phi(t-s))(x)ds.$$

Then one has $\mathcal{L} = AB$ and $\hat{\mathcal{L}} = BA$. It follows that $R_0 = r(\mathcal{L}) = r(\hat{\mathcal{L}})$, where $r(\hat{\mathcal{L}})$ is the spectral radius of the operator $\hat{\mathcal{L}}$.

As the previous discussion, there exist constants $\mathcal{Q} > 1$ and $c_i \in \mathbb{R}$ such that

$$\|V_i(t,s)\| \leq \mathcal{Q}e^{c_i(t-s)}, \ \forall t\geq s, \ t,s\in\mathbb{R}, \ i=1,2.$$

It follows that $c_i^* := \bar{\omega}(V_i) \leq c_i$, where

$$\bar{\omega}(V_i) = \inf\{\omega \mid \exists M\geq 1, \forall s\in\mathbb{R}, \ t\geq 0 : \|V_i(t+s,s)\| \leq Me^{\omega t}\}$$

is the exponent growth bound of the evolution operator $V_i(t,s)$. Define $c^* = \max\{c_1^*, c_2^*\}$. $r(V_i(T,0))$ is defined as the spectral radius of $V_i(T,0)$ for $i=1,2$. In addition, $V_i(t,0)$ is compact and strongly positive on $\mathbb{Y}$ for any $t>0$ and $i=1,2$. By the Krein–Rutman theorem [19, Theorem 7.2], we have $r(V_i(T,0))>0$ for $i=1,2$. It further follows from [19, Lemma 14.2] that $r(V_i(T,0))<1$ for $i=1,2$. According to [51, Proposition 5.6] with $s=0$, one has $c_i^*<0$ for $i=1,2$.

For any given $\lambda\in(c^*,\infty)$, we introduce an operator $\hat{\mathcal{L}}_\lambda$ on $\mathbf{C}_T$:

$$\hat{\mathcal{L}}_\lambda(\phi)(t,x) := \int_0^\infty e^{-\lambda s}\left(\tilde{V}(t,t-s)\left(\mathcal{C}(\phi)\right)(t-s)\right)(x)ds.$$

Clearly, $\hat{\mathcal{L}}_0 = \hat{\mathcal{L}}$. It follows that the operator $\hat{\mathcal{L}}_\lambda$ is bounded for $\lambda\in(c^*,\infty)$. Moreover, the compactness of $V_i(t,s)(i=1,2)$, $t>s$, implies that $\hat{\mathcal{L}}_\lambda$ is compact. Denote $\rho(\lambda)$ as the spectral radius of $\hat{\mathcal{L}}_\lambda$ for $\lambda\in(c^*,\infty)$. It is easy to see that $R_0 = r(\mathcal{L}) = r(\hat{\mathcal{L}}) = \rho(0)$. Similar to the arguments in [4, Lemma 1] and [67, Lemma 3.2], we can show the following properties of the function $\rho(\lambda)$.

Lemma 3.2. *For $\lambda\in(c^*,\infty)$, the following statements are true for $\rho(\lambda)$*

- **(i):** *$\rho(\lambda)$ is continuous and non–increasing;*
- **(ii):** *$\rho(\infty)=0$;*
- **(iii):** *$\rho(\lambda)=1$ has at most one solution; ρ is either strictly decreasing in $\lambda\in(c^*,\infty)$, or strictly decreasing in $\lambda\in(c^*,b)$ for some $b>c^*$, and $\rho(\lambda)=0$ in $\lambda\in[b,\infty)$.*

Let ϵ be a positive parameter. Consider the following periodic time–delayed nonlocal equations:

$$\begin{cases} \frac{\partial\omega_1^\epsilon(t,x)}{\partial t} = D_1\Delta\omega_1^\epsilon(t,x) - r_1(t,x)\omega_1^\epsilon(t,x) + \int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad\times\Big\{\Big(\beta_{11}(t-a,y)+\epsilon\Big)\partial_v g_{11}(u_{S_1}^*(t-a,y),0)\omega_1^\epsilon(t-a,y) \\ \qquad+\Big(\beta_{12}(t-a,y)+\epsilon\Big)\partial_v g_{12}(u_{S_1}^*(t-a,y),0)\omega_2^\epsilon(t-a,y)\Big\}dyda, \\ \qquad t>0,\ x\in\Omega, \\ \frac{\partial\omega_2^\epsilon(t,x)}{\partial t} = D_2\Delta\omega_2^\epsilon(t,x) - r_2(t,x)\omega_2^\epsilon(t,x) + \int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad\times\Big\{\Big(\beta_{21}(t-a,y)+\epsilon\Big)\partial_v g_{21}(u_{S_2}^*(t-a,y),0)\omega_1^\epsilon(t-a,y) \\ \qquad+\Big(\beta_{22}(t-a,y)+\epsilon\Big)\partial_v g_{22}(u_{S_2}^*(t-a,y),0)\omega_2^\epsilon(t-a,y)\Big\}dyda, \\ \qquad t>0,\ x\in\Omega, \\ \omega_i^\epsilon(s,x)=\psi_i(s,x),\ \psi=(\psi_1,\psi_2)\in\mathbb{D},\ s\in[-\tau_i,0],\ x\in\Omega,\ i=1,2, \\ \frac{\partial\omega_1^\epsilon(t,x)}{\partial n} = \frac{\partial\omega_2^\epsilon(t,x)}{\partial n} = 0,\ t>0,\ x\in\partial\Omega. \end{cases} \tag{14}$$

Define the Poincaré map of (14) $\mathcal{P}^\epsilon:\mathbb{D}\to\mathbb{D}$ by $\mathcal{P}^\epsilon(\psi)=\omega_T^\epsilon(\psi)$ for all $\psi\in\mathbb{D}$, where

$$\omega_T^\epsilon(\psi)(s,x) = \omega^\epsilon(s+T,x;\psi) = (\omega_1^\epsilon(s+T,x;\psi),\ \omega_2^\epsilon(s+T,x;\psi))$$

for all $(s,x)\in[-\tau,0]\times\bar{\Omega}$ ($\tau:=\max\{\tau_1,\tau_2\}$), and ω_t^ϵ is the solution map of (14). Let $n_0 := \min\{n\in\mathbb{N}: nT>2\tau\}$. $(\mathcal{P}^\epsilon)^{n_0}:\mathbb{D}\to\mathbb{D}$ is denoted by $(\mathcal{P}^\epsilon)^{n_0}(\phi) = \omega^\epsilon(n_0T+s,x;\phi)$ for all $(s,x)\in[-\tau,0]\times\bar{\Omega}$. Define r_0^ϵ as the spectral radius of $\mathcal{P}^\epsilon$. Without loss of generality, we replace $\mathcal{P}$ and r_0 with $\mathcal{P}^0$ and r_0^0, respectively. It follows from [25, Section 3] (see also [47, Section 5.3]) that $\omega^\epsilon(t,x;\phi)>0$ for $t>\tau$, $x\in\bar{\Omega}$, $\phi\in\mathbb{D}^+$ with $\phi\not\equiv 0$, and $\omega_t^\epsilon(\cdot,\cdot;\phi)$ is strongly positive for $t>2\tau$. Moreover, ω_t^ϵ is compact on $\mathbb{D}^+$ for all $t>2\tau$. Hence, $(\mathcal{P}^\epsilon)^{n_0} = \omega_{n_0T}^\epsilon(\cdot)$ is compact and strongly positive. By [30, Lemma 3.1], r_0^ϵ is a simple eigenvalue of $\mathcal{P}^\epsilon$ having a strong positive eigenvector $\psi \in \mathbb{D}^+$ and the modulus of any other eigenvalue is less than r_0^ϵ. Assume that $\omega^\epsilon(t,x;\psi)$ is the solution of system (14) with $\omega^\epsilon(s,x;\psi) = \psi(s,x)$ for all $s \in [-\tau, 0]$, $x \in \Omega$. We can conclude from the strong positivity of ψ that $\omega^\epsilon(\cdot,\cdot;\psi) \gg 0$. Let $\mu^\epsilon = \frac{\ln r_0^\epsilon}{T}$ and $\mathcal{V}^\epsilon(t,x) = e^{-\mu^\epsilon t}\omega^\epsilon(t,x;\psi)$ for all $t > \tau$ and $x \in \bar{\Omega}$. By arguments similar to those in [25, Lemma 3.2] and [62, Theorem 2.1], we have that $\mathcal{V}^\epsilon(t,x)$ is a nontrivial and nonnegative T–periodic function and $e^{\mu^\epsilon t}\mathcal{V}^\epsilon(t,x)$ is a solution of (14). Furthermore, by the strong positivity of $V_i(t,s)$, $t > s$, we have $\mathcal{V}^\epsilon(t,x) > 0$ for any $t \in \mathbb{R}$ and $x \in \bar{\Omega}$. Thus, we have the following lemma.

Lemma 3.3. *Let $\mu^\epsilon = \frac{\ln r_0^\epsilon}{T}$. Then there exists a positive T–periodic function $\mathcal{V}^\epsilon(t,x)$ such that $e^{\mu^\epsilon t}\mathcal{V}^\epsilon(t,x)$ is a solution of* (14).

In the following, by the methods similar to [71, Section 2] and [67, Section 3], we prove that $R_0 - 1$ has the same sign as $r_0 - 1$.

Lemma 3.4. *Let $\mu = \frac{\ln r_0}{T}$. If $r_0 > r(V_i(T,0))$ for $i = 1,2$, then $\rho(\mu) = 1$.*

Proof. By virtue of [51, Proposition A.2], we have $c_i^* := \frac{\ln r(V_i(T,0))}{T} < 0$. Since $r_0 > r(V_i(T,0))$, one has $\mu > c^*$. According to the continuity of solution with respect to parameter $\epsilon > 0$, we have $\lim_{\epsilon\to 0}\mathcal{P}^\epsilon = \mathcal{P}$. In view of the upper semi–continuity of the spectral [26, Section IV.3.1] and the continuity of a finite system of eigenvalues [26, Section IV.3.5], we get $\lim_{\epsilon\to 0^+} r_0^\epsilon = r_0$. It follows from Lemma 3.3 that $\lim_{\epsilon\to 0^+}\mu^\epsilon = \mu$.

Let $\epsilon_n = \frac{1}{n}$ for $n \geq 1$ such that $r_0^{\epsilon_n} > r(V_i(T,0))$. We define

$$
\begin{aligned}
&(C_{ij}^{\epsilon_n}\psi_j)(t,x)\\
&:= \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(t,t-a,x,y)\Big(\beta_{ij}(t-a,y) + \epsilon_n\Big)\\
&\quad\times \partial_v g_{ij}(u_{S_i}^*(t-a,y),0)\psi_j(t-a,y)dyda,\\
&\hat{\mathcal{L}}^{\epsilon_n}(\psi)(t,x) = (\hat{\mathcal{L}}_1^{\epsilon_n}(\psi)(t,x), \hat{\mathcal{L}}_2^{\epsilon_n}(\psi)(t,x)),\\
&\hat{\mathcal{L}}_\lambda^{\epsilon_n}(\psi)(t,x) = (\hat{\mathcal{L}}_{1\lambda}^{\epsilon_n}(\psi)(t,x), \hat{\mathcal{L}}_{2\lambda}^{\epsilon_n}(\psi)(t,x)),
\end{aligned}
$$

where

$$\hat{\mathcal{L}}_i^{\epsilon_n}(\psi)(t,x) = \int_0^\infty \Big\{V_i(t,t-s)\Big((C_{i1}^{\epsilon_n}\phi_1)(t-s) + (C_{i2}^{\epsilon_n}\phi_2)(t-s)\Big)\Big\}(x)ds,$$

$$\hat{\mathcal{L}}_{i\lambda}^{\epsilon_n}(\psi)(t,x) = \int_0^\infty e^{-\lambda s}\Big\{V_i(t,t-s)\Big((C_{i1}^{\epsilon_n}\phi_1)(t-s) + (C_{i2}^{\epsilon_n}\phi_2)(t-s)\Big)\Big\}(x)ds$$

for $i = 1,2$.

According to Lemma 3.3, there is a positive periodic function $\mathcal{V}^{\epsilon_n}(t,x)$ such that $\omega^{\epsilon_n}(t,x) = e^{\mu^{\epsilon_n}t}\mathcal{V}^{\epsilon_n}(t,x)$ is a solution of (14). That is, it satisfies for $t \geq s$ and $s \in \mathbb{R}$,

$$
\begin{cases}
\omega_1^{\epsilon_n}(t,\cdot) = V_1(t,s)\omega_1^{\epsilon_n}(s) + \int_s^t V_1(t,\eta)\Big(\left(C_{11}^{\epsilon_n}\omega_1^{\epsilon_n}\right)(\eta) + \left(C_{12}^{\epsilon_n}\omega_2^{\epsilon_n}\right)(\eta)\Big)d\eta,\\
\omega_2^{\epsilon_n}(t,\cdot) = V_2(t,s)\omega_2^{\epsilon_n}(s) + \int_s^t V_2(t,\eta)\Big(\left(C_{21}^{\epsilon_n}\omega_1^{\epsilon_n}\right)(\eta) + \left(C_{22}^{\epsilon_n}\omega_2^{\epsilon_n}\right)(\eta)\Big)d\eta,
\end{cases}
$$

which implies that

$$\begin{cases} e^{\mu^{\epsilon_n} t}\mathcal{V}_1^{\epsilon_n}(t,\cdot) = V_1(t,s)\Big(e^{\mu^{\epsilon_n} s}\mathcal{V}_1^{\epsilon_n}(s)\Big) + \int_s^t e^{\mu^{\epsilon_n}\eta}V_1(t,\eta)\Big(\left(C_{11}^{\epsilon_n}\mathcal{V}_1^{\epsilon_n}\right)(\eta) \\ \quad + \left(C_{12}^{\epsilon_n}\mathcal{V}_2^{\epsilon_n}\right)(\eta)\Big)d\eta, \\ e^{\mu^{\epsilon_n} t}\mathcal{V}_2^{\epsilon_n}(t,\cdot) = V_2(t,s)\Big(e^{\mu^{\epsilon_n} s}\mathcal{V}_2^{\epsilon_n}(s)\Big) + \int_s^t e^{\mu^{\epsilon_n}\eta}V_2(t,\eta)\Big(\left(C_{21}^{\epsilon_n}\mathcal{V}_1^{\epsilon_n}\right)(\eta) \\ \quad + \left(C_{22}^{\epsilon_n}\mathcal{V}_2^{\epsilon_n}\right)(\eta)\Big)d\eta, \end{cases} \tag{15}$$

where $t \geq s$ and $s \in \mathbb{R}$. Since $r_0^{\epsilon_n} > r(V_i(T,0))$, then we have $\mu^{\epsilon_n} := \frac{\ln r_0^{\epsilon_n}}{T} > c_i^*$ and $[V_i(t,s)(e^{\mu^{\epsilon_n} s}\mathcal{V}_i^{\epsilon_n}(s))] \to 0$ as $s \to -\infty$. Letting $s \to -\infty$ in the first equation of (15), we get

$$\begin{aligned} \mathcal{V}_1^{\epsilon_n}(t,\cdot) &= \int_{-\infty}^t e^{-\mu^{\epsilon_n}(t-\eta)}V_1(t,\eta)\Big(\left(C_{11}^{\epsilon_n}\mathcal{V}_1^{\epsilon_n}\right)(\eta) + \left(C_{12}^{\epsilon_n}\mathcal{V}_2^{\epsilon_n}\right)(\eta)\Big)d\eta. \\ &= \int_0^{+\infty} e^{-\mu^{\epsilon_n} s}V_1(t,t-s)\Big(\left(C_{11}^{\epsilon_n}\mathcal{V}_1^{\epsilon_n}\right)(t-s) + \left(C_{12}^{\epsilon_n}\mathcal{V}_2^{\epsilon_n}\right)(t-s)\Big)ds \end{aligned}$$

Similarly, one has

$$V_2^{\epsilon_n}(t,\cdot) = \int_0^{+\infty} e^{-\mu^{\epsilon_n} s}V_2(t,t-s)\Big(\left(C_{21}^{\epsilon_n}\mathcal{V}_1^{\epsilon_n}\right)(t-s) + \left(C_{22}^{\epsilon_n}\mathcal{V}_2^{\epsilon_n}\right)(t-s)\Big)ds,$$

which implies that $\hat{\mathcal{L}}_{\mu^{\epsilon_n}}(\mathcal{V}^{\epsilon_n})(t)(\cdot) = \mathcal{V}^{\epsilon_n}(t,\cdot)$. Denote $\rho^{\epsilon_n}(\lambda)$ as the spectral radius of $\hat{\mathcal{L}}_\lambda^{\epsilon_n}$ for $\lambda \in (c^*,\infty)$. Since $\beta_{ij}(t,x) + \epsilon_0 > 0$, it follows that $\hat{\mathcal{L}}_\lambda^{\epsilon_n} : \mathbf{C}_T \to \mathbf{C}_T$ is continuous, compact and strongly positive, and hence, the Krein–Rutmann theorem associated with the strongly positivity of $\mathcal{V}^{\epsilon_n}$ imply that $\rho^{\epsilon_n}(\mu^{\epsilon_n}) = 1$. It is easy to see that $\hat{\mathcal{L}}_\lambda^{\epsilon_n}\psi \geq \hat{\mathcal{L}}_\lambda^{\epsilon_{n+1}}\psi$ for all $\psi \in \mathbf{C}_T$. Let $f_n(\lambda) = \rho^{\epsilon_n}(\lambda)$. It then follows from [9, Theorem 1.1] that the sequence $\{f_n\}_{n\geq 1}$ is non–increasing. Similarly, according to the upper semi–continuity of the spectral [26, Section IV 3.1] and the continuity of a finite system of eigenvalues [26, Section IV.3.5], one has $\lim_{n\to\infty}\rho^{\epsilon_n}(\lambda) = \rho(\lambda)$ for any fixed $\lambda \in [a,b] \subset (c^*,\infty)$. Hence, Dini's theorem implies that $\lim_{n\to\infty}\rho^{\epsilon_n}(\lambda) = \rho(\lambda)$ uniformly for $\lambda \in [a,b]$. Choose a sufficient small $\delta > 0$ such that $\mu - \delta > c^*$. By the above analysis, it follows that there exists a constant $N_1 = N_1(\delta) \geq 1$ such that for any $n \geq N_1$,

$$\mu - \delta \leq \mu^{\epsilon_n} \leq \mu + \delta.$$

On the one hand, we obtain from the continuity of $\rho(\lambda)$ for $\lambda \in (c^*,\infty)$ that for any $\eta > 0$, there exists an $N_2 \in \mathbb{N}$ such that,

$$|\rho(\mu^{\epsilon_n}) - \rho(\mu)| < \frac{\eta}{2}.$$

On the other hand, for any $\eta > 0$, there is an $N_3 \geq 1$ such that for $n \geq N_3$,

$$|\rho^{\epsilon_n}(\mu^{\epsilon_n}) - \rho(\mu^{\epsilon_n})| < \frac{\eta}{2}.$$

Thus, for any $\eta > 0$, we have

$$|\rho^{\epsilon_n}(\mu^{\epsilon_n}) - \rho(\mu)| \leq |\rho^{\epsilon_n}(\mu^{\epsilon_n}) - \rho(\mu^{\epsilon_n})| + |\rho(\mu^{\epsilon_n}) - \rho(\mu)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

when $n \geq N := \max\{N_1,N_2,N_3\}$. Letting $n \to \infty$, we have $\rho^{\epsilon_n}(\mu^{\epsilon_n}) \to \rho(\mu)$. Therefore, $\rho(\mu) = 1$. This completes the proof. □

Next, we state the main result of this subsection.

Theorem 3.5. *one has:*

(i): $R_0 > 1$ *if and only if* $r_0 > 1$*;*

(ii): $R_0 = 1$ *if and only if* $r_0 = 1$*;*
(iii): $R_0 < 1$ *if and only if* $r_0 < 1$.

Proof. (i) Assume $R_0 > 1$. In this case one has $\rho(0) > 1$. In view of Lemma 3.2, there exists a constant $\lambda_0 > 0$ such that $\rho(\lambda_0) = 1$. Since $\hat{\mathcal{L}}_{\lambda_0}$ is compact and positive on $\mathbf{C}_T$, it follows from the Krein–Rutmann theorem [19, Theorem 7.1] that $\rho(\lambda_0)$ is an eigenvalue of $\hat{\mathcal{L}}_{\lambda_0}$ with a positive eigenfunction $\psi^* \in \mathbf{C}_T$, that is, $\hat{\mathcal{L}}_{\lambda_0}\psi^* = \psi^*$. Since $V_i(t,s) = V_i(t,r)V_i(r,s)$ for $t \geq r \geq s$, we have

$$\begin{aligned}
&\hat{\mathcal{L}}_{1\lambda_0}(\psi^*)(t,x)\\
= &\int_0^\infty e^{-\lambda_0 s}\left\{V_1(t,t-s)\Big((C_{11}\psi_1^*)(t-s) + (C_{12}\psi_2^*)(t-s)\Big)\right\}(x)ds\\
= &\int_{-\infty}^t e^{-\lambda_0(t-s)}\left\{V_1(t,s)\Big((C_{11}\psi_1^*)(s) + (C_{12}\psi_2^*)(s)\Big)\right\}(x)ds\\
= &\int_{-\infty}^m e^{-\lambda_0(t-s)}\left(V_1(t,s)\left((C_{11}\psi_1^*)(s) + (C_{12}\psi_2^*)(s)\right)\right)(x)ds\\
&+\int_m^t e^{-\lambda_0(t-s)}\left(V_1(t,s)\left((C_{11}\psi_1^*)(s) + (C_{12}\psi_2^*)(s)\right)\right)(x)ds\\
= &e^{-\lambda_0(t-m)}V_1(t,m)\int_{-\infty}^m e^{-\lambda_0(m-s)}\left(V_1(m,s)\sum_{i=1}^2 (C_{1i}\psi_i^*)(s)\right)(x)ds\\
&+\int_m^t e^{-\lambda_0(t-s)}\left(V_1(t,s)\sum_{i=1}^2 (C_{1i}\psi_i^*)(s)\right)(x)ds\\
= &e^{-\lambda_0(t-m)}\Big(V_1(t,m)\psi_1^*(m)\Big)(x)\\
&+e^{-\lambda_0 t}\int_m^t\left(V_1(t,s)\sum_{j=1}^2 C_{1j}\left(e^{\lambda_0 s}\psi_j^*\right)(s)\right)(x)ds,
\end{aligned}$$

namely,

$$\begin{aligned}
&e^{\lambda_0 t}\psi_1^*(t,x)\\
= &\left(V_1(t,m)\left(e^{\lambda_0 m}\psi_1^*(m)\right)\right)(x) + \int_m^t\left(V_1(t,s)\sum_{j=1}^2 C_{1j}\left(e^{\lambda_0 s}\psi_j^*\right)(s)\right)(x)ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&e^{\lambda_0 t}\psi_2^*(t,x)\\
= &\left(V_2(t,m)\left(e^{\lambda_0 m}\psi_2^*(m)\right)\right)(x) + \int_m^t\left(V_2(t,s)\sum_{j=1}^2 C_{2j}\left(e^{\lambda_0 s}\psi_j^*\right)(s)\right)(x)ds.
\end{aligned}$$

Set $\psi_{it}^*(\theta,x) = \psi_i^*(t+\theta,x)$, $\forall\theta \in [-\tau,0]$. It is obvious that

$$\omega(t,x) := (e^{\lambda_0 t}\psi_1^*(t,x), e^{\lambda_0 t}\psi_2^*(t,x))$$

is a solution of (13) with $\omega_0 := e^{\lambda_0 \cdot}\psi^*$. Note that

$$\begin{aligned}
\omega_t(\theta,x) =&(e^{\lambda_0(t+\theta)}\psi_1^*(t+\theta,x), e^{\lambda_0(t+\theta)}\psi_2^*(t+\theta,x))\\
=&e^{\lambda_0 t}(e^{\lambda_0\theta}\psi_{1t}^*(\theta,x), e^{\lambda_0\theta}\psi_{2t}^*(\theta,x))
\end{aligned}$$

for $\theta \in [-\tau, 0]$. Since $\omega(t,\cdot) \not\equiv 0$ on $[0,\infty)$, we have $e^{\lambda_0 \cdot}\psi^* \in \mathbb{D} \setminus \{0\}$. Due to the T-periodicity of $\psi^*(t,x)$, we get

$$\mathcal{P}(e^{\lambda_0 \cdot}\psi^*) = e^{\lambda_0 T}(e^{\lambda_0 \cdot}\psi_{10}^*, e^{\lambda_0 \cdot}\psi_{20}^*).$$

Obviously, $e^{\lambda_0 T}$ is an eigenvalue of $\mathcal{P}$. Thus, one has $r_0 \geq e^{\lambda_0 T} > 1$.

If $r_0 > 1$, then one has $\mu > 0$. Since $c^* < 0$, then $r_0 > r(V_i(T,0))$ for $i = 1,2$. It follows from Lemma 3.4 that $\rho(\mu) = 1$. By the monotonicity of $\rho(\lambda)$, we have $1 = \rho(\mu) < \rho(0) = R_0$.

(ii) Assume $R_0 = 1$. It then follows that $\rho(0) = 1$. By similar arguments to the proof of (i) with $\lambda_0 = 0$, we can prove $r_0 \geq 1$. It is easy to see that $r_0 \geq 1 > r(V_i(T,0))$. Due to Lemma 3.4, we obtain $\rho(\mu) = 1$. By virtue of Lemma 3.2, one has $\mu = 0$, and hence $r_0 = 1$.

Assume $r_0 = 1$. Then we have $r_0 > r(V_i(T,0))$. Thus, by Lemma 3.4, we obtain $\rho(\mu) = \rho(0) = R_0 = 1$.

The conclusions of (iii) is immediately followed from conclusion (i) and (ii). This completes the proof. ☐

3.3. **Persistence and extinction.**

In this subsection, we establish the threshold dynamics of system (6) with respect to R_0. Firstly, the following lemma holds.

Lemma 3.6. *Assume that $(u_{S_1}(t,x;\phi), u_{S_2}(t,x;\phi), u_1(t,x;\phi), u_2(t,x;\phi))$ are a solution of system (6) with $\phi = (\phi_{S_1}, \phi_{S_2}, \phi_1, \phi_2) \in \mathbb{C}_\tau^+$. Then we have*

(i): *If there exists some $t_0 \geq 0$ such that $u_i(t_0,\cdot;\phi) \not\equiv 0 (i = 1,2)$, then one has*

$$u_i(t,x;\phi) > 0, \ \forall t > t_0, \ x \in \bar{\Omega}, \ i = 1,2;$$

(ii): *For any $\phi \in \mathbb{C}_\tau^+$, we always have $u_{S_i}(t,\cdot;\phi) > 0 (i = 1,2), \ \forall t > 0$ and*

$$\liminf_{t\to\infty} u_{S_i}(t,x;\phi) \geq \mathcal{Q}, \ i = 1,2$$

uniformly for $x \in \bar{\Omega}$, where $\mathcal{Q}$ is a positive constant.

Proof. The proof of the lemma is similar to those of [67, Lemma 4.2], so we omit the details. ☐

Secondly, we present the main theorem of this paper.

Theorem 3.7. *Let $u(t,x;\phi)$ be the solution of (6) with $u_0 = \phi \in \mathbb{C}_\tau^+$, then the following two statements are valid:*

(i): *If $R_0 < 1$, then the disease free T–periodic solution $(u_{S_1}^*, u_{S_2}^*, 0, 0)$ is globally attractive in $\mathbb{C}_\tau^+$.*

(ii): *If $R_0 > 1$, then there exists an $\eta > 0$ such that for any $\phi \in \mathbb{C}_\tau^+$ with $\phi_1(0,\cdot) \not\equiv 0$ or $\phi_2(0,\cdot) \not\equiv 0$, one has*

$$\lim_{t\to\infty} \inf u_i(t,x) \geq \eta, \ i = 1,2$$

uniformly for all $x \in \bar{\Omega}$.

Proof. (i) Assume that $R_0 < 1$. It follows from Theorem 3.5 that $r_0 < 1$. Consider the following system with parameter $\epsilon > 0$:

$$\begin{cases} \frac{\partial \overline{u}_1^\epsilon(t,x)}{\partial t} = D_1\Delta \overline{u}_1^\epsilon(t,x) - r_1(t,x)\overline{u}_1^\epsilon(t,x) + \int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad \times\Big\{\Big(\beta_{11}(t-a,y)+\epsilon\Big)\partial_v g_{11}(u_{S_1}^*(t-a,y)+\epsilon,0)\overline{u}_1^\epsilon(t-a,y) \\ \qquad +\Big(\beta_{12}(t-a,y)+\epsilon\Big)\partial_v g_{12}(u_{S_1}^*(t-a,y)+\epsilon,0)\overline{u}_2^\epsilon(t-a,y)\Big\}dyda, \\ \qquad t \ge kT,\ x\in\Omega, \\ \frac{\partial \overline{u}_2^\epsilon(t,x)}{\partial t} = D_2\Delta \overline{u}_2^\epsilon(t,x) - r_2(t,x)\overline{u}_2^\epsilon(t,x) + \int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad \times\Big\{\Big(\beta_{21}(t-a,y)+\epsilon\Big)\partial_v g_{21}(u_{S_2}^*(t-a,y)+\epsilon,0)\overline{u}_1^\epsilon(t-a,y) \\ \qquad +\Big(\beta_{22}(t-a,y)+\epsilon\Big)\partial_v g_{22}(u_{S_2}^*(t-a,y)+\epsilon,0)\overline{u}_2^\epsilon(t-a,y)\Big\}dyda, \\ \qquad t \ge kT,\ x\in\Omega, \\ \frac{\partial}{\partial n}\overline{u}_1^\epsilon(t,x) = \frac{\partial}{\partial n}\overline{u}_2^\epsilon(t,x) = 0,\ t\ge kT,\ x\in\partial\Omega, \end{cases} \tag{16}$$

where k is an integer determined later. Define the Poincare map of (16) $\mathcal{T}^\epsilon : \mathbb{D}\to\mathbb{D}$ by

$$\mathcal{T}^\epsilon(\phi) = \overline{u}_T^\epsilon(\phi),\ \forall\phi\in\mathbb{D},$$

where

$$\overline{u}_T^\epsilon(\phi)(s,x) = \overline{u}^\epsilon(s+T,x;\phi),\ \forall(s,x)\in[-\tau,0]\times\bar{\Omega}$$

and $\overline{u}^\epsilon(t,x;\phi)$ is the solution of (16) with $\overline{u}^\epsilon(s,x) = \phi(s,x)$ for all $s\in[-\tau,0]$, $x\in\Omega$. Let $\overline{r}^\epsilon$ be the spectral radius of $\mathcal{T}^\epsilon$. Since $r_0 < 1$, then there exists a positive constant ϵ_0 such that $\overline{r}^\epsilon < 1$ for any $\epsilon\in[0,\epsilon_0)$. Fix $\epsilon\in[0,\epsilon_0)$. Then, one has $\overline{\mu}^\epsilon := \frac{\ln \overline{r}^\epsilon}{T} < 0$. By Lemma 3.3, there is a positive T–periodic function $(\bar{\mathcal{V}}_1^\epsilon(t,x),\bar{\mathcal{V}}_2^\epsilon(t,x))$ such that $(\overline{u}_1^\epsilon(t,x),\overline{u}_2^\epsilon(t,x)) = (e^{\bar{\mu}^\epsilon t}\bar{\mathcal{V}}_1^\epsilon(t,x), e^{\bar{\mu}^\epsilon t}\bar{\mathcal{V}}_1^\epsilon(t,x))$ is a solution of (16).

Since the $u_{S_i}(i=1,2)$ equations of (6) are dominated by (11), respectively, we obtain that there exists an integer $k>0$ such that $u_{S_i}(t,x)\le u_{S_i}^*(t,x)+\epsilon$ for any $t\ge kT$, $x\in\bar{\Omega}$ and $i=1,2$. According to (v) of (H1), for all $t\ge kT$ and $x\in\Omega$, we have

$$\begin{cases} \frac{\partial u_1(t,x)}{\partial t} \le D_1\Delta u_1(t,x) - r_1(t,x)u_1(t,x) + \int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t,t-a,x,y) \\ \qquad \times\Big\{\Big(\beta_{11}(t-a,y)+\epsilon\Big)\partial_v g_{11}(u_{S_1}^*(t-a,y)+\epsilon,0)u_1(t-a,y) \\ \qquad +\Big(\beta_{12}(t-a,y)+\epsilon\Big)\partial_v g_{12}(u_{S_1}^*(t-a,y)+\epsilon,0)u_2(t-a,y)\Big\}dyda, \\ \frac{\partial u_2(t,x)}{\partial t} \le D_2\Delta u_2(t,x) - r_2(t,x)u_2(t,x) + \int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t,t-a,x,y) \\ \qquad \times\Big\{\Big(\beta_{21}(t-a,y)+\epsilon\Big)\partial_v g_{21}(u_{S_2}^*(t-a,y)+\epsilon,0)u_1(t-a,y) \\ \qquad +\Big(\beta_{22}(t-a,y)+\epsilon\Big)\partial_v g_{22}(u_{S_2}^*(t-a,y)+\epsilon,0)u_2(t-a,y)\Big\}dyda. \end{cases}$$

For any given $\phi\in\mathbb{C}_\tau^+$, since $u_i(t,x;\phi)(i=1,2)$ are globally bounded, there exists some $\alpha>0$ such that $(u_1(t,x;\phi),u_2(t,x;\phi))\le\alpha(e^{\bar{\mu}^\epsilon t}\bar{\mathcal{V}}_1^\epsilon(t,x), e^{\bar{\mu}^\epsilon t}\bar{\mathcal{V}}_2^\epsilon(t,x))$ for any $t\in[kT,kT+\tau]$ and $x\in\bar{\Omega}$. By the similar arguments in [25, Section 2] and using the comparison theorem for the abstract functional differential equation [37, Proposition 3], one has $(u_1(t,x;\phi),u_2(t,x;\phi))\le\alpha\,(e^{\bar{\mu}^\epsilon t}\bar{\mathcal{V}}_1^\epsilon(t,x), e^{\bar{\mu}^\epsilon t}\bar{\mathcal{V}}_2^\epsilon(t,x))$ for any $t\ge kT$ and $x\in\bar{\Omega}$. It then follows from $\bar{\mu}^\epsilon<0$ that $u_i(t,x;\phi)\to0$ as $t\to\infty$ uniformly $x\in\bar{\Omega}$. In addition, the equations $u_{Si}(i=1,2)$ in system (6) are asymptotic to system (11). By [67, Lemma 2.1], we get that $u_{Si}^*(i=1,2)$ are global attractive solutions of (11).

Next, similar to the proof of [67, Theorem 4.3 (i)], we get

$$\lim_{t\to\infty}\Big(u_{S_i}(t,x;\phi)-u^*_{S_i}(t,x)\Big)=0$$

uniformly for $x\in\bar{\Omega}$.

(2) Assume $R_0>1$. In the case, one has $r_0>1$. Let

$$\mathbb{W}_0=\{\phi\in\mathbb{C}^+_\tau:\phi_1(0,\cdot)\not\equiv 0 \text{ or } \phi_2(0,\cdot)\not\equiv 0\}$$

and

$$\partial\mathbb{W}_0:=\mathbb{C}^+_\tau\backslash\mathbb{W}_0=\{\phi\in\mathbb{C}^+_\tau:\phi_1(0,\cdot)\equiv 0 \text{ and } \phi_2(0,\cdot)\equiv 0\}.$$

If $\phi\in\mathbb{W}_0$ with $\phi_1(0,\cdot)\not\equiv 0$, then Lemma 3.6 implies that $u_1(t,x;\phi)>0$ for any $x\in\bar{\Omega}$ and $t>0$. Thus, by (5), one gets $\int_0^{\tau_2} f_2(a)\int_\Omega\Gamma_2(t,t-a,x,y)\beta_{21}(t-a,y)g_{21}(u_{S_2},u_1)(t-a,y)dyda>0$ for $t>\tau_2$, which yields $u_2(t,x;\phi)>0$ for $t>\tau_2$ and $x\in\bar{\Omega}$. Similarly, if $\phi\in\mathbb{W}_0$ with $\phi_2\not\equiv 0$, then one has $u_i(t,x;\phi)>0 (i=1,2)$ for any $t>\tau_1$ and $x\in\bar{\Omega}$. Thus, there exists $k_0\in\mathbb{N}$ such that $\Phi^k_T(\mathbb{W}_0)\subseteq\mathbb{W}_0$ for each $k>k_0$, where $\Phi^k_T(\phi)=u_{kT}(\phi)=u(kT+s,x;\phi)$ for $s\in[-\tau,0]$, $x\in\bar{\Omega}$ and $\phi\in\mathbb{W}_0$ and $u(t,x;\phi)$ is a solution of system (6) for $t>0$, $x\in\bar{\Omega}$ and $\phi\in\mathbb{W}_0$. Define

$$M_\partial:=\{\phi\in\partial\mathbb{W}_0:\Phi^k_T(\phi)\in\partial\mathbb{W}_0,\forall k\in\mathbb{N}\}.$$

Let $\omega(\phi)$ be the omega limit set of the orbit $\gamma^+:=\{\Phi^k_T(\phi):\forall k\in\mathbb{N}\}$ and $M:=(u^*_{S_1,0},u^*_{S_2,0},\hat{0},\hat{0})$, where $\hat{0}$ is the constant function and identical to zero. For any given $\phi\in M_\partial$, we have $\Phi^k_T(\phi)\in\partial\mathbb{W}_0$, $\forall k\in\mathbb{N}$. Thus, one has $u_i(t,x;\phi)\equiv 0$ for $t\geq 0$, $x\in\bar{\Omega}$, $\phi\in M_\partial$ and $i=1,2$. It follows from [67, Lemma 2.1] that $\lim_{t\to\infty}\Big(u_{S_i}(t,x;\phi)-u^*_{S_i}(t,x)\Big)=0$ uniformly for $x\in\bar{\Omega}$ and $i=1,2$. Consequently, we have $\omega(\phi)=\{M\}$, $\forall\phi\in M_\partial$.

Next, we consider the following linear system with parameter $\theta>0$:

$$\begin{cases}\frac{\partial v^\theta_1(t,x)}{\partial t}=D_1\Delta v^\theta_1(t,x)-r_1(t,x)v^\theta_1(t,x)+\int_0^{\tau_1}f_1(a)\int_\Omega\Gamma_1(t,t-a,x,y)\\ \qquad\times\Big(\beta_{11}(t-a,y)\mathcal{N}_{11}(u^*_{S_1}(t-a,y)-\theta,\theta)v^\theta_1(t-a,y)\\ \qquad+\beta_{12}(t-a,y)\mathcal{N}_{12}(u^*_{S_1}(t-a,y)-\theta,\theta)v^\theta_2(t-a,y)\Big)dyda,\\ \qquad t>0,\ x\in\Omega,\\ \frac{\partial v^\theta_2(t,x)}{\partial t}=D_2\Delta v^\theta_2(t,x)-r_2(t,x)v^\theta_2(t,x)+\int_0^{\tau_2}f_2(a)\int_\Omega\Gamma_2(t,t-a,x,y)\\ \qquad\times\Big(\beta_{21}(t-a,y)\mathcal{N}_{21}(u^*_{S_2}(t-a,y)-\theta,\theta)v^\theta_1(t-a,y)\\ \qquad+\beta_{22}(t-a,y)\mathcal{N}_{22}(u^*_{S_2}(t-a,y)-\theta,\theta)v^\theta_2(t-a,y)\Big)dyda,\\ \qquad t>0,\ x\in\Omega,\\ \frac{\partial v^\theta_1(t,x)}{\partial n}=\frac{\partial v^\theta_2(t,x)}{\partial n}=0,\ t>0,\ x\in\partial\Omega,\\ v^\theta_i(s,x)=\phi_i(s,x),\ s\in[-\tau,0],\ x\in\bar{\Omega},\ \phi=(\phi_1,\phi_2)\in\mathbb{D},\ i=1,2.\end{cases}\tag{17}$$

Let $n_0:=\min\{n\in\mathbb{N}:nT>2\tau\}$. Define the Poincaré map of (17) $\mathcal{E}^{n_0}_\theta:\mathbb{D}\to\mathbb{D}$ by $\mathcal{E}^{n_0}_\theta(\phi)=v^\theta_{n_0T}(\phi)=(v^\theta_{1,n_0T}(\phi),v^\theta_{2,n_0T}(\phi))$, where $v^\theta_{i,n_0T}(\phi)(s,x)=v^\theta_i(s+n_0T,x;\phi)$ for $(s,x)\in[-\tau,0]\times\bar{\Omega}$, and $v^\theta(t,x;\phi)$ is the solution of (17) with $v^\theta(s,x)=\phi(s,x)=(\phi_1(s,x),\phi_2(s,x))$ for all $(s,x)\in[-\tau,0]\times\bar{\Omega}$. Let $r^{n_0}_\theta$ be the spectral radius of $\mathcal{E}^{n_0}_\theta$. It is obvious that $\mathcal{E}^{n_0}_\theta$ is compact and positive. The Krein–Rutman theorem [19, Theorem 7.1] implies that there exist a eigenvalue $r^{n_0}_\theta>0$ and a positive eigenfunction $\tilde{\varphi}=(\tilde{\varphi}_1,\tilde{\varphi}_2)\in\mathbb{D}$ such that $\mathcal{E}^{n_0}_\theta(\tilde{\varphi})=r^{n_0}_\theta\tilde{\varphi}$. Denote $r^{n_0}_0$ as the spectral radius of $\Phi^{n_0}_T$ and $\Phi^{n_0}_T$ is defined as in Lemma 3.1. Since $r_0>1$ which implies that $r_0^{n_0} > 1$, there exists a parameter $\theta_0 > 0$ small enough such that $r_\theta^{n_0} > 1$ for $\theta \in [0, \theta_0)$. Fix $\theta \in [0, \theta_0)$.

Claim. *M is a uniform weak repeller for $\mathbb{W}_0$ in the sense that*

$$\limsup_{k\to\infty} \left\|\Phi_T^k(\phi) - M\right\| \geq \theta, \ \forall \phi \in \mathbb{W}_0.$$

Applying a contradiction way, suppose that $\limsup_{k\to\infty} \left\|\Phi_T^k(\phi) - M\right\| < \theta$ for some $\phi_0 \in \mathbb{W}_0$. Namely, there exists $\tilde{k}_1 > 0$ large enough such that $0 < u_1(t, x, \phi_0) < \theta$ and $0 < u_2(t, x; \phi_0) < \theta$, $u_{S_1}(t, x; \phi_0) > u_{S_1}^*(t, x; \phi_0) - \theta$ and $u_{S_2}(t, x; \phi_0) > u_{S_2}^*(t, x; \phi_0) - \theta$ for any $t \geq \tilde{k}_1 T$ and $x \in \bar{\Omega}$. Furthermore, we select $\tilde{K} = \max\{n_0, \tilde{k}_1\}$. According to (v) of (H1), for any $t \geq \tilde{K}T$ and $x \in \Omega$, $u_1(t, x; \phi_0)$ and $u_2(t, x; \phi_0)$ satisfy

$$\begin{cases} \frac{\partial}{\partial t}u_1(t,x) \geq D_1\Delta u_1(t,x) - r_1(t,x)u_1(t,x) + \int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(t, t-a, x, y) \\ \qquad\qquad \times\Big(\beta_{11}(t-a,y)\mathcal{N}_{11}(u_{S_1}^*(t-a,y) - \theta, \theta)u_1(t-a,y) \\ \qquad\qquad +\beta_{12}(t-a,y)\mathcal{N}_{12}(u_{S_1}^*(t-a,y) - \theta, \theta)u_2(t-a,y)\Big)dyda, \\ \frac{\partial}{\partial t}u_2(t,x) \geq D_2\Delta u_2(t,x) - r_2(t,x)u_2(t,x) + \int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(t, t-a, x, y) \\ \qquad\qquad \times\Big(\beta_{21}(t-a,y)\mathcal{N}_{21}(u_{S_2}^*(t-a,y) - \theta, \theta)u_1(t-a,y) \\ \qquad\qquad +\beta_{22}(t-a,y)\mathcal{N}_{22}(u_{S_2}^*(t-a,y) - \theta, \theta)u_2(t-a,y)\Big)dyda. \end{cases} \tag{18}$$

Since $u_i(t, x; \phi_0) > 0 (i = 1, 2)$ for $t > \tau$ and $x \in \bar{\Omega}$, there exists a constant $\kappa > 0$ such that

$$u_i((\tilde{K}+1)T + s, x; \phi_0) \geq \kappa\tilde{\varphi}_i(s, x), \ s \in [-\tau, 0], \ x \in \bar{\Omega}, \ i = 1, 2.$$

Due to (17), (18) and the comparison principle, there exists $\kappa > 0$ such that

$$\begin{aligned} &(u_1(t, x; \phi_0), u_2(t, x; \phi_0))^T \\ \geq&\kappa(v_1^\theta(t - (\tilde{K}+1)T, x; \tilde{\varphi}), \ v_2^\theta(t - (\tilde{K}+1)T, x; \tilde{\varphi})), \ \forall t \geq (\tilde{K}+1)T, \ x \in \bar{\Omega}. \end{aligned}$$

Therefore, one has

$$u_i(KT, x; \phi_0) \geq \kappa v_i^\theta((K - \tilde{K} - 1)T, x; \tilde{\varphi})) = \kappa(r_\theta^{n_0})^m)\tilde{\varphi}_i(s, x), \tag{19}$$

where we select $K = (\tilde{K}+1) + mn_0$ and $i = 1, 2$. Since $\tilde{\varphi}_i(s, x)(i = 1, 2)$ are positive, there exist $s_i \in [-\tau, 0]$ and $x_i \in \bar{\Omega}$ such that $\tilde{\varphi}_i(s_i, x_i) > 0(i = 1, 2)$. Thus, it follows from (19) that $u_i(KT, x_i; \phi_0) \to +\infty$ as $K \to \infty$ (namely, $m \to \infty$) which contradicts the boundedness of $u_i(t, x; \phi)(i = 1, 2)$. The claim is proved.

It follows from the above claim that M is an isolated invariant set for Φ_T in $\mathbb{W}_0$, and $W^s(M) \cap \mathbb{W}_0 = \emptyset$, where $W^s(M)$ is the stable set of M. According to the acyclicity theorem on uniform persistence for maps (see [71, Theorem 1.3.1 and Remark 1.3.1]), one has that $\Phi_T : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ is uniformly persistence with respect to $(\mathbb{W}_0, \partial\mathbb{W}_0)$, namely, there exists a $\tilde{\delta} > 0$ such that

$$\liminf_{k\to\infty} \mathrm{d}(\Phi_T^k, \partial\mathbb{W}_0) \geq \tilde{\delta}, \forall \phi \in \mathbb{W}_0.$$

It then follows from [71, Theorem 3.1.1] that the periodic semiflow $\Phi_t : \mathbb{C}_\tau^+ \to \mathbb{C}_\tau^+$ is also uniformly persistent with respect to $(\mathbb{W}_0, \partial\mathbb{W}_0)$. It is easy to see that $\Phi_T^{n_0}$ is compact and point dissipative on $\mathbb{W}_0$. Therefore, according to [35, Theorem 2.9], one obtains that $\Phi_T^{n_0} : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor $\mathcal{Z}_0$.

In the following, we further prove the persistence stated in (ii). Define a continuous function $p : \mathbb{C}_\tau^+ \to \mathbb{R}_+$ by (similar to [31, Theorem 4.1])

$$p(\phi) = \min\{\min_{x\in\bar\Omega} \phi_1(0, x),\ \min_{x\in\bar\Omega} \phi_2(0, x)\},\ \forall\phi \in \mathbb{C}_\tau^+.$$

Since $\mathcal{Z}_0 = \Phi_T^{n_0}(\mathcal{Z}_0)$, we have that

$$\phi_i(0, \cdot) > 0,\ \phi \in \mathcal{Z}_0,\ i = 1, 2. \tag{20}$$

Let $\mathcal{B}_0 := \bigcup_{t\in[0,n_0T]} \Phi_t(\mathcal{Z}_0)$. It then follows that $\mathcal{B}_0 \subset \mathbb{W}_0$ and

$$\lim_{t\to\infty} d(\Phi_t(\phi), \mathcal{B}_0) = 0$$

for all $\phi \in \mathbb{W}_0$. Since $\mathcal{B}_0$ is a compact subset of $\mathbb{W}_0$, we have $\min_{\phi\in\mathcal{B}_0} p(\phi) > 0$. Thus, by Lemma 3.6, there exists a $\delta^* > 0$ such that $\liminf_{t\to\infty} u_i(t, \cdot; \phi) \geq \delta^*(i = 1, 2)$. Furthermore, there exists $0 < \delta < \delta^*$ such that

$$\liminf_{t\to\infty} u_i(t, \cdot; \phi) \geq \delta,\ \phi \in \mathbb{W}_0,\ i = 1, 2.$$

The proof is completed. □

4. **A special case.**

In this section, we investigate the special case where all the coefficients in (4) are independent of the time variable t and the spatial variable x. That is,

$$\begin{array}{l} \mu_i(t, x) \equiv \mu_i,\ d_i(t, x) \equiv d_i,\ \beta_{ij}(t, x) \equiv \beta_{ij}, \\ r_i(t, x) \equiv r_i,\ \Gamma_i(t, t-a, x, y) \equiv \Gamma_i(a, x, y), \end{array} t > 0,\ x \in \Omega,\ i, j = 1, 2.$$

In addition, $g_{ij}(u, v) \equiv p_i(u)q_j(v)$ and $p_i(u)$ and $q_i(v)$ satisfy

(A1): **(i):** $p_i(u), q_i(v) : \mathbb{R}^+ \to \mathbb{R}^+ (i, j = 1, 2)$ can be continuously differentiable for all $u, v \geq 0$;
- **(ii):** $q_i(0) = 0$ and $p_i(0) = 0$ for $i, j = 1, 2$. Furthermore, $p_i(u) > 0$ for $u > 0$;
- **(iii):** $p_i'(u) \geq 0$ and $q_i'(v) \geq 0$ for all $u, v \geq 0$ and $i, j = 1, 2$. In particular, $q_i'(0) > 0$ for $i = 1, 2$;
- **(iv):** there exist $\eta_i > 0 (i = 1, 2)$ such that $p_i(u)q_i(v) \leq \eta_i u$ for all $u, v \geq 0$.
- **(v):** $\mathcal{N}_j(v) := \frac{q_j(v)}{v} > 0$, $\mathcal{N}_j'(v) \leq 0$ for all $v > 0$ and $j = 1, 2$.

In short, we consider the following spatio–temporally homogeneous reaction –diffusion epidemic model with Neumann boundary condition:

$$\begin{cases} \frac{\partial S_i(t,x)}{\partial t} = D_{S_i}\Delta S_i(t, x) + \mu_i - d_i S_i(t, x) - \beta_{i1}p_i(S_i(t, x))q_1(I_1(t, x)) \\ \qquad\qquad -\beta_{i2}p_i(S_i(t, x))q_2(I_2(t, x)),\ t > 0,\ x \in \Omega, \\ \frac{\partial I_i(t,x)}{\partial t} = D_i\Delta I_i(t, x) - r_i I_i(t, x) + \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(a, x, y) \\ \qquad\qquad \times\Big(\beta_{i1}p_i(S_i(t-a, y))q_1(I_1(t-a, y)) \\ \qquad\qquad +\beta_{i2}p_i(S_i(t-a, y))q_2(I_2(t-a, y))\Big)dyda,\ t > 0,\ x \in \Omega, \\ \frac{\partial S_i(t,x)}{\partial n} = \frac{\partial I_i(t,x)}{\partial n} = 0,\ t > 0,\ x \in \partial\Omega. \end{cases} \tag{21}$$

By a straightforward computation, one has

$$S_i^0(t, x) \equiv \frac{\mu_i}{d_i},\ t \geq 0,\ x \in \Omega.$$

Next, we give the explicit expression with the basic reproduction number R_0. Let $\phi = (\phi_1, \phi_2)^T$ be the initial distribution of infective individuals such that

$\int_\Omega \phi_i(x)dx = 1$ for $i = 1, 2$. Then $T_i(t)\phi_i$ represents the solution of the following system

$$\begin{cases} \frac{\partial u_i(t,x)}{\partial t} = D_i \Delta u_i(t,x) - r_i u_i(t,x),\ t > 0,\ x \in \Omega, \\ \frac{\partial u_i(t,x)}{\partial n} = 0,\ t > 0,\ x \in \partial\Omega, \\ u_i(0,x) = \phi_i(x),\ x \in \Omega \end{cases}$$

for $i = 1, 2$. Thus, $T_i(t)\phi_i$ can be given by

$$(T_i(t)\phi_i)(x) = \int_\Omega \tilde{\Gamma}_i(t,x,z)\phi_i(z)dz,$$

where $\tilde{\Gamma}_i(t,x,z)$ for $t > 0$ and $x, z \in \Omega$ is the fundamental solution associated with the partial differential operator $\partial_t - D_i\Delta - r_i$ and Neumann boundary condition for $i = 1, 2$ and $\int_\Omega \tilde{\Gamma}_i(t,x,z)dz = \int_\Omega \tilde{\Gamma}_i(t,x,z)dx = e^{-r_i t}$ for $i = 1, 2$. Let $T(t)\phi = (T_1(t)\phi_1, T_2(t)\phi_2)^T$ be the remaining distribution of infective individuals at time t. Also in this case, V is the positive linear operator on $C(\bar{\Omega}, \mathbb{R} \times \mathbb{R})$ defined by

$$V(\phi)(x) = \begin{pmatrix} V_{11}(\phi)(x) & V_{12}(\phi)(x) \\ V_{21}(\phi)(x) & V_{22}(\phi)(x) \end{pmatrix},\ \forall \phi \in C(\bar{\Omega}, \mathbb{R} \times \mathbb{R}),\ x \in \bar{\Omega},$$

where $V_{ij}(\phi)(x) = \beta_{ij} p_i(S_i^0) q_j'(0) \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(a,x,y)\phi_j(y)dyda$, $i, j = 1, 2$. Thus, $V(T(t)\ \phi)$ is the distribution of newly infected individuals at time t. Thus, the next generate operator can be represented by

$$L(\phi) := \int_0^\infty V(T(t)\phi)dt = V\left(\int_0^\infty T(t)\phi dt\right).$$

Furthermore, the total number of infectious individuals is given by

$$\mathcal{L} := \int_\Omega L(\phi)(x)dx.$$

Let $\vartheta_1 := \int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(a,x,y)dxda$. According to (v) of (A1), we can obtain

$$\begin{aligned} &\int_\Omega \int_0^\infty V_{11}(T_1(t)\phi_1)(x)dtdx \\ =&\int_\Omega \beta_{11} p_1(S_1^0) q_1'(0) \int_0^\infty \int_0^{\tau_1} f_1(a) \int_\Omega \int_\Omega \Gamma_1(a,x,y)\tilde{\Gamma}_1(t,y,z)\phi_1(z)dzdydadtdx \\ =&\beta_{11} p_1(S_1^0) q_1'(0)\vartheta_1 \int_0^\infty e^{-r_1 t} \int_\Omega \phi_1(z)dzdt \\ =&\frac{\vartheta_1 \beta_{11} p_1(S_1^0) q_1'(0)}{r_1}. \end{aligned}$$

By the same methods, one has

$$\begin{aligned} \int_\Omega \int_0^\infty V_1(T_2(t)\phi_2)(x)dtdx &= \frac{\vartheta_1 \beta_{12} p_1(S_1^0) q_2'(0)}{r_2}, \\ \int_\Omega \int_0^\infty V_2(T_j(t)\phi_j)(x)dtdx &= \frac{\vartheta_2 \beta_{2j} p_2(S_2^0) q_j'(0)}{r_j},\ j = 1, 2. \end{aligned}$$

As a consequence, it follows that

$$\mathcal{L} = \begin{pmatrix} \frac{\vartheta_1 \beta_{11} p_1(S_1^0) q_1'(0)}{r_1} & \frac{\vartheta_1 \beta_{12} p_1(S_1^0) q_2'(0)}{r_2} \\ \frac{\vartheta_2 \beta_{21} p_2(S_2^0) q_1'(0)}{r_1} & \frac{\vartheta_2 \beta_{22} p_2(S_2^0) q_2'(0)}{r_2} \end{pmatrix}.$$

Let $r(\mathcal{L})$ be the spectral radius of $\mathcal{L}$. Finally, we can define the spectral radius of $\mathcal{L}$ as the basic reproduction number R_0, that is,

$$R_0 = r(\mathcal{L}).$$

We are in a position to show the main results in this section.

Theorem 4.1. *Let $u(t, x, \phi)$ be the solution of (21) with $u_0 = \phi \in \mathbb{C}_\tau^+$, then the following two statements are valid:*

- **(1):** *If $R_0 < 1$, then the disease free equilibrium $(\frac{\mu_1}{d_1}, \frac{\mu_2}{d_2}, 0, 0)$ is globally attractive.*
- **(2):** *If $R_0 > 1$, then the system (21) has a positive constant steady state $u^* = (S_1^*, S_2^*, I_1^*, I_2^*)$ which is globally attractive.*

Proof. The conclusion of (1) follows from Theorem 3.7(i).

In the following, we prove the conclusion (2) by using a Volterra like Lyapunov functional. Similarly to the proof of [48, Theorem 2.1], we can show that the corresponding ordinary differential equations of (21):

$$\begin{cases} \frac{dS_i(t)}{dt} = \mu_i - d_iS_i(t) - \beta_{i1}p_i(S_i(t))q_1(I_1(t)) - \beta_{i2}p_i(S_i(t))q_2(I_2(t)), \\ \qquad\qquad t > 0,\ i = 1, 2, \\ \frac{dI_i(t)}{dt} = -r_iI_i(t) + \beta_{i1}p_i(S_i(t))q_1(I_1(t)) + \beta_{i2}p_i(S_i(t))q_2(I_2(t)), \\ \qquad\qquad t > 0,\ i = 1, 2 \end{cases}$$

admits at least one endemic equilibrium $u^* = (S_1^*, S_2^*, I_1^*, I_2^*)(S_i^*, I_i^* > 0,\ i = 1, 2)$, which is also a positive constant steady state of (21).

Next, we are ready to prove the global attractivity of the endemic equilibrium u^* and hence, the endemic equilibrium u^* is unique. Set $V(x) = x - 1 - \ln x$. Then we define

$$W(t) = \int_\Omega \left\{ \frac{\overline{W}_1(t, x)}{\beta_{12}p_1(S_1^*)q_2(I_2^*)} + \frac{\overline{W}_2(t, x)}{\beta_{21}p_2(S_2^*)q_1(I_1^*)} \right\} dx,$$

where

$$\overline{W}_i(t, x) = \Phi_{S_i}(t, x) + \frac{1}{\vartheta_i}\Phi_{I_i}(t, x) + \frac{1}{\vartheta_i}\sum_{j=1}^{2} \beta_{ij}p_i(S_i^*)q_j(I_j^*)Q(S_{i,t}, I_{j,t}),$$

$$\Phi_{S_i}(t, x) = \int_{S_i^*}^{S_i(t,x)} \frac{p_i(s) - p_i(S_i^*)}{p_i(s)} ds,\ \ \Phi_{I_i} = \int_{I_i^*}^{I_i(t,x)} \frac{q_i(s) - q_i(I_i^*)}{q_i(s)} ds$$

and

$$\begin{aligned} &Q(S_{i,t}, I_{j,t}) \\ &= \int_0^{\tau_i} f_i(a) \int_\Omega \Gamma_i(a, x, y) \int_0^a V\left(\frac{p_i(S_i(t - \sigma, y))q_j(I_j(t - \sigma, y))}{f_i(S_i^*)q_j(I_j^*)} \right) d\sigma dy da \end{aligned}$$

for $i, j = 1, 2$. Let $\phi = (\phi_{S_1}, \phi_{S_2}, \phi_1, \phi_2) \in \mathcal{Z}_0$ ($\mathcal{Z}_0$ is defined as Theorem 3.7(ii)). Since $\mathcal{Z}_0$ is invariant, there exists a non–negative solution (S_1, S_2, I_1, I_2) of (21) that is defined for all $t \in \mathbb{R}$, takes all its value in $\mathcal{Z}_0$ and satisfies $S_i(0, x) = \phi_{S_i}(x)$ and $I_i(0, x) = \phi_i(x)$ for all $x \in \Omega$. It further follows from Lemma 3.6 and (20) that

$\inf_{t\in\mathbb{R},x\in\Omega} S_i(t,x) > 0$ and $\inf_{t\in\mathbb{R},x\in\Omega} I_i(t,x) > 0$ for $i = 1, 2$, which implies that $W(t)$ is defined for all $t \in \mathbb{R}$. We firstly compute the derivation of Φ_{S_i} and Φ_{I_i} on t

$$\begin{aligned}
\frac{\partial \Phi_{S_1}(t,x)}{\partial t} &= D_{S_1}\frac{p_1(S_1(t,x)) - p_1(S_1^*)}{p_1(S_1(t,x))}\Delta S_1(t,x) + F_1(t,x)\\
&\quad + \sum_{j=1}^{2}\beta_{1j}p_1(S_1^*)q_j(I_j^*)H_{1j}(t,x),\\
\frac{\partial \Phi_{I_1}(t,x)}{\partial t} &= D_1\frac{q_1(I_1(t,x)) - q_1(I_1^*)}{q_1(I_1(t,x))}\Delta I_1(t,x) + J_1(t,x) + \sum_{j=1}^{2}\beta_{1j}p_1(S_1^*)q_j(I_j^*)\\
&\quad \int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(a,x,y)T_{1j}(t,a,x,y)dyda,\\
\frac{\partial \Phi_{S_2}}{\partial t} &= D_{S_2}\left(\frac{p_2(S_2(t,x)) - p_2(S_2^*)}{p_2(S_2(t,x))}\right)\Delta S_2(t,x) + F_2(t,x)\\
&\quad + \sum_{j=1}^{2}\beta_{2j}p_2(S_2^*)q_j(I_j^*)H_{2j}(t,x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \Phi_{I_2}}{\partial t} &= D_2\left(1 - \frac{q_2(I_2^*)}{q_2(I_2(t,x))}\right)\Delta I_2(t,x) + J_2(t,x)\\
&\quad + \sum_{j=1}^{2}\beta_{2j}p_2(S_2^*)q_j(I_j^*)\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(a,x,y)T_{2j}(t,a,x,y)dyda,
\end{aligned}$$

where

$$\begin{aligned}
F_i(t,x) &= -d_i\int_\Omega \frac{(p_i(S_i(t,x)) - p_i(S_i^*))(S_i(t,x) - S_i^*)}{S_i(t,x)}dx,\ i = 1,2,\\
J_i(t,x) &= r_i\frac{I_i(t,x)I_i^*}{q(I_i(t,x))q(I_i^*)}\Big(q_i(I_i(t,x)) - q(I_i^*)\Big)\Big(\mathcal{N}_i(I_i(t,x)) - \mathcal{N}_i(I_i^*)\Big),\\
&\quad i = 1,2,\\
H_{ij}(t,x) &= 1 - \frac{p_i(S_i^*)}{p_i(S_i(t,x))} + \frac{q_j(I_j(t,x))}{q_j(I_j^*)} - \frac{p_i(S_i(t,x))q_j(I_j(t,x))}{p_i(S_i^*)q_j(I_j^*)},\ i,j = 1,2
\end{aligned}$$

and

$$\begin{aligned}
T_{ij}(t,a,x,y) &= 1 + \frac{p_i(S_i(t-a,y))q_j(I_j(t-a,y))}{p_i(S_i^*)q_j(I_j^*)} - \frac{q_i(I_i(t,x))}{q_i(I_i^*)}\\
&\quad - \frac{p_i(S_i(t-a,y))q_j(I_j(t-a,y))q_i(I_i^*)}{p_i(S_i^*)q_j(I_j^*)q_i(I_i(t,x))},\ i,j = 1,2.
\end{aligned}$$

Moreover, one has

$$\begin{aligned}
&\frac{\partial Q(S_{i,t}, I_{j,t})}{\partial t}\\
&= \int_0^{\tau_i} f_i(a)\int_\Omega \Gamma_i(a,x,y)\left(V\left(\frac{p_i(S_i(t,y))q_j(I_j(t,y))}{p_i(S_i^*)q_j(I_j^*)}\right)\right.\\
&\quad \left. - V\left(\frac{p_i(S_i(t-a,y))q_j(I_j(t-a,y))}{p_i(S_i^*)q_j(I_j^*)}\right)\right)dyda
\end{aligned}$$

for $i, j = 1, 2$. Secondly, let

$$W_i(t) = \int_\Omega \overline{W}_i(t,x)dx,\ \ i = 1, 2.$$

By a straightforward computation, we can get

$$\begin{aligned}
\frac{dW_1(t)}{dt} \quad = \quad & D_{S_1} \int_\Omega \left(1 - \frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right) \Delta S_1(t,x)dx \\
& + \int_\Omega \frac{D_1}{\vartheta_1} \left(1 - \frac{q_1(I_1^*)}{q_1(I_1(t,x))}\right) \Delta I_1(t,x)dx + \int_\Omega F_1(t,x)dx \\
& + \frac{1}{\vartheta_1} \int_\Omega J_1(t,x)dx - \frac{\beta_{11}}{\vartheta_1} p_1(S_1^*)q_1(I_1^*) \int_0^{\tau_1} f_1(a) \int_\Omega \int_\Omega \Gamma_1(a,x,y) \\
& \left(V\left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right) V\left(\frac{p_1(S_1(t-a,y))q_1(I_1(t-a,y))}{p_1(S_1^*)q_1(I_1(t,x))}\right)\right) dydxda \\
& + \frac{\beta_{12}}{\vartheta_1} p_1(S_1^*)q_2(I_2^*) \int_\Omega \int_0^{\tau_1} f_1(a) \int_\Omega \Gamma_1(a,x,y) \left(-V\left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right)\right. \\
& -V\left(\frac{p_1(S_1(t-a,y))q_2(I_2(t-a,y))q_1(I_1^*)}{p_1(S_1^*)q_2(I_2^*)q_1(I_1(t,x))}\right) \\
& \left. +V\left(\frac{q_2(I_2(t,x))}{q_2(I_2^*)}\right) - V\left(\frac{q_1(I_1(t,x))}{q_1(I_1^*)}\right)\right) dydxda
\end{aligned}$$

and

$$\begin{aligned}
\frac{dW_2(t)}{dt} \quad = \quad & D_{S_2} \int_\Omega \left(1 - \frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right) \Delta S_2(t,x)dx \\
& + \int_\Omega \frac{D_2}{\vartheta_2} \left(1 - \frac{q_2(I_2^*)}{q_2(I_2(t,x))}\right) \Delta I_2(t,x)dx + \int_\Omega F_2(t,x)dx \\
& + \frac{1}{\vartheta_2} \int_\Omega J_2(t,x)dx + \frac{\beta_{21}}{\vartheta_2} p_2(S_2^*)q_1(I_1^*) \int_0^{\tau_2} f_2(a) \int_\Omega \int_\Omega \Gamma_2(a,x,y) \\
& \left(-V\left(\frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right) + V\left(\frac{p_2(S_2(t-a,y))q_1(I_1(t-a,y))q_2(I_2^*)}{p_2(S_2^*)q_1(I_1^*)q_2(I_2(t,x))}\right)\right. \\
& \left. +V\left(\frac{q_1(I_1(t,x))}{q_1(I_1^*)}\right) - V\left(\frac{q_2(I_2(t,x))}{q_2(I_2^*)}\right)\right) dydxda \\
& + \frac{\beta_{21}}{\vartheta_2} p_2(S_2^*)q(I_2^*) \int_\Omega \int_0^{\tau_2} f_2(a) \int_\Omega \Gamma_2(a,x,y) \left(-V\left(\frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right)\right. \\
& \left. -V\left(\frac{p_2(S_2(t-a,y))q_2(I_2(t-a,y))}{p_2(S_2^*)q_2(I_2(t,x))}\right)\right) dydxda.
\end{aligned}$$

In addition, there holds

$$\begin{aligned}
& D_{S_j} \int_\Omega \Delta S_j(t,x) \left(1 - \frac{p_j(S_j^*)}{p_j(S_j)(t,x)}\right) dx \\
= & D_{S_j} \int_{\partial\Omega} \left(1 - \frac{p_j(S_j^*)}{p_j(S_j(t,x))}\right) \frac{\partial}{\partial n} S_j(t,x)dx - D_{S_j} \int_\Omega \frac{p_j'(S_j)p_j(S_j^*)(\nabla S_j(t,x))^2}{p_j^2(S_j)(t,x)} dx \\
= & -D_{S_j} \int_\Omega \frac{p_j'(S_j)p_j(S_j^*)(\nabla S_j(t,x))^2}{p_j^2(S_j(t,x))} dx,\ \ j = 1, 2.
\end{aligned}$$

According (iii) of (A1), one has $-D_{S_j}\int_\Omega \frac{p_j'(S_j)p_j(S_j^*)(\nabla S_j(t,x))^2}{p_j^2(S_j(t,x))}dx \le 0$ for $j=1,2$. In the same way, we can obtain

$$D_j\int_\Omega \Delta I_j(t,x)\left(1-\frac{q_j(I_j^*)}{q_j(I_j(t,x))}\right)dx = -D_j\int_\Omega \frac{q_j(I_j^*)q_j'(I_j)(\nabla I_j(t,x))^2}{q_j^2(I_j(t,x))}dx \le 0$$

for $j=1,2$. By virtue of (iii) of (A1), one has

$$\int_\Omega F_i(t,x)dx = -\int_\Omega d_i\frac{(p_i(S_i(t,x))-p_i(S_i^*))(S_i(t,x)-S_i^*)}{p_i(S_i(t,x))}dx \le 0,\ \ i=1,2. \tag{22}$$

Furthermore, in view of (iii) and (v) of (A1), it follows that

$$\int_\Omega J_i(t,x)dx \le 0,\ \ i=1,2. \tag{23}$$

As a consequence,

$$\begin{aligned}\frac{dW_1(t)}{dt} \;=\;& -D_{S_1}\int_\Omega \frac{p_1(S_1^*)p_1'(S_1)(\nabla S_1(t,x))^2}{p_1^2(S_1(t,x))}dx + \int_\Omega F_1(t,x)dx\\ &-\frac{D_1}{\vartheta_1}\int_\Omega \frac{q_1(I_1^*)q_1'(I_1)(\nabla I_1(t,x))^2}{q_1^2(I_1(t,x))}dx + \frac{1}{\vartheta_1}\int_\Omega J_1(t,x)dx\\ &+\frac{\beta_{11}p_1(S_1^*)q_1(I_1^*)}{\vartheta_1}\int_0^{\tau_1} f_1(a)\int_\Omega\int_\Omega \Gamma_1(a,x,y)\left(-V\left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right)\right.\\ &\left.-V\left(\frac{p_1(S_1(t-a,y))q_1(I_1(t-a,y))}{p_1(S_1^*)q_1(I_1(t,x))}\right)\right)dadydx\\ &+\frac{\beta_{12}p_1(S_1^*)q_2(I_2^*)}{\vartheta_1}\int_\Omega\int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(a,x,y)\left(-V\left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right)\right.\\ &+V\left(\frac{q_2(I_2(t,x))}{q_2(I_2^*)}\right)-V\left(\frac{p_1(S_1(t-a,y))q_2(I_2(t-a,y))q_1(I_1^*)}{p_1(S_1^*)q_2(I_2^*)q_1(I_1(t,x))}\right)\\ &\left.-V\left(\frac{q_1(I_1(t,x))}{q_1(I_1^*)}\right)\right)dadydx\end{aligned}$$

and

$$\begin{aligned}\frac{dW_2(t)}{dt} \;=\;& -D_{S_2}\int_\Omega \frac{p_2(S_2^*)p_2'(S_2)(\nabla S_2(t,x))^2}{p_2^2(S_2)(t,x)}dx + \int_\Omega F_2(t,x)dx\\ &+\frac{D_2}{\vartheta_2}\int_\Omega \frac{q_2(I_2^*)q_2'(I_2)(\nabla I_2(t,x))^2}{q_2^2(I_2(t,x))}dx + \frac{1}{\vartheta_2}\int_\Omega J_2(t,x)dx\\ &+\frac{\beta_{22}p_2(S_2^*)q_2(I_2^*)}{\vartheta_2}\int_\Omega\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(a,x,y)\left(-V\left(\frac{q_2(S_2^*)}{q_2(S_2(t,x))}\right)\right.\\ &\left.-V\left(\frac{p_2(S_2(t-a,y))q_2(I_2(t-a,y))}{p_2(S_2^*)q_2(I_2(t,x))}\right)\right)dydadx\\ &+\frac{\beta_{21}p_2(S_2^*)q_1(I_1^*)}{\vartheta_2}\int_\Omega\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(a,x,y)\left(-V\left(\frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right)\right.\\ &+V\left(\frac{q_1(I_1(t,x))}{q_1(I_1^*)}\right)-V\left(\frac{p_2(S_2(t-a,y))q_1(I_1(t-a,y))q_2(I_2^*)}{p_2(S_2^*)q_1(I_1^*)q_2(I_2(t,x))}\right)\\ &\left.-V\left(\frac{q_2(I_2(t,x))}{q_2(I_2^*)}\right)\right)dydadx.\end{aligned}$$

Set

$$W = \frac{1}{\beta_{12}p_1(S_1^*)q_2(I_2^*)}W_1 + \frac{1}{\beta_{21}p_2(S_2^*)q_1(I_1^*)}W_2.$$

Then we have

$$\begin{aligned}
&\frac{dW}{dt}\\
= &-\frac{D_{S_1}}{\beta_{12}p_1(S_1^*)q_2(I_2^*)}\int_\Omega \frac{p_1(S_1^*)p_1'(S_1)(\nabla S_1(t,x))^2}{p_1^2(S_1(t,x))}dx\\
&-\frac{D_1}{\vartheta_1\beta_{12}p_1(S_1^*)q_2(I_2^*)}\int_\Omega \frac{q_1(I_1^*)q_1'(I_1)(\nabla I_1(t,x))^2}{q_1^2(I_1(t,x))}dx\\
&-\frac{D_{S_2}}{\beta_{21}p_2(S_2^*)q_1(I_1^*)}\int_\Omega \frac{p_2(S_2^*)p_2'(S_2)(\nabla S_2(t,x))^2}{p_2^2(S_2(t,x))}dx\\
&-\frac{D_2}{\vartheta_2\beta_{21}p_2(S_2^*)q_1(I_1^*)}\int_\Omega \frac{q_2(I_2^*)q_2'(I_2)(\nabla I_2(t,x))^2}{q_2^2(I_2(t,x))}dx\\
&+\int_\Omega \frac{F_2(t,x)}{\beta_{21}p_2(S_2^*)q_1(I_1^*)}dx + \int_\Omega \frac{F_1(t,x)}{\beta_{12}p_1(S_1^*)q_2(I_2^*)}dx\\
&+\int_\Omega \frac{J_1(t,x)}{\vartheta_1\beta_{12}p_1(S_1^*)q_2(I_2^*)}dx + \int_\Omega \frac{J_2(t,x)}{\vartheta_2\beta_{21}p_2(S_2^*)q_1(I_1^*)}dx\\
&+\frac{\beta_{11}q_1(I_1^*)}{\beta_{12}q_2(I_2^*)\vartheta_1}\int_\Omega\int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(a,x,y)\\
&\left(-V\left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right) - V\left(\frac{p_1(S_1(t-a,y))q_1(I_1(t-a,y))}{p_1(S_1^*)q_1(I_1(t,x))}\right)\right)dydadx\\
&-\frac{1}{\vartheta_1}\int_\Omega\int_0^{\tau_1} f_1(a)\int_\Omega \Gamma_1(a,x,y)\\
&\left(V\left(\frac{p_1(S_1^*)}{p_1(S_1(t,x))}\right) + V\left(\frac{p_1(S_1(t-a,y))q_2(I_2(t-a,y))q_1(I_1^*)}{p_1(S_1^*)q_2(I_2^*)q_1(I_1(t,x))}\right)\right)dydadx\\
&-\frac{\beta_{22}q_2(I_2^*)}{\beta_{21}q_1(I_1^*)\vartheta_2}\int_\Omega\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(a,x,y)\\
&\left(V\left(\frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right) + V\left(\frac{p_2(S_2(t-a,y))q_2(I_2(t-a,y))}{p_2(S_2^*)q_2(I_2(t,x))}\right)\right)dydadx\\
&-\frac{1}{\vartheta_2}\int_\Omega\int_0^{\tau_2} f_2(a)\int_\Omega \Gamma_2(a,x,y)\\
&\left(V\left(\frac{p_2(S_2^*)}{p_2(S_2(t,x))}\right) + V\left(\frac{p_2(S_2(t-a,y))q_1(I_1(t-a,y))q_2(I_2^*)}{p_2(S_2^*)q_1(I_1^*)q_2(I_2(t,x))}\right)\right)dydadx.
\end{aligned}$$

On the basis of (22) and (23), we have $\frac{dW}{dt} \le 0$. By using the similar arguments introduced in [52, Theorem 12.1], we can prove that the attractor $\mathcal{Z}_0$ in Theorem 3.7 is a singleton set which is formed by the endemic equilibrium u^*. This completes the proof. □

Acknowledgments. The authors would like to thank the an anonymous reviewer and the handling editor for their helpful comments and suggestions which helped us in improving the paper. This work was supported by NNSF of China (11371179) and the Fundamental Research Funds for the Central Universities.

REFERENCES

[1] S. Altizer, A. Dobson, P. Hosseini, P. Hudson, M. Pascual and P. Rohani, Seasonality and the dynamics of infectious disease, *Ecol. Lett.*, **9** (2006), 467–484.

[2] R. M. Anderson, Discussion: the Kermack−McKendrick epidemic threshold theorem, *Bull. Math. Biol.*, **53** (1991), 3–32.

[3] R. M. Anderson and R. May, *Infectious Diseases of Humanns: Dynamics and Control,* Oxford University Press, Oxford, 1991.

[4] N. Bacaër, D. Ait and H. El, Genealogy with seasonality, the basic reproduction number, and the influenza pandemic, *J. Math. Biol.*, **62** (2011), 741–762.

[5] N. Bacaër and S. Guernaoui, The epidemic threshold of vector–borne disease with seasonality, *J. Math. Biol.*, **53** (2006), 421–436.

[6] E. Beretta, T. Hara, W. Ma and Y. Takeuchi, Global asymptotic stability of an SIR epidemic model with distributed time delay, *Nonlinear Anal.*, **47** (2001), 4107–4115.

[7] B. Bonzi, A. A. Fall, A. Iggidr and G. Sallet, Stability of differential susceptibility and infectivity epidemic models, *J. Math. Biol.*, **62** (2011), 39–64.

[8] F. Brauer, Compartmental models in epidemiology, *Mathematical Epidemiology*, Springer, **1945** (2008), 19–79.

[9] L. Burlando, Monotonicity of spectral radius for positive operators on ordered Banach space, *Arch. Math.*, **56** (1991), 49–57.

[10] L. Cai, M. Martcheva and X.-Z. Li, Competitive exclusion in a vector-host epidemic model with distributed delay, *J. Biol. Dyn.*, **7** (2013), 47–67.

[11] D. Dancer and P. Koch Medina, Abstract ecolution equations, Periodic problem and applications, Longman, Harlow, UK, 1992.

[12] O. Diekmann, J. Heesterbeek and J. Metz, On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, *J. Math. Biol.*, **28** (1990), 365–382.

[13] W. E. Fitzgibbon, M. Langlais, M. E. Parrott and G. F. Webb, A diffusive system with age dependency modeling FIV, *Nonlinear Anal.*, **25** (1995), 975–989.

[14] W. E. Fitzgibbon, C. B. Martin and J. J. Morgan, A diffusive epidemic model with criss–cross dynamics, *J. Math. Anal. Appl.*, **184** (1994), 399–414.

[15] W. E. Fitzgibbon, M. E. Parrott and G. F. Webb, Diffusion epidemic models with incubation and crisscross dynamics, *Math. Biosci.*, **128** (1995), 131–155.

[16] D. Gao and S. Ruan, Malaria models with spatial effects, John Wiley & Sons. (in press)

[17] I. Gudelj, K. A. J. White and N. F. Britton, The effects of spatial movement and group interactions on disease dynamics of social animals, *Bull. Math. Biol.*, **66** (2004), 91–108.

[18] Z. Guo, F.-B. Wang and X. Zou, Threshold dynamics of an infective disease model with a fixed latent period and non–local infections, *J. Math. Biol.*, **65** (2012), 1387–1410.

[19] P. Hess, *Periodic–Parabolic Boundary Value Problems and Positivity*, Longman Scientific and Technical, Harlow, UK, 1991.

[20] H. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42** (2000), 599–653.

[21] W. Huang, K. Cooke and C. Castillo-Chavez, Stability and bifurcation for a multiple–group model for the dynamics of HIV/AIDS transmission, *SIAM J. Appl. Math.*, **52** (1992), 835–854.

[22] G. Huang and A. Liu, A note on global stability for a heroin epidemic model with distributed delay, *Appl. Math. Lett.*, **26** (2013), 687–691.

[23] J. M. Hyman and J. Li, Differential susceptibility epidemic models, *J. Math. Biol.*, **50** (2005), 626–644.

[24] H. Inaba, On a new perspective of the basic reproduction number in heterogeneous environments, *J. Math. Biol.*, **65** (2012), 309–348.

[25] Y. Jin and X.-Q. Zhao, Spatial dynamics of a nonlocal periodic reaction–diffusion model with stage structure, *SIAM J. Math. Anal.*, **40** (2009), 2496–2516.

[26] T. Kato, *Peturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelerg, 1976.

[27] J. Li and X. Zou, Generalization of the Kermack–McKendrick SIR model to a patchy environment for a disease with latency, *Math. Model. Nat. Phenom.*, **4** (2009), 92–118.

[28] J. Li and X. Zou, Dynamics of an epidemic model with non–local infections for diseases with latency over a patchy environment, *J. Math. Biol.*, **60** (2010), 645–686.

[29] M. Li, Z. Shuai and C. Wang, Global stability of multi–group epidemic models with distributed delays, *J. Math. Anal. Appl.*, **361** (2010), 38–47.

[30] X. Liang and X.-Q. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, *Comm. Pure Appl. Math.*, **60** (2007), 1–40.

[31] Y. Lou and X.-Q. Zhao, Threshold dynamics in a time–delayed periodic SIS epidemic model, *Discrete Contin. Dyn. Syst. Ser. B*, **12** (2009), 169–186.

[32] Y. Lou and X.-Q. Zhao, A reaction–diffusion malaria model with incubation period in the vector population, *J. Math. Biol.*, **62** (2011), 543–568.

[33] Y. Lou and X.-Q. Zhao, A theoretical approach to understanding population dynamics with deasonal developmental durations, *J Nonlinear Sci.*, **27** (2017), 573–603.

[34] P. Magal and C. McCluskey, Two–group infection age model including an application to nosocomial infection, *SIAM J. Appl. Math.*, **73** (2013), 1058–1095.

[35] P. Magal and X.-Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Math. Anal.*, **37** (2005), 251–275.

[36] M. Martcheva, *An Introduction to Mathematical Epidemiology, Texts in Applied Mathematics*, Springer, New York, 2015.

[37] R. Martain and H. L. Smith, Abstract functional differential equations and reaction–diffusion system, *Trans. Amer. Math. Soc.*, **321** (1990), 1–44.

[38] C. McCluskey, Complete global stability for an SIR epidemic model with delay-distributed or discrete, *Nonlinear Anal. Real World Appl.*, **11** (2010), 55–59.

[39] C. McCluskey and Y. Yang, Global stability of a diffusive virus dynamics model with general incidence function and time delay, *Nonlinear Anal. Real World Appl.*, **25** (2015), 64–78.

[40] J. D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1989.

[41] R. Peng and X.-Q. Zhao, A reaction–diffusion SIS epidemic model in a time–periodic environment, *Nonlinearity*, **25** (2012), 1451–1471.

[42] B. Perthame, *Parabolic Equations in Biology*, Springer, Cham, 2015.

[43] L. Rass and J. Radcliffe, *Spatial Deterministic Epidemics*, Mathematical Surveys and Monographs, 102. American Mathematical Society, Providence, RI, 2003.

[44] R. Ross, An application of the theory of probabilities to the study of a priori pathometry: I, *Proc. R. Soc. Lond.*, **92** (1916), 204–230.

[45] S. Ruan, Spatial−temporal dynamics in nonlocal epidemiological models, *Mathematics for Life Science and Medicine,* Springer−Verlag, Berlin, (2007), 99–122.

[46] S. Ruan and J. Wu, *Modeling Spatial Spread of Communicable Diseases Involving Animal Hosts*, Chapman & Hall/CRC, Boca Raton, FL, (2009), 293–316.

[47] H. L. Smith, *Monotone Dynamical System: An Introduction to the Theory of Competitive and Cooperative Systems,* Math. Surveys and Monogr. vol 41, American Mathematical Society, Providence, 1995.

[48] R. Sun, Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence, *Comput. Math. Appl.*, **60** (2010), 2286–2291.

[49] Y. Takeuchi, W. Ma and E. Beretta, Global asymptotic properties of a delay SIR epidemic model with finite incubation times, *Nonlinear Anal.*, **42** (2000), 931–947.

[50] H. R. Thieme, Mathematics in population biology, Princeton University Press, Princeton, NJ, 2003.

[51] H. R. Thieme, Spectral bound and reproduction number for infinite–dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211.

[52] H. R. Thieme, Global stability of the endemic equilibrium in infinite dimension: Lyapunov functions and positive operators, *J. Differential Equations*, **250** (2011), 3772–3801.

[53] P. van den Driessche and X. Zou, Modeling relapse in infectious diseases, *Math. Biosci.*, **207** (2007), 89–103.

[54] B.-G. Wang, W.-T. Li and Z.-C. Wang, A reaction–diffusion SIS epidemic model in an almost periodic environment, *Z. Angew. Math. Phys.*, **66** (2015), 3085–3108.

[55] B.-G. Wang and X.-Q. Zhao, Basic reproduction ratios for almost periodic compartmental epidemic models, *J. Dynam. Differential Equations*, **25** (2013), 535–562.

[56] L. Wang, Z. Liu and X. Zhang, Global dynamics of an SVEIR epidemic model with distributed delay and nonlinear incidence, *Appl. Math. Comput.*, **284** (2016), 47–65.

[57] W. Wang and X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, *J. Dynam. Differential Equations*, **20** (2008), 699–717.

[58] W. Wang and X.-Q. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, *SIAM J. Appl. Math.*, **71** (2011), 147–168.

[59] W. Wang and X.-Q. Zhao, Basic reproduction numbers for reaction–diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673.

[60] W. Wang and X.-Q. Zhao, Spatial invasion threshold of Lyme disease, *SIAM J. Appl. Math.*, **75** (2015), 1142–1170.

[61] J. Wu, Spatial structure: Partial differential equations models, *Mathematical Epidemiology,* Springer, Berlin, **1945** (2008), 191–203.

[62] D. Xu and X.-Q. Zhao, Dynamics in a periodic competitive model with stage structure, *J. Math. Anal. Appl.*, **311** (2005), 417–438.

[63] Z. Xu and X.-Q. Zhao, A vector–bias malaria model with incubation period and diffusion, *Discrete Contin. Dyn. Syst. Ser. B*, **17** (2012), 2615–2634.

[64] L. Zhang and J.-W. Sun, Global stability of a nonlocal epidemic model with delay, *Taiwanese J. Math.*, **20** (2016), 577–587.

[65] L. Zhang and Z.-C. Wang, A time-periodic reaction-diffusion epidemic model with infection period, *Z. Angew. Math. Phys.*, **67** (2016), Art. 117, 14 pp.

[66] L. Zhang, Z.-C. Wang and Y. Zhang, Dynamics of a reaction-diffusion waterborne pathogen model with direct and indirect transmission, *Comput. Math. Appl.*, **72** (2016), 202–215.

[67] L. Zhang, Z.-C. Wang and X.-Q. Zhao, Threshold dynamics of a time periodic reaction–diffusion epidemic model with latent period, *J. Differential Equations*, **258** (2015), 3011–3036.

[68] Y. Zhang and X.-Q. Zhao, A reaction–diffusion Lyme disease model with seasonality, *SIAM J. Appl. Math.*, **73** (2013), 2077–2099.

[69] X.-Q. Zhao, *Dynamical System in Population Biology*, Spring-Verlag, New York. 2003.

[70] X.-Q. Zhao, Global dynamics of a reaction and diffusion model for Lyme disease, *J. Math. Biol.*, **65** (2012), 787–808.

[71] X.-Q. Zhao, Basic reproduction ratios for periodic compartmental models with time delay, *J. Dyman. Differential Equations*, **29** (2017), 67–82.

Received May 14, 2016; Accepted December 31, 2016.

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