

ON THE CONTINUITY OF THE FUNCTION DESCRIBING THE TIMES OF MEETING IMPULSIVE SET AND ITS APPLICATION

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ABSTRACT. The properties of the limit sets of orbits of planar impulsive semi-dynamic system strictly depend on the continuity of the function, which describes the times of meeting impulsive sets. In this note, we will show a more realistic counter example on the continuity of this function which has been proven and widely used in impulsive dynamical system and applied in life sciences including population dynamics and disease control. Further, what extra condition should be added to guarantee the continuity of the function has been addressed generally, and then the applications and shortcomings have been discussed when using the properties of this function.

1. Introduction. The definition of impulsive semi-dynamical system and its properties including the limit sets of orbits have been investigated [1, 9]. The generalized planar impulsive dynamical semi-dynamical system can be described as follows

$$\begin{cases} \frac{dx}{dt} = P(x, y), & \frac{dy}{dt} = Q(x, y), & (x, y) \notin M, \\ \Delta x = a(x, y), & \Delta y = b(x, y), & (x, y) \in M, \end{cases} \quad (1)$$

where $(x, y) \in R^2$, $\Delta x = x^+ - x$ and $\Delta y = y^+ - y$. P, Q, a, b are continuous functions from R^2 into R , $M \subset R^2$ denotes the impulsive set. For each point $z(x, y) \in M$, the map or impulsive function $I : R^2 \rightarrow R^2$ is defined as

$$I(z) = z^+ = (x^+, y^+) \in R^2, \quad x^+ = x + a(x, y), \quad y^+ = y + b(x, y)$$

and z^+ is called as an impulsive point of z .

Let $N = I(M)$ be the phase set (i.e. for any $z \in M, I(z) = z^+ \in N$), and $N \cap M = \emptyset$. Let (X, Π, R_+) or (X, Π) be a semi-dynamical system, where $X = R^2$ is a metric space, R_+ is the set of all non-negative reals. For any $z \in X$, the function $\Pi_z : R_+ \rightarrow X$ defined by $\Pi_z(t) = \Pi(z, t)$ is clearly continuous such that $\Pi(z, 0) = z$ for all $z \in X$, and $\Pi(\Pi(z, t), s) = \Pi(z, t + s)$ for all $z \in X$ and $t, s \in R_+$. The set

$$C^+(z) = \{\Pi(z, t) | t \in R_+\}$$

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is called the positive orbit of z . For all $t \geq 0$ and $z \in X$, we define $F(z, t) = \{w : \Pi(w, t) = z\}$, and further for any set $M \subset X$, let

$$M^+(z) = C^+(z) \cap M - \{z\}.$$

Based on above notations, the definition of impulsive semi-dynamical system is defined as follows [1, 9, 23].

Definition 1.1. An planar impulsive semi-dynamic system $(R^2, \Pi; M, I)$ consists of a continuous semi-dynamic system (R^2, Π) together with a nonempty closed subset M (or impulsive set) of R^2 and a continuous function $I : M \rightarrow R^2$ such that for every $z \in M$, there exists a $\epsilon_z > 0$ such that

$$F(z, (0, \epsilon_z)) \cap M = \emptyset \text{ and } \Pi(z, (0, \epsilon_z)) \cap M = \emptyset.$$

Definition 1.2. Let (R^2, Π) be a dynamical system. An open set V in R^2 is called a tube if there exist an $s > 0$ and a subset $S \subset V$ such that

1. $\Pi(z, t) \in V$ for all $z \in S$ and $-s < t < s$, and
2. for each $z \in V$ there is a unique $s(z)$, $|s(z)| < s$ such that $\Pi(z, s(z)) \in S$.

It is clear that $V = [\Pi(z, t) | z \in S, -s < t < s]$. The set S is called an (s, V) -section of the tube V . It is well known that if $z_0 \in R^2$ is not a rest point, that is, $\Pi(z_0, t) \neq z_0$ for all t , then there exists a tube containing z_0 .

Definition 1.3. Let (R^2, Π) be a dynamical system. A subset $M \subset R^2$ is said to be well placed in R^2 if every point $z \in M$ lies in some tube whose section lies in M .

Denote the points of discontinuity of Π_z by $\{z_n^+\}$ and call z_n^+ an impulsive point of z_n . A function Φ from R^2 into the extended positive reals $R_+ \cup \{\infty\}$ is defined as follows: let $z \in X$, if $M^+(z) = \emptyset$ then $\Phi(z) = \infty$, otherwise if $M^+(z) \neq \emptyset$ then $\Phi(z) = s$, where $\Pi(z, t) \notin M$ for $0 < t < s$ but $\Pi(z, s) \in M$. Obviously, this function describes the times of meeting impulsive sets, and its continuity plays an important role in impulsive semi-dynamical system. In 1990 [9], Kual defined the time function (denoted as Φ)-the time of reaching impulsive points in the given impulsive semi-dynamical system $(X, \pi; M, I)$. And he also provided a theorem that the function Φ is continuous on X when the impulsive set M is well placed. That is, the following main results on the continuity of function Φ have been provided in [1, 9] (i.e. Theorem 2.1 in [1] and Theorem 4 in [9]).

Theorem 1.4. *Let (R^2, Π) be a dynamical system. Let M be a closed and well placed subset of R^2 . Suppose that (R^2, Π, M, I) is an impulsive semi-dynamical system. Then Φ is continuous on R^2 .*

2. Example and main results. In 2004 [2], the author pointed out some errors on Theorem 1.4, that is, it need not be continuous under the assumptions. And the main aspect concerned in the paper [2] is the continuity of Φ on the impulsive set M . In another word, the author has proved that the function Φ sometimes is not continuous (actually not low semicontinuous) on some points in the impulsive set M and more conditions are needed to guarantee its continuity. They concluded that Φ just continuous on $X \setminus M$ under the assumptions in [9]. And in paper [1], the authors have adopted this result. In this note, we have make further efforts that the function would not be continuous on $X \setminus M$ under the assumption in the paper [9]. We first address this in the following by a realistic example in pest control and discuss generally what extra condition should be added to guarantee the continuity of Φ on $X \setminus M$.

In the following we will provide an example to show this Theorem is not true for some special cases. Considering the following model with state-dependent feedback control

$$\left\{ \begin{array}{l} \frac{dx(t)}{dt} = ax(t) \left[1 - \frac{x(t)}{K} \right] - \frac{\beta x(t)y(t)}{1 + \omega x(t)}, \\ \frac{dy(t)}{dt} = \frac{\eta\beta x(t)y(t)}{1 + \omega x(t)} - \delta y(t), \\ x(t^+) = (1 - \theta)x(t), \\ y(t^+) = y(t) + \tau, \end{array} \right\} \quad \begin{array}{l} x < ET, \\ x = ET. \end{array} \tag{2}$$

where $x(t^+) \doteq x^+$ and $y(t^+) \doteq y^+$ denote the numbers of pests and natural enemies after an integrated control strategy is applied at time t [11, 17]. The definitions of all parameters and their biological meanings please refer to literatures [11, 17]. For convenience, we assume that the initial density of the pest population is always less than the economic threshold (ET) (i.e. $x(0^+) \doteq x_0^+ < ET$, $y(0^+) \doteq y_0^+ > 0$) and $ET < K$. In model (2), $0 \leq \theta < 1$ is the proportion by which the pest density is reduced by killing and/or trapping once the number of pests reaches ET , while τ ($\tau \geq 0$) is the constant number of natural enemies released at this time t .

Define four curves as follows

$$L_0 : x = \frac{\delta}{\eta\beta - \delta\omega}; \quad L_1 : y = \frac{r}{\beta} \left[1 - \frac{x}{K} \right] (1 + \omega x);$$

$$L_2 : x = ET; \quad \text{and} \quad L_3 : x = (1 - \theta)ET.$$

The intersection points of two lines L_1 and L_2 , L_1 and L_3 are $Q_{ET} = (ET, y_{ET})$ and $Q_{\theta ET} = ((1 - \theta)ET, y_{\theta ET})$, respectively, where

$$y_{ET} = \frac{r}{\beta} \left[1 - \frac{ET}{K} \right] (1 + \omega ET), \quad y_{\theta ET} = \frac{r}{\beta} \left[1 - \frac{(1 - \theta)ET}{K} \right] (1 + \omega(1 - \theta)ET).$$

Define the open set in R_+^2 as follows

$$\Omega = \{(x, y) | x > 0, y > 0, x < ET\} \subset R_+^2 = \{(x, y) | x \geq 0, y \geq 0\}. \tag{3}$$

In the following we assume that model (2) without impulsive effects exists an unstable focus E^* with

$$E^* = (x_e, y_e) = \left(\frac{\delta}{\eta\beta - \delta\omega}, \frac{r\eta(K\eta\beta - K\delta\omega - \delta)}{K(\eta\beta - \delta\omega)^2} \right),$$

which means that model (2) without impulsive effects has a unique stable limit cycle (denoted by Γ) [11, 17], as shown in Fig.1. Note that the Γ intersects with the isocline L_1 at two points $E_{\Gamma_1}(x_{\Gamma_1}, y_{\Gamma_1})$ and $E_{\Gamma_2}(x_{\Gamma_2}, y_{\Gamma_2})$ with $x_{\Gamma_1} < x_{\Gamma_2}$.

In the following we show that model (2) defines an impulsive semi-dynamical system. From a biological point of view, we focus on the space R_+^2 to discuss model (2). As an example, we only focus on the special case, i.e. $x_{\Gamma_1} < (1 - \theta)ET < ET < x_{\Gamma_2}$, as shown in Fig.1, and the trajectory initiating from the point $Q_{\theta ET}$ (denoted by Γ_1) plays a key role in defining the impulsive set and determining the continuity of the function Φ . There are three cases: (A) Γ_1 reaches the section $S_{ET} = \{(x, y) | x = ET, y \geq 0\}$ without crossing the curve L_1 ; (B) Γ_1 reaches the section $S_{ET} = \{(x, y) | x = ET, y \geq 0\}$ just on the point Q_{ET} ; and (C) Γ_1 will cross the curve L_1 several times before reaching the section S_{ET} , shown in Fig.2(A), (B) and (C), respectively.

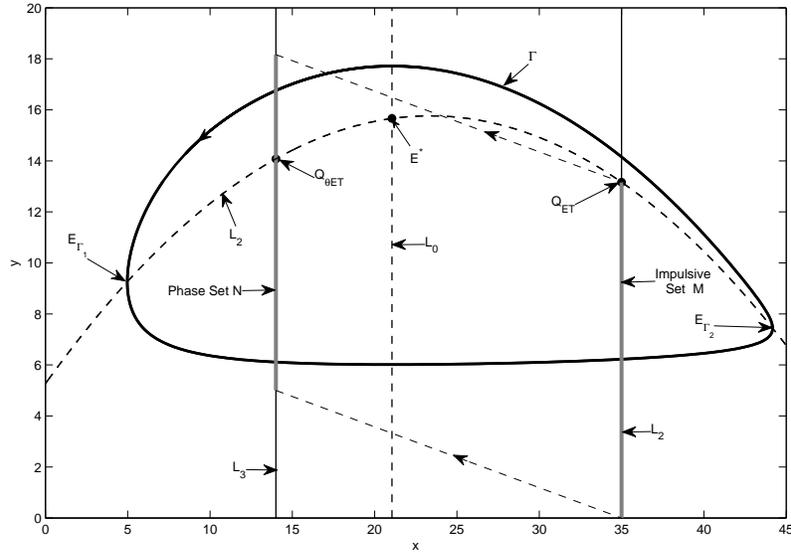


FIGURE 1. Illustrations of impulsive set, phase set and definition of impulsive semi-dynamical system for model (2). The parameter values are fixed as: $r = 1, K = 52, \beta = 0.19, \eta = 0.45, \omega = 0.19, \delta = 0.36, \tau = 5, ET = 35, \theta = 0.7$.

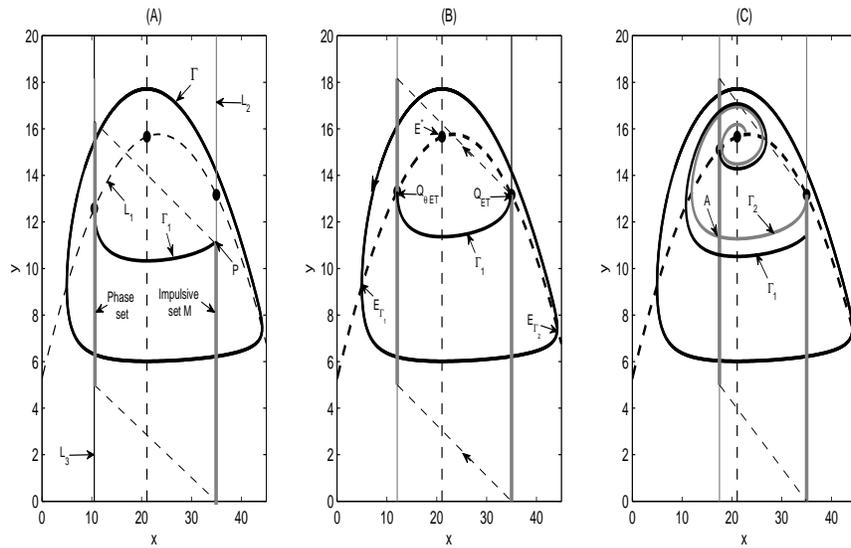


FIGURE 2. Three possible trajectories of model (2) with $x_{\Gamma_1} < (1 - \theta)ET < ET < x_{\Gamma_2}$ for model (2). The parameter values are fixed as: $r = 1, K = 52, \beta = 0.19, \eta = 0.45, \omega = 0.19, \delta = 0.36, \tau = 5, ET = 35, \theta = 0.7(A), 0.659(B)$ and $0.5(C)$.

Further, we define the section $S_{\theta ET} = \{(x, y) | x = (1 - \theta)ET, y \geq 0\}$ and assume that it does not involve the rest point E^* in it. Choosing the section $S_{\theta ET}$ as a Poincaré section. Assume that the point $P_k^+ = ((1 - \theta)ET, y_k^+)$ lies in the section $S_{\theta ET}$ (here $E^* \notin S_{\theta ET}$), and the trajectory $\Psi(t, t_0, (1 - \theta)ET, y_k^+) = (x(t, t_0, (1 - \theta)ET, y_k^+), y(t, t_0, (1 - \theta)ET, y_k^+))$ initiating from P_k^+ will reach at the section S_{ET} in a finite time t_1 (i.e. $x(t_1, t_0, (1 - \theta)ET, y_k^+) = ET$ and $\Phi(y_k^+) = t_1$), denote the intersection point as $P_{k+1} = (ET, y_{k+1})$. This indicates that y_{k+1} is determined by y_k^+ , i.e. we have $y_{k+1} = y(t_1, t_0, (1 - \theta)ET, y_k^+) \doteq \mathcal{P}(y_k^+)$. One time state dependent feedback control action is implemented at point P_{k+1} such that it jumps to point $P_{k+1}^+ = ((1 - \theta)ET, y_{k+1}^+)$ with $y_{k+1}^+ = y_{k+1} + \tau$ on $S_{\theta ET}$. Therefore, we can define the Poincaré map P_M as

$$y_{k+1}^+ = \mathcal{P}(y_k^+) + \tau = y(t_1, t_0, (1 - \theta)ET, y_k^+) + \tau \doteq P_M(y_k^+), \text{ and } \Phi(y_k^+) = t_1. \quad (4)$$

Now define the impulsive set M for three cases shown in Fig.1 as

$$\mathcal{M} = \{(x, y) | x = ET, 0 \leq y \leq Y_M\}, \quad (5)$$

which is a closed subset of R_+^2 , where $Y_M = \mathcal{P}(y_{\theta ET})$ for cases (A) and (B), and $Y_M = y_{ET}$ for case (C). It is easy to see that any solution of model (2) without impulsive effects initiating from Ω will reach in the impulsive set M in a finite time except for the rest point E^* . Define the continuous function $I : (ET, y) \in \mathcal{M} \rightarrow (x^+, y^+) = ((1 - \theta)ET, y + \tau) \in \Omega$. Thus, the phase set \mathcal{N} can be defined as follows

$$\mathcal{N} = I(\mathcal{M}) = \{(x^+, y^+) \in \Omega | x^+ = (1 - \theta)ET, \tau \leq y^+ \leq \mathcal{P}(y_{\theta ET}) + \tau\}. \quad (6)$$

Therefore, $(R_+^2, \Pi; \Omega, \mathcal{M}, I)$ or $(R_+^2, \Pi; \mathcal{M}, I)$ with respect to model (2) defines an impulsive semi-dynamical system.

According to the Definition 1.3 and topological structure of orbits of model (2) without impulsive effects, it is easy to see that M defined in (5) is a closed and well placed subset of R_+^2 , and further no initial point in (R^2, Π) belongs to the impulsive set M . This indicates that according to Theorem 1 the function Φ is continuous for the three cases shown in Fig.2. For the cases shown in Fig.2(A) and (B) it is easy to see that the results are true.

However, this is not true for case (C) shown in Fig.2(C). In fact, for case (C) there exists a trajectory (denoted by Γ_2) initiating from Ω will tangent to the section S_{ET} at point Q_{ET} , and the orbit Γ_2 will intersect with the section $S_{\theta ET}$ at several points, for example, point $A = ((1 - \theta)ET, y_A)$ as shown in Fig.2(C). Now we consider this point A and choose an open set U which contains the point A . Taking any two points $A_i \in U \cap S_{\theta ET} (i = 1, 2)$ with $y_{A_1} < y_A < y_{A_2}$, we see that the trajectory starting from the point A_2 will cross the curve L_1 at least one time before hitting the impulsive set M , and the trajectory starting from the point A_1 will reach the impulsive set M directly. This confirms that both trajectories initiating from the set $U \cap S_{\theta ET}$ can not simultaneously reach the tube V of the impulsive set M for small s , and consequently the function Φ is not continuous at the point A .

If we fixed all the parameter values as those shown in Fig.3, then we can see that the continuities of the Poincaré map and the function Φ depend on the parameter θ . For larger θ values both the Poincaré map and the function Φ are continuous, and for smaller θ values both the Poincaré map and the function Φ exist several discontinuous points, as shown in Fig.3. Moreover, we can show that the function Φ is continuous when the E^* is stable equilibrium or E^* does not exist. Thus, this counter example shows that more exact conditions are needed to show the continuity of the function Φ [1, 9]. Based on above discussion, we can see that if

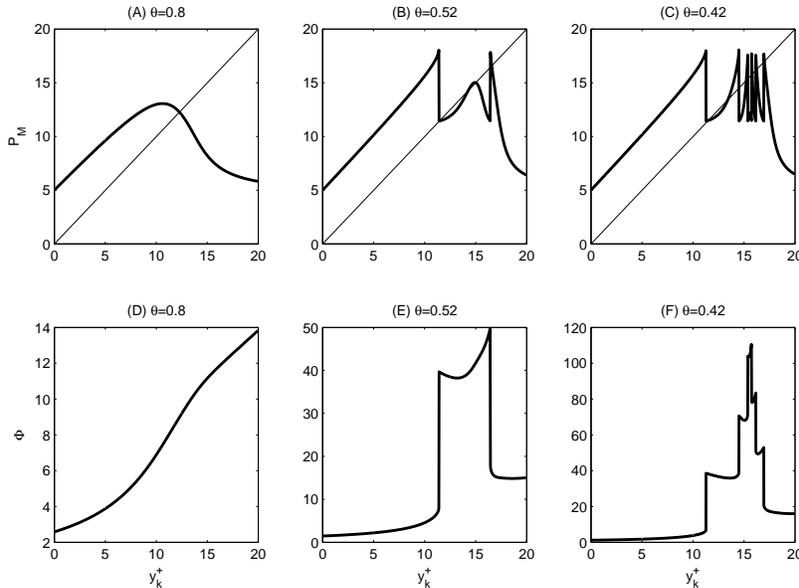


FIGURE 3. Continuity of Poincaré map and time function without impulse of model (2) for different θ . The other parameter values are fixed as: $r = 1, K = 52, \beta = 0.19, \eta = 0.45, \omega = 0.19, \delta = 0.36, \tau = 5, ET = 35$.

the impulsive set M is transversal, i.e. there is no trajectory of model (2) without impulsive effect which tangents to the set M , then the function Φ is continuous. Therefore, we have the following result on the continuity of the function Φ which can be proved by using the similar method as those in [1, 2, 9].

Theorem 2.1. *Let (R^2, Π) be a dynamical system and consider an impulsive semi-dynamical system (R^2, Π, M, I) . Let M be a closed and well placed subset of R^2 which is transversal. Then Φ is continuous on R^2 if and only if $z \notin M$.*

Note that the transversality condition in Theorem 2.1 may exclude the case (B) in Fig.2(B). In fact, based on our example we can conclude that the function Φ is continuous on phase set provided that the Poincaré map reaches its maximal value at $y_{\theta ET}$. This confirms that the conditions shown in Theorem 2.1 could be further improved.

3. Discussion. Recently, impulsive semi-dynamical systems or state dependent feedback control systems arise from many important applications in life sciences including biological resource management programmes and chemostat cultures [5, 6, 10, 12, 17, 18, 19, 20, 21, 22, 24], diabetes mellitus and tumor control [8, 13], vaccination strategies and epidemiological control [14, 15], and neuroscience [3, 4, 7]. In those fields, the threshold policies such as ET and threshold glucose level for closed-loop treatment play the pivotal role in the pest control and medications to treat diabetes. For example, control actions such as spraying pesticides and releasing natural enemies for pest control and drug administrations (such as insulin

and glucose-lowering pills) for disease treatment depend on the threshold levels including density of pest population and glucose level [8, 16, 17, 24].

The above state-dependent feedback control strategies can be defined in broad terms in real biological problems, which are usually modeled by the impulsive semi-dynamical systems. The continuity of the function Φ describing the times of meeting impulsive set which is equivalent to the continuity of the Poincaré map determined by the impulsive points in phase set (as shown in Fig.3) plays an important role in investigating the qualitative theories of the impulsive semi-dynamical system. In particular, it is very useful for determining the existence and stability of order one periodic solution or order one limit cycle [21, 23]. However, in order to guarantee the continuity of this function, only some special cases for the proposed models have been discussed in literatures [11, 12, 16, 23]. For example, the proposed models without impulsive effects do not have any positive equilibrium in the first quadrant, or the threshold level must be less than the values of the steady state once the models exist the interior equilibrium. Moreover, the Poincaré map can also help us to study the dynamic complexity of impulsive semi-dynamical system and its application in reality. Therefore, determining the continuity of the function describing the times of meeting impulsive set and then the Poincaré map are crucial for proposed models arising from different fields.

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