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BOGDANOV-TAKENS BIFURCATIONS IN THE ENZYME-CATALYZED REACTION COMPRISING A BRANCHED NETWORK

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Abstract. There have been some results on bifurcations of codimension one (such as saddle-node, transcritical, pitchfork) and degenerate Hopf bifurcations for an enzyme-catalyzed reaction system comprising a branched network but no further discussion for bifurcations at its cusp. In this paper we give conditions for the existence of a cusp and compute the parameter curves for the Bogdanov-Takens bifurcation, which induces the appearance of homoclinic orbits and periodic orbits, indicating the tendency to steady-states or a rise of periodic oscillations for the concentrations of the substrate and the product.

1. Introduction.

Many differential equations have been proposed (see [8, 11, 13], [17]-[19], [21]-[22], [24, 27] and references therein) to model the dynamic changes of substrate concentration and product one in enzyme-catalyzed reactions. Among those models, a typical form ([7]) is the following skeletal system

$$\begin{cases} \dot{x} = v - V_1(x,y) - V_3(x), \\ \dot{y} = q(V_1(x,y) - V_2(y)), \end{cases} \tag{1}$$

where x and y denote the concentrations of the substrate and the product respectively, v and q are both positive constants, $V_1(x,y)$ and $V_2(y)$ denote the enzyme reaction rate and the output rate of the product respectively and satisfy that

$$V_1(0,y) = 0,\ \partial V_1/\partial x > 0,\ \partial V_1/\partial y > 0,\ V_2(y) \geq 0,\ \forall x, y > 0,$$

and $V_3(x)$ denotes the branched-enzyme reaction rate. Figure 1 shows the scheme of the enzyme-catalyzed reaction which comprises a branched network from the substrate. In Figure 1, S and P represent the substrate and product, respectively, and E_1, E_2 and E_3 are the three enzymes.

The case that $V_3(x) \equiv 0$ in system (1), which represents an unbranched reaction, has been discussed extensively in [1, 6, 7, 9, 20]. Recently, more efforts were made

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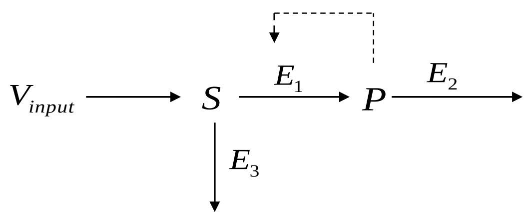

FIGURE 1. Reaction scheme.

to the case that $V_3(x) \not\equiv 0$. One of the efforts ([12, 23]) is made for $V_1(x, y) = x^m y^n, V_2(y) = y$ and $V_3(x) = lx$ and $v = 1$, with which system (1) reduces to

$$\begin{cases} \dot{x} = 1 - x^m y^n - lx, \\ \dot{y} = q(x^m y^n - y), \end{cases}$$

called the multi-molecular reaction model sometimes, where $m, n \geq 1$ are integers and $l \geq 0$ is real. All local bifurcations of this system such as saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation were discussed in [12] and [23]. Reference [15] is concerned with the case that $V_1(x, y) = \gamma x^m y^n$, $V_3(x) = \beta x$, $q = 1$ and $V_2(y)$ is a saturated reaction rate, i.e., $V_2(y) = v_2 y/(\mu_2 + y)$, with which (1) reduces to

$$\begin{cases} \dot{x} = v - \gamma x^m y^n - \beta x, \\ \dot{y} = \gamma x^m y^n - \frac{v_2 y}{\mu_2 + y}, \end{cases}$$

where $v, \gamma > 0, \mu_2, v_2$ and $\beta \geq 0$. Results on existence and nonexistence of periodic solutions on Hopf bifurcation were obtained in [15] with $n = 1$ and $\beta = 0$. When $V_2(y)$ and $V_3(x)$ are both saturated reaction rates, system (1) was considered in [16] as

$$\begin{cases} \dot{x} = v - V_1(x, y) - \frac{v_3 x}{u_3 + x}, \\ \dot{y} = q(V_1(x, y) - \frac{v_2 y}{u_2 + y}) \end{cases}$$

with $V_1(x, y) = v_1 x(1 + x)(1 + y)^2/[L + (1 + x)^2(1 + y)^2]$, where L is the allosteric constant of E_1. Varying the parameter v_2 but fixing the other parameters, Liu ([16]) investigated numerically how the enzyme saturation affects the emergence of dynamical behaviors such as the change from a stable oscillatory state to a divergent state. Later, Davidson and Liu ([3]) discussed the saddle-node bifurcation, Hopf bifurcation and the global bifurcation corresponding to the appearance of homoclinic orbit. When $V_2(y)$ and $V_3(x)$ are both saturated reaction rates, system (1) was also considered in [4] as

$$\begin{cases} \dot{x} = v - v_1 xy - \frac{v_3 x}{u_3 + x}, \\ \dot{y} = q(v_1 xy - \frac{v_2 y}{u_2 + y}) \end{cases} \tag{2}$$

with $V_1(x, y) = v_1 xy$. With a change of variables $x = u_3 \tilde{x}$, $y = u_2 \tilde{y}$ and the time rescaling $t \to v_1^{-1} \mu_2^{-1} t$, system (2) can be written as

$$\begin{cases} \dot{x} = a - xy - \frac{bx}{1 + x}, \\ \dot{y} = \kappa y(x - \frac{c}{1 + y}), \end{cases} \tag{3}$$

where we still use x, y to present $\tilde{x}$, $\tilde{y}$ and take notations $a := v_1^{-1} u_3^{-1} u_2^{-1} v$, $b := v_1^{-1} u_3^{-1} u_2^{-1} v_3$, $c := v_1^{-1} u_3^{-1} u_2^{-1} v_2$ and $\kappa := u_2^{-1} q u_3$ for positive constants. Actually, system (3) is orbitally equivalent to the following quartic polynomial differential system

$$\begin{cases} \dot{x} = (1+y)\{(1+x)(a-xy)-bx\}, \\ \dot{y} = \kappa(1+x)y\{(1+y)x-c\}, \end{cases} \tag{4}$$

in the first quadrant $\mathcal{Q}_+ := \{(x,y) : x \geq 0, y \geq 0\}$ by a time scaling $d\tau = (x+1)(y+1)dt$. In [4] Davidson, Xu and Liu discussed the case that $k=1$ and $a<c$, where the system has at most two equilibria, giving the existence of limit cycles (by the Poincaré-Bendixson Theorem seen in [10] or [26]) and the non-existence of periodic orbits (by the Dulac Criterion seen in [10] or [26]), proving the uniqueness of limit cycles (by reducing to the form of Liénard system) with some restrictions, and illustrating with the software AUTO saddle-node bifurcation, transcritical bifurcation and Hopf bifurcation for fixed $\kappa=1, b=1.5$ and $c=5$. Recently, the general case that $\kappa, a, b, c > 0$ was discussed in [27], where all codimension-one bifurcations such as saddle-node, transcritical and pitchfork bifurcations were investigated and the weak focus was proved to be of at most order 2.

In this paper we continue the work of [27] to give conditions for the existence of a cusp and compute the parameter curves for the Bogdanov-Takens bifurcation, which induces the appearance of homoclinic orbits and periodic orbits, indicating the tendency to steady-states or a rise of periodic oscillations for the concentrations of the substrate and product.

2. **Condition for cusp.**

It is proved in [27] that system (4) has at most 3 equilibria, i.e., $E_0 : (a/(b-a), 0)$, $E_1 : (p_1, c/p_1 - 1)$ and $E_2 : (p_2, c/p_2 - 1)$, where

$$\begin{aligned} p_1 &:= -\tfrac{1}{2}\big\{(a-b-c+1) - [(a-b-c+1)^2 - 4(a-c)]^{1/2}\big\}, \\ p_2 &:= -\tfrac{1}{2}\big\{(a-b-c+1) + [(a-b-c+1)^2 - 4(a-c)]^{1/2}\big\}. \end{aligned} \tag{5}$$

Moreover, if $a = a_* := c + (b^{1/2}-1)^2$, then E_1 and E_2 coincide into one, i.e., the equilibrium $E_* : (b^{1/2}-1, c(b^{1/2}+1)/(b-1)-1)$. There are found in [27] totally 6 bifurcation surfaces

$$\begin{aligned}
\mathcal{T}_{E_0} &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | a = bc/(1+c), b \neq (c+1)^2\} := \bigcup_{i=1}^{4} \mathcal{T}^{(i)}_{E_0}, \\
\mathcal{P}_{E_0} &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | a = bc/(1+c), b = (c+1)^2\}, \\
\mathcal{H}_{E_1} &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | \kappa = \kappa_1, bc/(1+c) < a < c, 0 < b \leq 1\} \\
&\quad \cup \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | \kappa = \kappa_1, bc/(1+c) < a < c + (b^{1/2}-1)^2, 1 < b < (c+1)^2\}, \\
\mathcal{SN}_{E_*} &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | a = a_*,\ 1 < b < (c+1)^2, \kappa \neq \kappa_*\} := \bigcup_{i=1}^{4} \mathcal{SN}^{(i)}_{E_*}, \\
\mathcal{B}_1 &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | a = c\}, \\
\mathcal{B}_2 &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | a = b\},
\end{aligned}$$

which divide $\mathbb{R}^4_+ := \{(a,b,c,\kappa) : a > 0, b > 0, c > 0, \kappa > 0\}$ into 8 subregions

$$\begin{aligned}
\mathcal{R}_1 &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | c < a < a_*, 1 < b < c, c > 1, \\
&\qquad \text{or } b < a < a_*, c < b < (c+1)^2/4, c > 1\}, \\
\mathcal{R}_2 &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | b < a < c, 0 < b < c\} \\
\mathcal{R}_3 &:= \{(a,b,c,\kappa) \in \mathbb{R}^4_+ | bc/(1+c) < a < b, 0 < b < c \text{ or } bc/(1+c) < a < c, c < b < c+1\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_4 &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | 0<a<bc/(1+c), 0<b<c+1 \text{ or } 0<a<c, b>c+1\},\\
\mathcal{R}_5 &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | c<a<bc/(1+c), b>c+1\},\\
\mathcal{R}_6 &:= \{(a,b,c,\kappa)\in\mathbb{R}^4 | c<a<b, c<b<(c+1), c>3\\
&\qquad \text{or } bc/(1+c)<a<b, c+1<b<(c+1)^2/4, c>3\\
&\qquad \text{or } bc/(1+c)<a<a_*, (c+1)^2/4<b<(c+1)^2, c>3\\
&\qquad \text{or } c<a<b, c<b<(c+1)^2/4, 1<c\le 3\\
&\qquad \text{or } c<a<a_*, (c+1)^2/4<b<c+1, 1<c\le 3\\
&\qquad \text{or } bc/(1+c)<a<c+(b^{1/2}-1)^2, (c+1)<b<(c+1)^2, c\le 3\\
&\qquad \text{or } c<a<a_*, 1<b<c+1, c\le 1\},\\
\mathcal{R}_7 &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | c+(b^{1/2}-1)^2<a<b, (c+1)^2/4<b<(c+1)^2, c>1\\
&\qquad \text{or } bc/(1+c)<a<b, b>(c+1)^2 \text{ or } c<a<b, c<b<1, c\le 1\\
&\qquad \text{or } c+(b^{1/2}-1)^2<a<b, 1<b<(c+1)^2, c\le 1\},\\
\mathcal{R}_0 &:= \mathbb{R}_+^4 \backslash \{\mathcal{P}_{E_0}\cup \mathcal{SN}_{E_*}\cup \mathcal{T}_{E_0}\cup (\bigcup_{i=1}^{2}\mathcal{B}_i)\cup\mathcal{B}\cup(\bigcup_{i=1}^{7}\mathcal{R}_i)\},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_{E_0}^{(1)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=bc/(1+c), 0<b<c+1\},\\
\mathcal{T}_{E_0}^{(2)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=bc/(1+c), c+1<b<(c+1)^2\},\\
\mathcal{T}_{E_0}^{(3)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=bc/(1+c), b>(c+1)^2\},\\
\mathcal{T}_{E_0}^{(4)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=bc/(1+c), b=c+1\},\\
\mathcal{SN}_{E_*}^{(1)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=a_*,\ 1<b<(c+1)^2/4, c>1, \kappa\ne\kappa_*\},\\
\mathcal{SN}_{E_*}^{(2)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=a_*,\ b=(c+1)^2/4, c>1, \kappa\ne\kappa_*\},\\
\mathcal{SN}_{E_*}^{(3)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=a_*,\ (c+1)^2/4<b<(c+1)^2, c>1, \kappa\ne\kappa_*\},\\
\mathcal{SN}_{E_*}^{(4)} &:= \{(a,b,c,\kappa)\in\mathbb{R}_+^4 | a=a_*,\ 1<b<(c+1)^2, c\le 1, \kappa\ne\kappa_*\},\\
\kappa_1 &:= p_1^{-2}\{(p_1+1)(c-p_1)\}^{-1}c\{p_1(c-p_1)+a\},\\
\kappa_* &:= (c-b^{1/2}+1)^{-1}(b^{1/2}-1)^{-2}c^2.
\end{aligned} \tag{6}$$

The following lemma is a summary of Theorems 1, 2 and 3 of [27].

Lemma 2.1. **(i)** *System (4) has a saddle-node E_0 if $(a,b,c,\kappa)\in\mathcal{T}_{E_0}\cup\mathcal{P}_{E_0}$. Moreover, as (a,b,c,κ) crosses either $\mathcal{T}_{E_0}^{(1)}$ from $\mathcal{R}_3$ to $\mathcal{R}_4$, $\mathcal{T}_{E_0}^{(2)}$ from $\mathcal{R}_6$ to $\mathcal{R}_5$, or $\mathcal{T}_{E_0}^{(4)}$ from $\mathcal{R}_6$ to $\mathcal{R}_4$, a saddle E_0 and a stable (resp. unstable) node E_1 merge into a stable node E_0 on the boundary of the first quadrant $\mathcal{Q}_+$ for $\kappa<\kappa_1$ (resp. $\kappa>\kappa_1$) through a transcritical bifurcation; as (a,b,c,κ) crosses $\mathcal{T}_{E_0}^{(3)}$ from $\mathcal{R}_5$ to $\mathcal{R}_7$, a stable node E_0 and a saddle E_2 merge into a saddle E_0 on the boundary of $\mathcal{Q}_+$ through a transcritical bifurcation; as (a,b,c,κ) crosses $\mathcal{P}_{E_0}$ from $\mathcal{R}_7$ to $\mathcal{R}_5$, a saddle E_0 changes into a stable node E_0, a saddle E_2 through a pitchfork bifurcation at E_0 on the boundary of $\mathcal{Q}_+$.*
(ii) *System (4) has a weak focus E_1 of at most order 2 for $(a,b,c,\kappa)\in\mathcal{H}_{E_1}$, which is of order ℓ exactly and produces at most ℓ limit cycles through Hopf bifurcations as $(a,b,c,\kappa)\in\mathcal{H}_{E_1}^{(\ell)}$, $\ell=1,2$, where $\mathcal{H}_{E_1}^{(1)}:=\mathcal{H}_{E_1}\backslash\mathcal{H}_{E_1}^{(2)}$ and*

$$\begin{aligned}
\mathcal{H}_{E_1}^{(2)} := \{&(a,b,c,\kappa)\in\mathcal{H}_{E_1} : 2p_1(p_1+1)a^3+\{(p_1^2+p_1+1)c^2+p_1(2p_1^2+p_1-2)c\\
&-3p_1^3(p_1+1)\}a^2-(c-p_1)\{(p_1^3+3p_1^2+p_1+1)c^2+2p_1^2(p_1^2+3p_1+3)c\\
&+3p_1^4(p_1+1)\}a+p_1^2\{(p_1+2)c+p_1^2\}\{c-p_1(p_1+1)\}(c-p_1)^2=0\}.
\end{aligned}$$

(iii) *System (4) has a saddle-node E_* if $(a,b,c,\kappa)\in\mathcal{SN}_{E_*}$. Moreover, as (a,b,c,κ) crosses either $\mathcal{SN}_{E_*}^{(1)}$ from $\mathcal{R}_0$ to $\mathcal{R}_1$, $\mathcal{SN}_{E_*}^{(2)}$ from $\mathcal{R}_0$ to $\mathcal{R}_6$, or $\mathcal{SN}_{E_*}^{(3)}\cup\mathcal{SN}_{E_*}^{(4)}$*

from $\mathcal{R}_7$ to $\mathcal{R}_6$, a stable (resp. unstable) node E_1 and a saddle E_2 arise through a saddle-node bifurcation for $\kappa < \kappa_1$(resp. $\kappa > \kappa_1$).

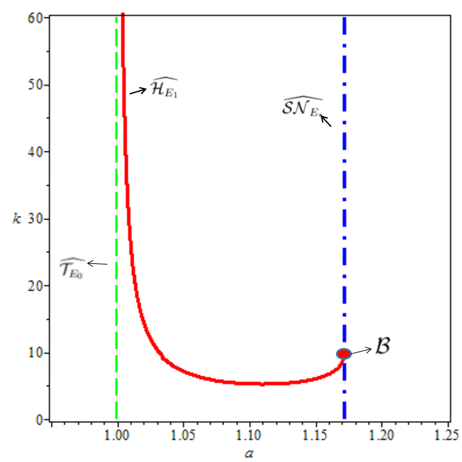

FIGURE 2. Bifurcation surfaces projection on the (a, κ)-plane.

The above Lemma 2.1 does not consider parameters in the set

$$\mathcal{B} := \{(a, b, c, \kappa) \in \mathbb{R}_+^4 | a = a_*,\ 1 < b < (c+1)^2, \kappa = \kappa_*\}, \tag{7}$$

where a_* is given below (5) and κ_* is given in (6). $\mathcal{B}$ is actually the intersection of the saddle-node bifurcation surface $\mathcal{SN}_{E_*}$ and the Hopf bifurcation surface $\mathcal{H}_{E_1}$, which are described by the curves $\widehat{\mathcal{SN}_{E_*}}$ and $\widehat{\mathcal{H}_{E_1}}$ respectively on the section $\{(a, b, c, \kappa) \in \mathbb{R}_+^4 | b = 2,\ c = 1\}$ in Figure 2. The intersection of $\widehat{\mathcal{SN}_{E_*}}$ and $\widehat{\mathcal{H}_{E_1}}$ indicates $\mathcal{B}$.

This paper is devoted to bifurcations in $\mathcal{B}$. For $(a, b, c, \kappa) \in \mathcal{B}$, equilibrium E_* is degenerate with two zero eigenvalues. In the following lemma we prove that E_* is a cusp.

Lemma 2.2. *If $(a, b, c, \kappa) \in \mathcal{B} \backslash \mathcal{C}$, where*

$$\mathcal{C} := \Big\{(a, b, c, \kappa) \in \mathcal{B} | c = \varsigma(b) := \frac{1}{4b^{1/2}}(b^{1/2} - 1)\{b^{1/2} + 2 + (17b - 12b^{1/2} + 4)^{1/2}\}\Big\},$$

then equilibrium E_* is a cusp in system (4).

Proof. For simplicity in statements, we use the notation

$$p := b^{1/2} - 1. \tag{8}$$

For $(a, b, c, \kappa) \in \mathcal{B}$, system (4) can be transformed into the form

$$\left\{\begin{array}{rl} \dot{x} = & y + \frac{c(p^2+cp+c)}{p^3}x^2 + \frac{1}{p+1}xy - \frac{p}{c^2(p+1)}y^2 - \frac{c(p^2+c)}{p^4}x^3 - \frac{p^2+2pc+2c}{p^2c(p+1)}x^2y \\ & -\frac{2p+1}{c^2(p+1)^2}xy^2 - \frac{c^2}{p^4}x^4 - \frac{2}{p^2(p+1)}x^3y - \frac{1}{c^2(p+1)^2}x^2y^2, \\ \dot{y} = & -\frac{c^3(p+1)}{p^3}x^2 - \frac{c^2(p+1)}{p^2(c-p)}xy - \frac{1}{c-p}y^2 - \frac{(p+1)(p^2+c)}{p^5(c-p)}x^3 - \frac{c(p^2+2pc+2c)}{p^3(c-p)}x^2y \\ & -\frac{2p+1}{p(p+1)(c-p)}xy^2 - \frac{c^4(p+1)}{p^5(c-p)}x^4 - \frac{2c^2}{p^3(c-p)}x^3y - \frac{1}{p(p+1)(c-p)}x^2y^2, \end{array}\right. \tag{9}$$

by translating E_* to the origin O and Jordanizing the linear part of system (4). For convenience, introducing new variables $(x, y) \mapsto (u, v)$, where $u = x$ and v denotes the right-hand side of the first equation in (9), we change (9) into the Kukles form

$$\left\{\begin{array}{rl}\dot u= & v,\\ \dot v= & -\frac{c^3(p+1)}{p^3}u^2+\frac{c\{(2p+2)c^2-(p^2+3p)c-2p^3\}}{p^3(c-p)}uv+\frac{c-2p-1}{(p+1)(c-p)}v^2+\frac{c^3(p^2+c)}{p^4(c-p)}u^3\\ & -\frac{c\{(p+1)(p+3)c^2+p(p^2-3p-3)c-p^3(3p+2)\}}{p^4(p+1)(c-p)}u^2v-\frac{(5p^2+8p+4)c+2p^2(p+1)}{cp^2(p+1)^2}uv^2\\ & -\frac{1}{c^2(p+1)}v^3-\frac{c^2(c^2+2p^2c-p^3)}{p^5(c-p)}u^4+\frac{1}{p^5(p+1)^2(c-p)}\{(p+4)(p+1)^2c^3\\ & +p(7p^3+7p^2-3p-4)c^2-p^3(8p^2+15p+8)c-2p^5(p+1)\}u^3v\\ & +\frac{(3p^3+6p^2+6p+2)c^2+p(2p+1)(2p^2+2p-1)c-p^3(p+1)(7p+4)}{cp^3(p+1)^3(c-p)}u^2v^2\\ & -\frac{(3p+4)c^2-3p(p+2)c-2p^3}{c^3p(p+1)^2(c-p)}uv^3-\frac{2c-3p}{c^4(p+1)^2(c-p)}v^4+O(|u,v|^5).\end{array}\right. \tag{10}$$

Since the linear part is nilpotent, by Theorem 8.4 in [14] system (10) is conjugated to the Bogdanov-Takens normal form, i.e., the right-hand side of the second equation is a sum of terms of the form $au^k+bu^{k-1}v$. Hence, one can use the transformation $u\to u,\ v\to v-\frac{c-2p-1}{(p+1)(c-p)}uv$ together with the time-rescaling $dt=(1-\frac{c-2p-1}{(p+1)(c-p)}u)d\tau$ to change system (10) into the following

$$\left\{\begin{array}{rl}\dot u= & v,\\ \dot v= & -\frac{c^3(p+1)}{p^3}u^2+\frac{c\{(2p+2)c^2-(p^2+3p)c-2p^3\}}{p^3(c-p)}uv+O(|u,v|^3),\end{array}\right. \tag{11}$$

where the term of v^2 is eliminated and terms of degree 2 are normalized. The term of u^2 exists since $-c^3(p+1)/p^3\neq 0$. For the existence of the term of uv, we need to discuss on the quadratic equation

$$c^2-\frac{p^2+3p}{2(p+1)}c-\frac{p^3}{p+1}=0, \tag{12}$$

which comes from the numerator of the coefficient of uv. Since the constant term is negative for $p>0$, the quadratic equation (12) has exactly one positive root

$$c=\frac{1}{4}(p+1)^{-1}p\{p+3+(17p^2+22p+9)^{1/2}\},$$

which defines the function $\varsigma(b)$ as shown in Lemma 2.2 with the replacement (8). It implies by Theorem 8.4 of [14] that for $c\neq\varsigma(b)$, i.e., $(a,b,c,\kappa)\in\mathcal{S}\backslash\mathcal{C}$, O is a cusp of system (11). The proof of this lemma is completed. □

3. **Bogdanov-Takens bifurcation.** In this section we discuss in the case that $(a,b,c,\kappa)\in\mathcal{B}\backslash\mathcal{C}$, in which system (4) is of codimension 2. We choose a,κ as the bifurcation parameters and unfold the Bogdanov-Takens normal forms of codimensions 2 when (a,κ) is perturbed near the point (a_*,κ_*), where a_* is given below (5) and κ_* is given in (6).

Theorem 3.1. *If $(a,b,c,\kappa)\in\mathcal{B}\backslash\mathcal{C}$, where $\mathcal{B}$ is defined in (7) and $\mathcal{C}$ is defined as in Lemma 2.2, then there are a neighborhood U of the point (a_*,κ_*) in the (a,κ)-parameter space and four curves*

$$\begin{aligned}\mathcal{SN}^+ &:= \Big\{(a,\kappa)\in U|a=a_*,\ \kappa>\kappa_*,0<c<\varsigma(b)\Big\}\cup\Big\{(a,\kappa)\in U|a=a_*,\ \kappa<\kappa_*,c>\varsigma(b)\Big\},\\ \mathcal{SN}^- &:= \Big\{(a,\kappa)\in U|a=a_*,\ \kappa<\kappa_*,0<c<\varsigma(b)\Big\}\cup\Big\{(a,\kappa)\in U|a=a_*,\ \kappa>\kappa_*,c>\varsigma(b)\Big\},\\ \mathcal{H} &:= \Big\{(a,\kappa)\in U|a=a_*-\Big((2b^{1/2}+1)c^2-((b^{1/2}-1)^2+3(b^{1/2}-1))c\\ &\qquad -2(b^{1/2}-1)^3\Big)^{-2}b^{1/2}(b^{1/2}-1)^6(c-b^{1/2}+1)^4(\kappa-\kappa_*)^2+O(|\kappa-\kappa_*|^3),\end{aligned}$$

$$
\begin{aligned}
&\kappa > \kappa_*, 0 < c < \varsigma(b)\Big\}\\
&\cup\Big\{(a,\kappa)\in U | a = a_* - \Big((2b^{1/2}+1)c^2 - ((b^{1/2}-1)^2 + 3(b^{1/2}-1))c\\
&-2(b^{1/2}-1)^3\Big)^{-2} b^{1/2}(b^{1/2}-1)^6(c-b^{1/2}+1)^4(\kappa-\kappa_*)^2 + O(|\kappa-\kappa_*|^3),\\
&\kappa < \kappa_*, c > \varsigma(b)\Big\},\\
\mathcal{L} := &\Big\{(a,\kappa)\in U | a = a_* - 49/25\Big((2b^{1/2}+1)c^2 - ((b^{1/2}-1)^2 + 3(b^{1/2}-1))c\\
&-2(b^{1/2}-1)^3\Big)^{-2} b^{1/2}(b^{1/2}-1)^6(c-b^{1/2}+1)^4(\kappa-\kappa_*)^2 + O(|\kappa-\kappa_*|^3),\\
&\kappa > \kappa_*, 0 < c < \varsigma(b)\Big\}\\
&\cup\Big\{(a,\kappa)\in U | a = a_* - 49/25\Big((2b^{1/2}+1)c^2 - ((b^{1/2}-1)^2 + 3(b^{1/2}-1))c\\
&-2(b^{1/2}-1)^3\Big)^{-2} b^{1/2}(b^{1/2}-1)^6(c-b^{1/2}+1)^4(\kappa-\kappa_*)^2 + O(|\kappa-\kappa_*|^3),\\
&\kappa < \kappa_*, c > \varsigma(b)\Big\},
\end{aligned}
$$

such that system (4) produces a saddle-node bifurcation near E_* as (a, c) acrosses $\mathcal{SN}^+ \cup \mathcal{SN}^-$, a Hopf bifurcation near E_* as (a, κ) acrosses $\mathcal{H}$, and a homoclinic bifurcation near E_* as (a, κ) acrosses $\mathcal{L}$, where κ_* and $\varsigma(b)$ are given in (6) and Lemma 2.2 respectively.

The above bifurcation curve $\mathcal{H}$ is exactly the same as $\mathcal{H}_{E_1}$ given in Lemma 2.1, and the union $\mathcal{SN}^+ \bigcup \mathcal{SN}^+$ is exactly the bifurcation curves $\mathcal{SN}_{E_*}$ given in Lemma 2.1.

Proof. Let $p = b^{1/2} - 1$ and

$$\varepsilon_1 := a - a_*, \quad \varepsilon_2 := \kappa - \kappa_*, \tag{13}$$

and consider $|\varepsilon_1|$ and $|\varepsilon_2|$ both to be sufficiently small. Expanding system (4) at E_*, we get

$$
\left\{
\begin{aligned}
\dot{x} = \;& \tfrac{c(p+1)}{p}\varepsilon_1 + (\tfrac{-c^2(p+1)}{p^2} + \tfrac{c}{p}\varepsilon_1)x + (-c(p+1) + (p+1)\varepsilon_1)y\\
& -\tfrac{c(c-p)}{p^2}x^2 + (-\tfrac{c(2+3p)}{p} + \varepsilon_1)xy - p(p+1)y^2 + O(\|(x,y)\|^3),\\
\dot{y} = \;& (\tfrac{c^3(p+1)}{p^4} + \tfrac{c(p+1)(c-p)}{p^2}\varepsilon_2)x + (\tfrac{c^2(p+1)}{p^2} + (p+1)(c-p)\varepsilon_2)y\\
& +(\tfrac{c^3}{p^4} + \tfrac{c(c-p)}{p^2}\varepsilon_2)x^2 + (\tfrac{c^3(2+3p)-c^2p(2p+1)}{(c-p)p^3} + \tfrac{c(3p+2)-p(2p+1)}{p}\varepsilon_2)xy\\
& +(\tfrac{c^2(p+1)}{(c-p)p} + p(p+1)\varepsilon_2)y^2 + O(\|(x,y)\|^3).
\end{aligned}
\right.
\tag{14}
$$

Introducing new variables $(x, y) \mapsto (\xi_1, \eta_1)$, where $\xi_1 = x$ and η_1 denotes the right-hand side of the first equation in (14), we change (14) into the Kukles form, whose second order truncation is the following

$$
\left\{
\begin{aligned}
\dot{\xi}_1 = \;& \eta_1,\\
\dot{\eta}_1 = \;& E_{00}(\varepsilon_1,\varepsilon_2) + E_{10}(\varepsilon_1,\varepsilon_2)\xi_1 + E_{20}(\varepsilon_1,\varepsilon_2)\xi_1^2\\
& +F(\xi_1,\varepsilon_1,\varepsilon_2)\eta_1 + E_{02}(\varepsilon_1,\varepsilon_2)\eta_1^2,
\end{aligned}
\right.
\tag{15}
$$

where $F(\xi_1, \varepsilon_1, \varepsilon_2) := E_{01}(\varepsilon_1, \varepsilon_2) + E_{11}(\varepsilon_1, \varepsilon_2)\xi_1$ and E_{ij} s $(i, j = 0, 1, 2)$ are given in Appendix. Notice that $(a, b, c, \kappa) \in \mathcal{B}\backslash\mathcal{C}$ implies that $c \neq \varsigma(b)$. From (12) we see that the quadratic equation has exactly one positive root $c = \varsigma(b)$. Thus, for $c \neq \varsigma(b)$ we can check that

$$F(0,0,0) = 0, \; \frac{\partial F}{\partial \xi_1}(0,0,0) = E_{11}(0,0) = (2p+2)(c^2 - \frac{p^2+3p}{2(p+1)}c - \frac{p^3}{p+1}) \neq 0.$$

By the Implicit Function Theorem, there exists a function $\xi_1 = \xi_1(\varepsilon_1, \varepsilon_2)$ defined in a small neighborhood of $(\varepsilon_1, \varepsilon_2) = (0, 0)$ such that $\xi_1(0, 0) = 0$ and $F(\xi_1(\varepsilon_1, \varepsilon_2), \varepsilon_1, \varepsilon_2) = 0$. Thus, from the definition of F we obtain $\xi_1(\varepsilon_1, \varepsilon_2) = -E_{01}(\varepsilon_1, \varepsilon_2)/E_{11}(\varepsilon_1, \varepsilon_2)$ near $(0, 0)$. Then, we use a parameter-dependent shift

$$\xi_2 = \xi_1 - \xi_1(\varepsilon_1, \varepsilon_2), \quad \eta_2 = \eta_1$$

to vanish the term proportional to η_2 in the equation for η_2 from system (15), which leads to the following system

$$\begin{cases} \dot{\xi_2} = & \eta_2, \\ \dot{\eta_2} = & \psi_1(\varepsilon_1, \varepsilon_2) + \psi_2(\varepsilon_1, \varepsilon_2)\xi_2 + E_{20}(\varepsilon_1, \varepsilon_2)\xi_2^2 + E_{11}(\varepsilon_1, \varepsilon_2)\xi_2\eta_2 + E_{02}(\varepsilon_1, \varepsilon_2)\eta_2^2, \end{cases} \tag{16}$$

where

$$\begin{aligned} \psi_1(\varepsilon_1, \varepsilon_2) \quad &:= \quad E_{00}(\varepsilon_1, \varepsilon_2) + E_{10}(\varepsilon_1, \varepsilon_2)\xi_1(\varepsilon_1, \varepsilon_2) + E_{20}(\varepsilon_1, \varepsilon_2)\xi_1^2(\varepsilon_1, \varepsilon_2), \\ \psi_2(\varepsilon_1, \varepsilon_2) \quad &:= \quad E_{10}(\varepsilon_1, \varepsilon_2) + 2\xi_1(\varepsilon_1, \varepsilon_2)E_{20}(\varepsilon_1, \varepsilon_2). \end{aligned}$$

In order to eliminate the η_2^2 term, one can use the transformation

$$\xi_3 = \xi_2, \ \eta_3 = \eta_2 - E_{02}(\varepsilon_1, \varepsilon_2)\xi_2\eta_2$$

together with the time-rescaling $dt = (1 - E_{02}(\varepsilon_1, \varepsilon_2)\xi_2)d\tau$ to change system (16) into the following

$$\begin{cases} \dot{\xi_3} = & \eta_3, \\ \dot{\eta_3} = & \zeta_1(\varepsilon_1, \varepsilon_2) + \zeta_2(\varepsilon_1, \varepsilon_2)\xi_3 + \tilde{E}_{20}(\varepsilon_1, \varepsilon_2)\xi_3^2 + E_{11}(\varepsilon_1, \varepsilon_2)\xi_3\eta_3, \end{cases} \tag{17}$$

where

$$\begin{aligned} \zeta_1(\varepsilon_1, \varepsilon_2) \quad &:= \quad \psi_1(\varepsilon_1, \varepsilon_2), \ \zeta_2(\varepsilon_1, \varepsilon_2) := \psi_2(\varepsilon_1, \varepsilon_2) - \psi_1(\varepsilon_1, \varepsilon_2)E_{02}(\varepsilon_1, \varepsilon_2), \\ \tilde{E}_{20}(\varepsilon_1, \varepsilon_2) \quad &:= \quad E_{20}(\varepsilon_1, \varepsilon_2) - E_{10}(\varepsilon_1, \varepsilon_2)E_{02}(\varepsilon_1, \varepsilon_2). \end{aligned}$$

Further, in order to reduce coefficient of ξ_3^2 to 1, we apply the transformation

$$u = \frac{\tilde{E}_{20}(\varepsilon_1, \varepsilon_2)}{E_{11}^2(\varepsilon_1, \varepsilon_2)}\xi_3, \quad v = \text{sign}\Big(\frac{E_{11}(\varepsilon_1, \varepsilon_2)}{\tilde{E}_{20}(\varepsilon_1, \varepsilon_2)}\Big)\frac{\tilde{E}_{20}^2(\varepsilon_1, \varepsilon_2)}{E_{11}^3(\varepsilon_1, \varepsilon_2)},$$

where $\tilde{E}_{20}(0, 0) = -\frac{c^3(p+1)}{p^3} < 0$, and the time-scaling $dt = |\frac{E_{11}(\varepsilon_1, \varepsilon_2)}{\tilde{E}_{20}(\varepsilon_1, \varepsilon_2)}|d\tau$ to system (17) and obtain

$$\begin{cases} \dot{u} = & v, \\ \dot{v} = & \phi_1(\varepsilon_1, \varepsilon_2) + \phi_2(\varepsilon_1, \varepsilon_2)u + u^2 + \vartheta uv, \end{cases} \tag{18}$$

where $\vartheta = \text{sign}\Big(\frac{E_{11}(0,0)}{\tilde{E}_{20}(0,0)}\Big)$,

$$\begin{aligned} \phi_1(\varepsilon_1, \varepsilon_2) &:= \frac{E_{11}^4(\varepsilon_1, \varepsilon_2)}{\tilde{E}_{20}^3(\varepsilon_1, \varepsilon_2)}\zeta_1(\varepsilon_1, \varepsilon_2) \\ &= \frac{\{(2p+2)c^2 - (p^2+3p)c - 2p^3\}^4\varepsilon_1\phi_{11}(\varepsilon_1, \varepsilon_2)}{p^4(c-p)^4\phi_{12}^2(\varepsilon_1, \varepsilon_2)}, \\ \phi_2(\varepsilon_1, \varepsilon_2) &:= \frac{E_{11}^2(\varepsilon_1, \varepsilon_2)}{\tilde{E}_{20}^2(\varepsilon_1, \varepsilon_2)}\zeta_2(\varepsilon_1, \varepsilon_2) \\ &= \frac{\sqrt{2}\{(2p+2)c^2 - (p^2+3p)c - 2p^3\}\phi_{21}(\varepsilon_1, \varepsilon_2)}{c^{3/2}(c-p)^2(p+1)^{1/2}p\phi_{12}^{3/2}(\varepsilon_1, \varepsilon_2)}, \end{aligned}$$

and polynomials ϕ_{ij} s are given in the Appendix.

Let

$$\mu_1 = \phi_1(\varepsilon_1, \varepsilon_2), \quad \mu_2 = \phi_2(\varepsilon_1, \varepsilon_2), \tag{19}$$

where ϕ_1 and ϕ_2 are defined just below (18). Clearly, $\phi_1(0,0) = \phi_2(0,0) = 0$. Compute the Jacobian determinant of (19) at the point $(0,0)$

$$\begin{vmatrix} \frac{\partial\phi_1(\varepsilon_1,\varepsilon_2)}{\partial\varepsilon_1} & \frac{\partial\phi_1(\varepsilon_1,\varepsilon_2)}{\partial\varepsilon_2} \\ \frac{\partial\phi_2(\varepsilon_1,\varepsilon_2)}{\partial\varepsilon_1} & \frac{\partial\phi_2(\varepsilon_1,\varepsilon_2)}{\partial\varepsilon_2} \end{vmatrix}_{(\varepsilon_1,\varepsilon_2)=(0,0)} = -\frac{\{(2p+2)c^2-(p^2+3p)c-2p^3\}^5}{p^6c^4(c-p)^4(p+1)} \neq 0, \quad (20)$$

implying that (19) is a locally invertible transformation of parameters. This transformation makes a local equivalence between system (18) and the versal unfolding system

$$\begin{cases} \dot{\widetilde{u}} = & \widetilde{v}, \\ \dot{\widetilde{v}} = & \mu_1 + \mu_2\widetilde{u} + \widetilde{u}^2 + \vartheta\widetilde{u}\widetilde{v}, \end{cases} \quad (21)$$

where ϑ is given in (18). As indicated in Section 7.3 of [10], system (21) has the following bifurcation curves

$$\begin{aligned} \mathcal{SN}^+ &:= \{(\mu_1,\mu_2) \in V_0 \mid \mu_1 = 0,\ \mu_2 > 0\}, \\ \mathcal{SN}^- &:= \{(\mu_1,\mu_2) \in V_0 \mid \mu_1 = 0,\ \mu_2 < 0\}, \\ \mathcal{H} &:= \{(\mu_1,\mu_2) \in V_0 \mid \mu_1 = -\mu_2^2,\ \mu_2 > 0\}, \\ \mathcal{L} &:= \{(\mu_1,\mu_2) \in V_0 \mid \mu_1 = -\tfrac{49}{25}\mu_2^2 + o(|\mu_2|^2),\ \mu_2 > 0\}, \end{aligned} \quad (22)$$

where V_0 is a small neighborhood of $(0,0)$ in $\mathbb{R}^2$.

In what follows, we present above bifurcation curves in parameters ε_1 and ε_2 in explicit forms. For this purpose, we need the relation between $(\varepsilon_1,\varepsilon_2)$ and (μ_1,μ_2). Note that ϕ_1 and ϕ_2 defined just below (18) are C^k near the origin (0,0)(k is an arbitrary integer). By condition (20), the well-known Implicit Function Theorem implies that there are two C^k functions

$$\varepsilon_1 = \omega_1(\mu_1,\mu_2),\ \varepsilon_2 = \omega_2(\mu_1,\mu_2) \quad (23)$$

in a small neighborhood of $(0,0,0,0)$ such that $\omega_1(0,0) = \omega_2(0,0) = 0$ and

$$\mu_1 = \phi_1(\omega_1(\mu_1,\mu_2),\omega_2(\mu_1,\mu_2)),\ \mu_2 = \phi_2(\omega_1(\mu_1,\mu_2),\omega_2(\mu_1,\mu_2)). \quad (24)$$

Substitute the second order formal Taylor expansions of ω_1 and ω_2 in (24) while expand ϕ_1 and ϕ_2 in (24) to the second order

$$\begin{aligned} \phi_1(\varepsilon_1,\varepsilon_2) = {} & \{(2p+2)c^2-(p^2+3p)c-2p^3\}^4\varepsilon_1/\{p^6c^2(c-p)^4(p+1)\} - \{(2p+2)c^2 \\ & -(p^2+3p)c-2p^3\}^4(24p^2c^4+42c^4p+21c^4-8p^3c^3-54c^3p^2-44c^3p \\ & -36c^2p^4-12p^3c^2+27p^2c^2+8p^5c+32cp^4+16p^6)\varepsilon_1^2/\{2c^4p^8(c-p)^6 \\ & (p+1)^2\} - \{(2p+2)c^2-(p^2+3p)c-2p^3\}^4\varepsilon_1\varepsilon_2/\{(c^4p^4(c-p)^3(p+1)\} \\ & +o(|\varepsilon_1,\varepsilon_2|^2), \end{aligned} \quad (25)$$

$$\begin{aligned} \phi_2(\varepsilon_1,\varepsilon_2) = {} & \{(2p+2)c^2-(p^2+3p)c-2p^3\}\varepsilon_1/\{2c^2(p^3-2cp+p^2+c^2p+c^2-2cp^2)p^4\} \\ & -\{(2p+2)c^2-(p^2+3p)c-2p^3\}\varepsilon_2/c^2 - \{(2p+2)c^2-(p^2+3p)c-2p^3\} \\ & (-243p^3c^3+832p^3c^4+513p^2c^4+455p^4c^3-594p^5c^2-1347p^3c^5-1209p^2c^5 \\ & +165p^4c^4+1138p^5c^3-324p^6c^2-424p^7c-200p^5c^4+382p^6c^3+512p^7c^2 \\ & -520cp^8-396c^5p-48p^9+108c^6-48p^{10}+384c^6p^3+414c^6p-104cp^9 \\ & +264c^2p^8+594c^6p^2-672c^5p^4+96c^6p^4-136c^5p^5-44c^4p^6-76c^3p^7)\varepsilon_1^2 \\ & /\{4c^3(p+1)^2(c-p)^4p^6\} - \{(2p+2)c^2-(p^2+3p)c-2p^3\}(8p^2c^4+23c^4p \\ & +12c^4+30p^3c^3+8c^3p^2-22c^3p-58c^2p^4-85p^3c^2+6p^2c^2-8p^5c+46cp^4 \\ & +24p^6)\varepsilon_1\varepsilon_2/\{4c^4p^2(p+1)(c-p)^2\} + (c-p)p^2\{(2p+2)c^2-(p^2+3p)c \\ & -2p^3\}\varepsilon_2^2/c^4 + o(|\varepsilon_1,\varepsilon_2|^2). \end{aligned} \quad (26)$$

Then, comparing the coefficients of terms of the same degree in (24), we obtain the second order approximations

$$\begin{aligned}\varepsilon_1 \;=\; & c^2p^6(c-p)^4(p+1)\mu_1/\{(2p+2)c^2-(p^2+3p)c-2p^3\}^4+c^2p^{10}(c-p)^6(p+1)(32p^2c^4\\ & +56c^4p+27c^4-16p^3c^3-79c^3p^2-59c^3p-48c^2p^4-19p^3c^2+36p^2c^2+12p^5c\\ & +50cp^4+24p^6)\mu_1^2/\{2\{(2p+2)c^2-(p^2+3p)c-2p^3\}^8\}+c^2p^8(c-p)^5(p+1)\\ & \mu_1\mu_2/\{(2p+2)c^2-(p^2+3p)c-2p^3\}^5+o(|\mu_1,\mu_2|^2), \qquad (27)\\ \varepsilon_2 \;=\; & c^2p^2(c-p)^2(-8p^5-12cp^4-18cp^3+8c^3p^2-11p^2c^2-9c^2p+14c^3p+6c^3)\mu_1\\ & /\{2\{(2p+2)c^2-(p^2+3p)c-2p^3\}^4\}-c^2\mu_2/\{(2p+2)c^2-(p^2+3p)c-2p^3\}\\ & +c^2p^6(c-p)^4(1314c^7p^2+630pc^7-270p^3c^4+2068p^3c^5+612p^2c^5+677p^4c^4\\ & -1134p^5c^3+4387p^5c^4-1056p^6c^3-1804p^7c^2-3741c^6p^3+756c^5p^4+1160c^3p^8\\ & -2268c^6p^4+1176c^7p^3-1272c^5p^6-352c^6p^5+384c^7p^4-320p^{11}+108c^7-704cp^{10}\\ & +224c^2p^9-2046c^5p^5+4258c^4p^6+832p^7c^4-1464p^8c^2-2289c^6p^2+1544p^7c^3\\ & -450c^6p-1344cp^9)\mu_1^2/\{8\{(2p+2)c^2-(p^2+3p)c-2p^3\}^8\}+c^2p^4(c-p)^2(40p^2c^4\\ & +61c^4p+24c^4-78p^3c^3-158c^3p^2-68c^3p-14c^2p^4+43p^3c^2+48p^2c^2+32p^5c\\ & +62cp^4+24p^6)\mu_1\mu_2/\{4\{(2p+2)c^2-(p^2+3p)c-2p^3\}^5\}+c^2p^2(c-p)\mu_2^2\\ & /\{(2p+2)c^2-(p^2+3p)c-2p^3\}^2+o(|\mu_1,\mu_2|^2). \qquad (28)\end{aligned}$$

Then we are ready to express those bifurcation curves in parameters ε_1 and ε_2.

For curves $\mathcal{S}N^{\pm}$, we need to consider $\mu_1=0$. From the first equality of (19) we see that $\mu_1=0$ if and only if $\varepsilon_1=0$ because in the expression of $\phi_1(\varepsilon_1,\varepsilon_2)$ we have $\phi_{11}(0,0)/\phi_{12}^2(0,0)=1/p^2c^2(p+1)\neq 0$. Thus, for $\mu_1=0$ we obtain from (28) that

$$\varepsilon_2=-\frac{c^2}{(2p+2)\Psi(c)}\mu_2+O(|\mu_2|^2), \qquad (29)$$

where $\Psi(c)$ is the same quadratic polynomial as given in (12). It follows that the inequality $\mu_2>0$ (or <0) together with the sign of $\Psi(c)$ determines the sign of ε_2. From the analysis of the quadratic equation (12) we see that $\Psi(c)<0$ (or >0) if $0<c<\varsigma(b)$ (or $c>\varsigma(b)$), where $\varsigma(b)$ is defined in Lemma 2.2. Hence from (22) we obtain that

$$\mathcal{SN}^+: \;=\; \{(\varepsilon_1,\varepsilon_2)\mid \varepsilon_1=0,\varepsilon_2>0,0<c<\varsigma(b)\}\cup\{(\varepsilon_1,\varepsilon_2)\mid \varepsilon_1=0,\varepsilon_2<0,c>\varsigma(b)\},$$

$$\mathcal{SN}^-: \;=\; \{(\varepsilon_1,\varepsilon_2)\mid \varepsilon_1=0,\varepsilon_2<0,0<c<\varsigma(b)\}\cup\{(\varepsilon_1,\varepsilon_2)\mid \varepsilon_1=0,\varepsilon_2>0,c>\varsigma(b)\}.$$

For curve $\mathcal{H}$, we need to consider $\mu_1=-\mu_2^2$, which is equivalent to $\Upsilon(\varepsilon_1,\varepsilon_2):=\phi_1(\varepsilon_1,\varepsilon_2)+\phi_2^2(\varepsilon_1,\varepsilon_2)=0$ by (19). Clearly, $\Upsilon(0,0)=0$ and

$$\frac{\partial\Upsilon}{\partial\varepsilon_1}\Big|_{(\varepsilon_1,\varepsilon_2)=(0,0)}=\{(2p+2)\Psi(c)\}^4/\{p^6c^2(c-p)^4(p+1)\}\neq 0.$$

By the Implicit Function Theorem, there exists a unique C^k function $\varepsilon_1=\epsilon_1(\varepsilon_2)$ such that $\epsilon_1(0)=0$ and $\Upsilon(\epsilon_1(\varepsilon_2),\varepsilon_2)=0$. Similarly to (27) and (28), expanding Υ at $(\varepsilon_1,\varepsilon_2)=(0,0)$ and substituting with a formal expansion of $\epsilon_1(\varepsilon_2)$ of order 2, we obtain by comparison of coefficients that

$$\varepsilon_1=\epsilon_1(\varepsilon_2)=-\frac{p^6(c-p)^4}{4(p+1)\Psi^2(c)}\varepsilon_2^2+o(|\varepsilon_2|^2). \qquad (30)$$

Further, replacing μ_1 with $\mu_1=-\mu_2^2$ in (28), we get

$$\varepsilon_2=-\frac{c^2}{(2p+2)\Psi(c)}\mu_2+o(|\mu_2|).$$

Similarly to (29), from (22) we obtain that

$$\mathcal{H} := \big\{(\varepsilon_1,\varepsilon_2) \mid \varepsilon_1 = -\frac{p^6(c-p)^4}{4(p+1)\Psi^2(c)}\varepsilon_2^2 + o(|\varepsilon_2|^2),\ \varepsilon_2>0, 0<c<\varsigma(b)\big\}$$
$$\cup\big\{(\varepsilon_1,\varepsilon_2) \mid \varepsilon_1 = -\frac{p^6(c-p)^4}{4(p+1)\Psi^2(c)}\varepsilon_2^2 + o(|\varepsilon_2|^2),\ \varepsilon_2<0, c>\varsigma(b)\big\}.$$

For curve $\mathcal{L}$, we need to consider $\mu_1 = -\frac{49}{25}\mu_2^2 + o(|\mu_2|^2)$, i.e., $\phi_1(\varepsilon_1,\varepsilon_2) = -\frac{49}{25}\phi_2^2(\varepsilon_1,\varepsilon_2) + o(|\phi_2|^2)$. Similarly to $\mathcal{H}$, we apply the Implicit Function Theorem to obtain

$$\varepsilon_1 = -\frac{49p^6(c-p)^4}{100(p+1)\Psi^2(c)}\varepsilon_2^2 + o(|\varepsilon_2|^2).$$

Similarly to (29), from (22) we obtain that

$$\mathcal{L} := \big\{(\varepsilon_1,\varepsilon_2) \mid \varepsilon_1 = -\frac{49p^6(c-p)^4}{100(p+1)\Psi^2(c)}\varepsilon_2^2 + o(|\varepsilon_2|^2),\ \varepsilon_2>0, 0<c<\varsigma(b)\big\}$$
$$\cup\big\{(\varepsilon_1,\varepsilon_2) \mid \varepsilon_1 = -\frac{49p^6(c-p)^4}{100(p+1)\Psi^2(c)}\varepsilon_2^2 + o(|\varepsilon_2|^2),\ \varepsilon_2<0, c>\varsigma(b)\big\}.$$

Finally, with the replacement (13) we can rewrite the above bifurcation curves $\mathcal{S}N^{\pm}, \mathcal{H}$ and $\mathcal{L}$ expressed in parameters $(\varepsilon_1,\varepsilon_2)$ in expressions in the original parameters (a,b,c,κ) as shown in Theorem 3.1. □

4. **Conclusions.** In this paper we analyzed the dynamics of system (4) near the equilibrium E_* when parameters lie near $\mathcal{B}\backslash\mathcal{C}$. We proved that E_* is a cusp when parameters lie on $\mathcal{B}\backslash\mathcal{C}$. We investigated the Bogdanov-Takens bifurcation near the cusp and compute in Theorem 3.1 the four bifurcation curves $\mathcal{SN}^+$, $\mathcal{SN}^-$, $\mathcal{H}$ and $\mathcal{L}$ in the practical parameters. Those bifurcation curves can be observed in Figure 3 in the case that $c>1$ and $b=(c+1)^2/4$ (which implies $p=(c-1)/2$). They display the merge of equilibria and the rise of homoclinic orbits and periodic orbits.

More concretely, in this case,

$$a_* = \frac{(c+1)^2}{4},\ \kappa_* = \frac{8c^2}{(c+1)(c-1)^2}.$$

Moreover, the four bifurcation curves divide the neighborhood U of (a_*,κ_*) into the following regions:

$$\mathcal{D}_I := \Big\{(a,\kappa)\in U|\ a<\frac{(c+1)^2}{4},\ \kappa\le\frac{8c^2}{(c+1)(c-1)^2}\Big\}$$
$$\bigcup\Big\{(a,\kappa)\in U|\ a<\frac{(c+1)^2}{4} - \frac{49(c-1)^6(c+1)^3}{3200(2c^2+c+1)^2}\big\{\kappa-\frac{8c^2}{(c+1)(c-1)^2}\big\}^2$$
$$+O(|\kappa-\frac{8c^2}{(c+1)(c-1)^2}|^3),\ \kappa>\frac{8c^2}{(c+1)(c-1)^2}\Big\},$$
$$\mathcal{D}_{II} := \Big\{(a,\kappa)\in U|\ \frac{(c+1)^2}{4} - \frac{49(c-1)^6(c+1)^3}{3200(2c^2+c+1)^2}\big\{\kappa-\frac{8c^2}{(c+1)(c-1)^2}\big\}^2$$
$$+O(|\kappa-\frac{8c^2}{(c+1)(c-1)^2}|^3) < a < \frac{(c+1)^2}{4} - \frac{(c-1)^6(c+1)^3}{128(2c^2+c+1)^2}$$
$$\big\{\kappa-\frac{8c^2}{(c+1)(c-1)^2}\big\}^2 + O(|\kappa-\frac{8c^2}{(c+1)(c-1)^2}|^3),\ \kappa>\frac{8c^2}{(c+1)(c-1)^2}\Big\},$$
$$\mathcal{D}_{III} := \Big\{(a,\kappa)\in U|\ \frac{(c+1)^2}{4} - \frac{(c-1)^6(c+1)^3}{128(2c^2+c+1)^2}\big\{\kappa-\frac{8c^2}{(c+1)(c-1)^2}\big\}^2$$

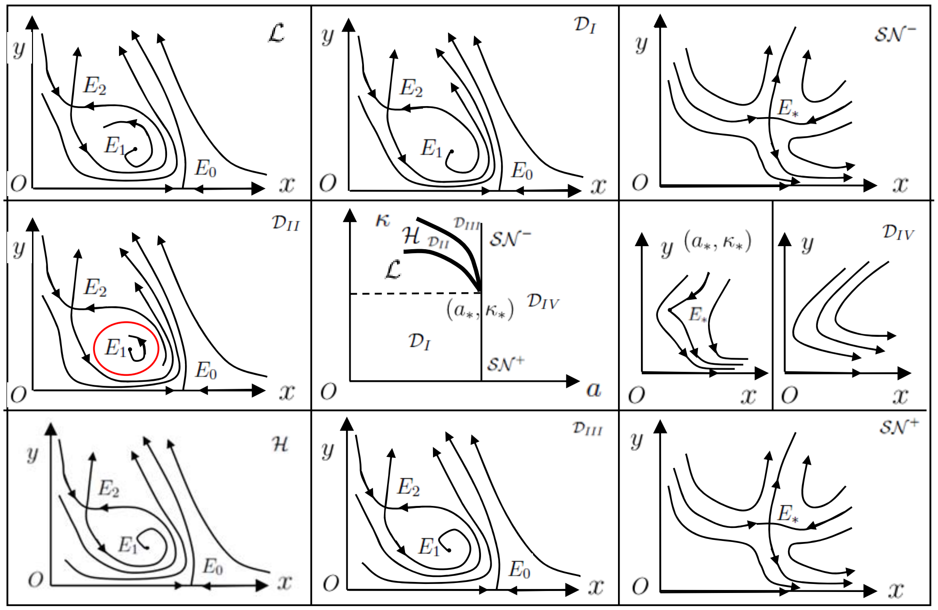

FIGURE 3. Bifurcation diagrams of system (4) for the case that $c > 1$ and $b = (c+1)^2/4$.

$$+O(|\kappa - \frac{8c^2}{(c+1)(c-1)^2}|^3) < a < \frac{(c+1)^2}{4},\ \kappa > \frac{8c^2}{(c+1)(c-1)^2}\Big\},$$
$$\mathcal{D}_{IV} := \Big\{(a,\kappa) \in U |\ a > \frac{(c+1)^2}{4}\Big\}.$$

Theorem 3.1 gives dynamical behaviors of system (4) near E_* in the first quadrant in Table 4. The coordinates of equilibria $E_0 : (x_0, 0)$, $E_1 : (p_1, q_1)$ and $E_2 : (p_2, q_2)$ are given by $x_0 := a/(b-1)$ and

$$p_1 := -\frac{1}{2}\{(a-b-c+1) - \{(a-b-c+1)^2 - 4(a-c)\}^{1/2}\},$$
$$p_2 := -\frac{1}{2}\{(a-b-c+1) + \{(a-b-c+1)^2 - 4(a-c)\}^{1/2}\}$$

as in [27]. E_0 exists in the first quadrant when $(a,\kappa) \in \mathcal{D}_I \cup \mathcal{L} \cup \mathcal{D}_{II} \cup \mathcal{H} \cup \mathcal{D}_{III}$ but disappears when $(a,\kappa) \in \mathcal{D}_{IV}$ (appearing in other quadrants) or $(a,\kappa) \in \mathcal{SN}^+ \cup \{(a_*,\kappa_*)\} \cup \mathcal{SN}^-$ (not existing).

Table 4. Dynamics of system (4) in various cases of parameter (a,κ)

Parameters	Equilibria				Limit cycles and homoclinic orbits	Region in bifurcation diagram
(a,κ)	E_0	E_1	E_2	E_*		
$\mathcal{D}_I$	saddle	unstable focus	saddle			$\mathcal{D}_I$
$\mathcal{L}$	saddle	unstable focus	saddle		one homoclinic rrbit	$\mathcal{L}$
$\mathcal{D}_{II}$	saddle	unstable focus	saddle		one limit cycle	$\mathcal{D}_{II}$
$\mathcal{H}$	saddle	stable focus	saddle			$\mathcal{H}$
$\mathcal{D}_{III}$	saddle	stable focus	saddle			$\mathcal{D}_{III}$
$\mathcal{SN}^+$				saddle-node		$\mathcal{SN}^+$
$\mathcal{D}_{IV}$						$\mathcal{D}_{IV}$
(a_*,κ_*)				cusp		(a_*,κ_*)
$\mathcal{SN}^-$				saddle-node		$\mathcal{SN}^-$

The appearance of limit cycle displays a rise of oscillatory phenomenon in system (4). Choosing parameters $a = 3.99999, b = 4, c = 3$ and $\kappa = 4.495$ in $\mathcal{D}_{II}$, we used

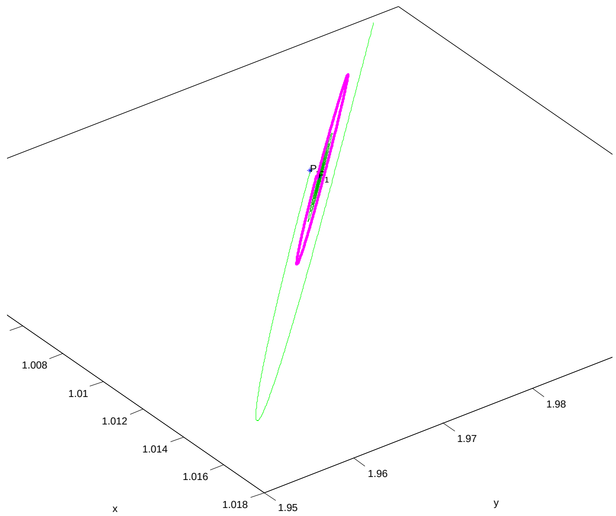

FIGURE 4. An attracting limit cycle.

the command *ODE45* in the software *Matlab* Version R2014a to simulate the orbit initiated from $(x_0, y_0) = (1.00432, 1.98662845)$ numerically, which plots an attractive limit cycle in Figure 4 and shows a dynamic balance and permanence of the substrate and the product in the enzyme-catalyzed reaction. The homoclinic loop actually gives a boundary for the break of the dynamic balance and permanence.

In this paper we only considered parameters in $\mathcal{B}\backslash\mathcal{C}$. When parameters lie in $\mathcal{C}$, higher degeneracy may happen at E_*. Although efforts have been made for higher degeneracies, for example, versal unfolding was discussed in [5] for a normal form of cusp system of codimension 3, it is still difficult to compute bifurcation curves in original parameters in the case of codimension 3. Such a computation with original parameters is indispensible for practical systems and for system (4) it will be our next work.

Appendix: Some coefficients. The functions in system (15) are

$$
\begin{aligned}
E_{00} &:= \{(2p+2)c^2-(p^2+3p)c-2p^3\}^4\varepsilon_1/\{c^2(p+1)p^6(c-p)^4\},\\
E_{10} &:= -\{(2p+2)c^2-(p^2+3p)c-2p^3\}^2\varepsilon_1\{(-6c^3p-4c^3p^2-4p^3c^2+3p^2c^2+4cp^4+4c^4p\\
&\quad +3c^4)-(p^2c^2-3c^3p-3c^2p+cp^2+2cp^3-2p^4)\varepsilon_1-(p^3c^2-2cp^4+p^5+4c^2p^4\\
&\quad -5p^5c-p^3c^3+2p^6)\varepsilon_2\}/\{(p+1)p^4c^3(c-p)^4\},\\
E_{20} &:= \{(-2c^6(p+1)^2(c-p)^2)+(9c^3p^2+4c^2p^4-13c^4p+4p^5c^2+6p^3c^3+9c^5p-15p^2c^4\\
&\quad -2p^4c^3+4p^2c^5-4p^3c^4+6c^5)\varepsilon_1-(2p^7c-6p^7c^2-6p^6c^2-2p^5c^4+6p^6c^3+2cp^8\\
&\quad -2p^4c^4+6p^5c^3)\varepsilon_2+(6p^5c^2-2p^4c^3-6p^6c-6p^7c+6p^6c^2-2p^5c^3+2p^7+2p^8)\varepsilon_1\varepsilon_2\\
&\quad +(6p^3c^3-4p^2c^4-2c^2p^4-10p^3c^2-9c^4p-2cp^4+17c^3p^2-2p^5c+13c^3p-9p^2c^2\\
&\quad -6c^4)\varepsilon_1^2\}/\{2c^3p^2(c-p)^2(p+1)\},\\
E_{01} &:= -\{(2p+2)c^2-(p^2+3p)c-2p^3\}\{2c^3\varepsilon_1+(cp^4-2p^3c^2+c^3p^2)\varepsilon_2+(2p^4-6cp^3\\
&\quad +4p^2c^2)\varepsilon_1\varepsilon_2+(12c^2-6cp)\varepsilon_1^2\}/\{p^2(c-p)^2c^3\},\\
E_{11} &:= \{(3c^3p^2-8p^2c^4-p^4c^3+2c^5+2c^2p^4+4c^5p+2p^2c^5-5c^4p+2p^5c^2+2p^3c^3\\
&\quad -3p^3c^4)+(3c^2p^4+3c^3p+p^2c^2+2p^5c+3p^2c^4+3p^3c^2+2c^4p+2cp^4-4p^3c^3\\
&\quad +c^3p^2)\varepsilon_1+(5p^6c^2-2p^7c-3p^6c+7p^5c^2+2c^2p^4-p^5c-5p^4c^3-p^3c^3+p^3c^4
\end{aligned}
$$

$$
\begin{aligned}
&+p^4c^4-4p^5c^3)\varepsilon_2-(5p^6c-4p^5c^2+p^4c^3-5c^2p^4-p^3c^2+7p^5c+p^3c^3+2cp^4\\
&-2p^7-p^5-3p^6)\varepsilon_1\varepsilon_2+(13cp^2-8cp^4+9c^3p^2-38p^2c^2+5cp^3+10p^4+10p^5\\
&+19c^3p+10c^3-13p^3c^2-25c^2p)\varepsilon_1^2\}/\{c^2(p+1)(-p+c)\},\\
E_{02} :=\ &\{(c-2p-1)+(5c^3-2c^2p)\varepsilon_1-(3p^3c^2-2c^3p^2-cp^4)\varepsilon_2+(p^4-cp^3)\varepsilon_1\varepsilon_2-(2cp\\
&-c^2)\varepsilon_1^2\}/\{(p+1)^2(c-p)^2\}.
\end{aligned}
$$

The functions below system (18) are

$$
\begin{aligned}
\phi_{11} :=\ &24c^6p^5+4c^8p^2-16c^7p^4+4c^8p^3-16c^5p^6+4p^7c^4+24c^6p^4-16c^7p^3-16c^5p^5\\
&+4c^4p^6+(9p^4c^4-16p^6c^3+40c^3p^7+68p^5c^4-26p^3c^5+3c^8-6c^8p+42c^6p^3+36c^6p^4\\
&-94c^5p^4+6c^7p^2-4c^4p^6-16c^2p^8-56c^5p^5-8c^8p^2+8c^7p^3+28c^6p^2-14c^7p)\varepsilon_1\\
&+(4c^7p^4+40c^5p^7-4c^2p^9-4c^2p^{10}+20c^3p^8+20c^3p^9-20c^6p^5+40c^5p^6-20c^6p^6\\
&-40c^4p^8+4c^7p^5-40p^7c^4)\varepsilon_2-(40p^2c^5+12p^4c^3+32c^7p^2+8p^5c^3+12c^7+92p^3c^5\\
&+8p^6c^2-32p^3c^4-12p^6c^3-28p^7c^2+4c^5p^4-88c^6p^2-56p^4c^4+36c^7p+48p^5c^4\\
&-60c^6p^3+16cp^8-32c^6p)\varepsilon_1^2+(12cp^9-24p^7c^4-8c^7p^4-6c^7p^3-88c^5p^5-32c^2p^8\\
&+20cp^{10}-24c^5p^4+6c^6p^3+2c^3p^7-24p^6c^3+36c^6p^5+72c^4p^6+96c^3p^8-76c^2p^9\\
&+36p^5c^4+40c^6p^4-44c^5p^6+6p^7c^2)\varepsilon_1\varepsilon_2+(8p^7c-9p^2c^4-16p^5c^2+6p^3c^3-c^2p^4\\
&+11p^4c^4+6p^3c^5+10p^4c^3-16p^5c^3-18p^2c^5+12p^3c^4-9c^6p^2+4p^6c-4p^8)\varepsilon_1^3\\
&+(-34c^4p^6+2c^3p^7+4p^9-28cp^9-16cp^8+6p^4c^4+8p^{10}-2p^7c+32c^2p^8+32p^7c^2\\
&-32p^6c^3-14p^5c^3+10p^6c^2-6c^6p^4+26c^5p^5+12p^5c^4)\varepsilon_1^2\varepsilon_2+(4c^3p^7-c^6p^6+44c^3p^9\\
&-41c^2p^{10}-c^4p^6+4cp^9+2c^5p^6-4p^{11}+28c^3p^8-32c^2p^9+8c^5p^7-26c^4p^8+20p^{11}c\\
&-12p^7c^4-p^{10}-4p^{12}+18cp^{10}-6c^2p^8)\varepsilon_1\varepsilon_2^2,\\
\phi_{12} :=\ &(2p^5c^3-4p^3c^4+2p^4c^3+2p^3c^5+2p^2c^5-4p^4c^4)+(9c^3p^2+4c^2p^4-13c^4p+4p^5c^2\\
&+6p^3c^3+9c^5p-15p^2c^4-2p^4c^3+4p^2c^5-4p^3c^4+6c^5)\varepsilon_1+(-2p^7c+6p^7c^2+6p^6c^2\\
&+2p^5c^4-6p^6c^3-2cp^8+2p^4c^4-6p^5c^3)\varepsilon_2+(6p^3c^3-4p^2c^4-2c^2p^4-10p^3c^2-9c^4p\\
&-2cp^4+17c^3p^2-2p^5c+13c^3p-9p^2c^2-6c^4)\varepsilon_1^2+(6p^5c^2-2p^4c^3-6p^6c-6p^7c\\
&+6p^6c^2-2p^5c^3+2p^7+2p^8)\varepsilon_1\varepsilon_2,\\
\phi_{21} :=\ &(6c^{10}+12c^8p^5+69c^8p^4-77c^9p^3+20c^7p^6+9c^6p^4-33c^7p^3+18c^5p^6-34c^6p^5\\
&+45c^8p^2-26c^7p^4+102c^8p^3-27c^9p-80c^9p^2+27c^7p^5+6c^5p^7+8c^4p^8-12c^5p^8\\
&-55c^6p^6-12c^6p^7+8c^4p^9+22c^{10}p^2-24c^9p^4+8c^{10}p^3+20c^{10}p)\varepsilon_1+(4p^{10}c^4+20p^9c^6\\
&-10p^{10}c^5-4p^5c^9+2p^{11}c^4-2c^9p^4-2p^6c^9+10p^7c^8+20p^6c^8+10c^8p^5-20p^9c^5\\
&+2c^4p^9-40p^7c^7-20c^7p^8-10c^5p^8-20c^7p^6+40c^6p^8+20c^6p^7)\varepsilon_2+(-12c^9+12c^3p^9\\
&-47c^8p^4+10c^9p^3-86c^6p^4-19c^7p^3+102c^5p^6-220c^6p^5+60c^8p^2+159c^7p^4-40c^8p^3\\
&+61c^5p^5+2c^4p^6-16p^7c^4-18c^9p+3c^9p^2+92c^7p^5+12c^3p^8+26c^5p^7-14c^4p^8\\
&+53c^8p+35c^6p^3-76c^7p^2-79c^6p^6)\varepsilon_1^2+(2p^5c^9-34c^8p^5+2c^3p^9+19c^8p^4-10c^9p^3\\
&+151c^7p^6-17c^5p^6+39c^6p^5-45c^7p^4+26c^8p^3-2c^3p^{10}+3p^7c^4-6c^9p^2+23c^7p^5\\
&+77c^5p^7-26c^4p^8+145c^5p^8-85c^6p^6-227c^6p^7-31c^4p^9-2c^9p^4-103c^6p^8+83p^7c^7\\
&+51p^9c^5-2p^{10}c^4-4p^{11}c^3-27p^6c^8)\varepsilon_1\varepsilon_2+(-4p^7c^8-2p^6c^8-30p^{10}c^4-60p^{11}c^4\\
&+40p^9c^5-2p^8c^8+12p^{11}c^3+24c^7p^8+12c^7p^9-60p^9c^6+12p^{13}c^3-30p^{10}c^6+40p^{11}c^5\\
&-30p^{12}c^4-4p^{13}c^2-2p^{14}c^2-2p^{12}c^2+24p^{12}c^3-30c^6p^8+80p^{10}c^5+12p^7c^7)\varepsilon_2^2\\
&+(-30c^8+69p^3c^5-16p^4c^4-212p^5c^4+58p^6c^3+331c^5p^4-232c^6p^4+79c^7p^3\\
&+117c^5p^6-65c^6p^5-21c^8p^2-3c^7p^4+5c^8p^3+379c^5p^5-187c^4p^6+4p^7c^4-94c^3p^8\\
&+44c^2p^8+44c^2p^9+91c^7p-53c^8p-263c^6p^3+163c^7p^2-106c^6p^2-38c^3p^7)\varepsilon_1^3\\
&+(36c^2p^{10}-10c^8p^5-199c^3p^9-41c^8p^4+9c^7p^6+18c^2p^{11}-166c^6p^4+84c^7p^3\\
&+165c^5p^6-297c^6p^5-18c^8p^2+193c^7p^4-48c^8p^3-110c^3p^{10}+164c^5p^5-76c^4p^6\\
&+47p^7c^4+123c^7p^5-78c^3p^8+18c^2p^9-208c^5p^7+351c^4p^8-219c^5p^8-62c^6p^6\\
&+79c^6p^7+233c^4p^9+12c^3p^7)\varepsilon_1^2\varepsilon_2+(-2p^{14}c+c^2p^{10}-4c^3p^9-8c^7p^6+21c^2p^{11}\\
&-2p^{12}c-72c^3p^{10}-4p^{13}c-4c^5p^7+6c^4p^8-102c^5p^8+c^6p^6+45c^6p^7+118c^4p^9
\end{aligned}
$$

$$+58p^9c^6 - 114p^{10}c^5 + 121p^{11}c^4 + 102c^6p^8 - 22p^7c^7 - 212p^9c^5 + 233p^{10}c^4 - 138p^{11}c^3$$
$$+40p^{12}c^2 - 70p^{12}c^3 + 20p^{13}c^2 - 14c^7p^8 + p^7c^8 + p^6c^8)\varepsilon_1\varepsilon_2^2(-176p^3c^4 + 41p^4c^3$$
$$+769p^3c^5 + 293p^2c^5 - 388p^4c^4 + 27p^5c^3 + 28p^6c^2 - 178p^5c^4 - 58p^6c^3 + 4p^7c^2 + 20cp^8$$
$$+20cp^9 + 603c^5p^4 - 192c^6p^4 + 75c^7p^3 + 127c^5p^5 + 34c^4p^6 + 72c^7 - 234c^6p - 24c^2p^8$$
$$+210c^7p - 616c^6p^3 + 213c^7p^2 - 658c^6p^2 - 44c^3p^7)\varepsilon_1^4 + (-286p^5c^4 + 154p^6c^3 - 32p^7c^2$$
$$+136c^2p^{10} - 56cp^{10} - 198c^3p^9 - 24cp^9 + 262c^5p^4 - 32cp^{11} - 330c^6p^4 + 70c^7p^3$$
$$+438c^5p^6 - 284c^6p^5 + 68c^7p^4 + 636c^5p^5 - 580c^4p^6 - 210p^7c^4 + 22c^7p^5 - 154c^3p^8$$
$$+30c^2p^8 + 198c^2p^9 + 64c^5p^7 + 84c^4p^8 - 122c^6p^3 + 24c^7p^2 - 76c^6p^6 + 198c^3p^7)\varepsilon_1^3\varepsilon_2$$
$$+(4p^{12} + 102c^2p^{10} - cp^{10} - 158c^3p^9 - c^7p^6 + 198c^2p^{11} - 64p^{12}c - 33cp^{11} - c^5p^6$$
$$-313c^3p^{10} - 32p^13c + 4p^7c^4 + 8p^{13} - 6c^3p^8 + 4c^2p^9 - 57c^5p^7 + 132c^4p^8 - 126c^5p^8$$
$$+10c^6p^6 + 26c^6p^7 + 272c^4p^9 + 4p^{14} + 16c^6p^8 - p^7c^7 - 70p^9c^5 + 144p^{10}c^4 - 161p^{11}c^3$$
$$+100p^{12}c^2)\varepsilon_1^2\varepsilon_2^2.$$

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