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# GLOBAL STABILITY IN A TUBERCULOSIS MODEL OF IMPERFECT TREATMENT WITH AGE-DEPENDENT LATENCY AND RELAPSE

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Abstract. In this paper, an $SEIR$ epidemic model for an imperfect treatment disease with age-dependent latency and relapse is proposed. The model is well-suited to model tuberculosis. The basic reproduction number $R_0$ is calculated. We obtain the global behavior of the model in terms of $R_0$. If $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, whereas if $R_0 > 1$, a Lyapunov functional is used to show that the endemic equilibrium is globally stable amongst solutions for which the disease is present.

1. **Introduction.** Mathematical modeling is a very important tool in analyzing the propagation and controlling of infectious diseases. Age structure is an important characteristic in the modeling of some infectious diseases. The first formulation of a partial differential equation(PDE) for the age distribution of a population was due to McKendrick [21]. Since the seminar papers by Kermack and McKendrick [13]-[15], age structure models have been used extensively to study the transmission dynamics of infectious diseases, we refer to the monographs by Hoppensteadt [11], Iannalli [12] and Webb [30] on this topic.

As an ancient disease, TB peaked and declined by 1940's before it became curable, while the downtrend stopped in the middle 1980's and 1990's. As one of the top 3 deadly infectious diseases, TB would cause a higher death rate if not treated, while the disease would be latent in an individual body for months, years or even decades before it outbreaks. McCluskey [20] pointed out that the risk of activation can be modeled as a function of duration age, and this form can be used to describe more general latent period via introducing the duration age in the latent class as a variable.

On the other hand, for the infectious tuberculosis, the removed individuals often have a higher relapse rate. Actually, the recurrence as an important feature of some animal and human diseases has been studied extensively, see [4], [23]. For instance,

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van den Driessche and coauthors in [4], [5] established two models with a constant relapse period and a general relapse distribution respectively, which showed the threshold property of the basic reproduction number. It is interest to investigate the model with age-dependent relapse rate and to determine whether the threshold property can be preserved or not.

Recently, Wang et al. [24]-[27] considered the global stability of nonlinear age-structured models, Liu et al. [17] introduced age-dependent latency and relapse into an $SEIR$ epidemic model and the local stability and global stability of equilibria are obtained by analyzing the corresponding characteristic equations and constructing the proper Volterra-type Lyapunov functionals, respectively. Wang et al. [28] proposed an $SVEIR$ epidemic model with media impact, age-dependent vaccination and latency, and discussed the global dynamics of the age-structure model.

However, most of the models assumed that TB would show neither its clinical symptoms nor its infectivity during its latent period, while in fact, TB has many early clinical symptoms such as fever, fatigability, night sweat, chest pain, hemoptysis and so on. Here we formulate and analyze an $SEIR$ epidemic model with continuous age dependent latency and relapse. We assumed, as the development of the disease, TB is infectious during its latent period with less infectivity and incomplete treatment comparing with outbreak period. Although epidemic models with age-dependent have been studied extensively, there have been still inadequate results on the full global stability. In this paper, we employ the method developed by Webb [30] for age-dependent models, namely integrating solutions along the characteristics to obtain an equivalent integral equation. We obtain the basic reproduction number in virtue of the method in [7]. Moreover, we study the asymptotic smoothness of the semi-flow generated by the system and the existence of a global attractor [3], [19]. Finally, we show the global stability of equilibria via constructing the proper Volterra-type Lyapunov functionals. For more details concerning the current Lyapunov functionals approach, we refer the reader to recent work [2].

This paper will be organized as follows: In Section 2 , we formulate our general $SEIR$ tuberculosis model with latent age and relapse age which is described by a coupled system of ODEs and PDEs. In Section 3, we investigate the existence of equilibria and obtain the expression of the basic reproduction number $R_0$. In Section 4, the local asymptotic stability of the equilibria will be derived. In Section 5, we present the results about uniform persistence. In section 6, we deal with the global stability of equilibria. Finally, some numerical simulations and useful discussions are made in the last section.

## 2. **Model formulation.**

The total population is decomposed into four disjoint subclasses, susceptible class $S$, latent class $E$, infectious class $I$, and removed class $R$. More precisely, let $S(t)$ denote the number of susceptible individuals at time $t$. Susceptible individuals would become new infected ones after they contact with infectious individuals at a rate $\beta$, while they enter a stage when they are infected with the disease but have little infectivity. This stage is often called latent stage, which maybe enter into the stage of removed class $R$ by receiving treatment at a rate $\mu$. The density of individuals in the latent class is denoted by $e(t,a)$ where $t$ is the duration time spenting in this class and $a$ is called the latent-stage progression age, denoting $E(t) = \int_0^{+\infty} e(t,a)da$ the total density of latent individuals. The number of individuals in the class $I$ at time $t$ is $I(t)$. The removal rate from latent class $E$ to infectious class $I$ is given by the function $\sigma(a)$. Thus, the total rate at which individuals progress into the infectious class alive is $\int_0^{+\infty}\sigma(a)e(t,a)da$. Infectious individuals come into the removed class after recovery due to complete treatment. Let $r_1$ be the recovery rate from the infectious class. The density of individuals in removed class is denoted by $r(t,c)$ , where c represents the relapse age, denoting $R(t) = \int_0^{\infty} r(t,c)dc$ the total density of removed individuals. In fact, infectious individuals might come into the latent class $E$ due to incomplete treatment at the rate $r_2$. Due to the relapse of the disease, the age-dependent relapse rate in the removed class is given by the function $k(c)$. The total rate at which individuals relapse into the infectious class alive is given by the quantity $\int_0^{+\infty} k(c)r(t,c)dc$. We also denote $\Lambda$ , $\delta_e, \delta_i$ as the density of the recruitment into the susceptible class, the additional death rates induced by the latent disease and infectious disease. The parameter $b$ is the natural death rate of all individuals. All recruitment of the population enters the susceptible class and occurs with constant flux $\Lambda$. Further, all parameters are assumed to be positive. FIGURE 1 shows the schematic flow diagram of our model which can be described by a system of ordinary and partial differential equations

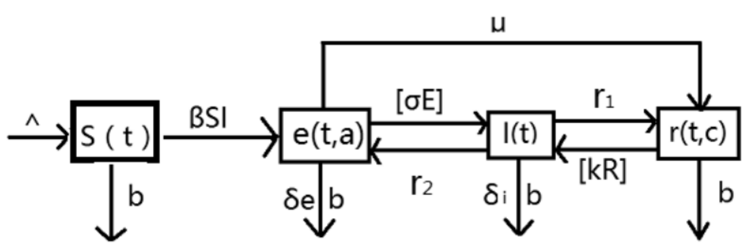


Figure 1. Here is the Model of TB

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - bS(t) - \beta S(t)I(t), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})e(t,a) = -(b+\delta_e+\mu(a)+\sigma(a))e(t,a), \\ \frac{dI(t)}{dt} = -(r_1+r_2+b+\delta_i)I(t) + \int_0^{\infty}\sigma(a)e(t,a)da + \int_0^{\infty}k(c)r(t,c)dc, \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial c})r(t,c) = -(k(c)+b)r(t,c), \end{cases} \tag{1}$$

with boundary conditions

$$\begin{cases} e(t,0) = \beta S(t)I(t) + r_2 I(t), \\ r(t,0) = r_1 I(t) + \int_0^{\infty}\mu(a)e(t,a)da, \end{cases} \tag{2}$$

and initial conditions

$$S(0) = S_0,\ e(0,a) = e_0(a),\ I(0) = I_0,\ r(0,c) = r_0(c), \tag{3}$$

where $S_0, I_0 \in R_+$, and $e_0(a), r_0(c) \in L^1_+(0,\infty)$ which is the nonnegative and Lebesgue integrable space of functions on $[0,+\infty)$.

In order to simplify the later derivation, we make the following hypotheses about the parameters of the system (1)

$(\mathbf{H1})\sigma(a), k(c), \mu(a) \in L^1_+(0,+\infty), with\ respective\ essential\ upper\ bounds\ \bar{\sigma}, \bar{k}, \bar{\mu};$
$(\mathbf{H2})\sigma(a), k(c), \mu(a) \in L^1_+(0,+\infty)\ are\ Lipschitz\ continuous\ on\ R_+,\ with$
$Lipschitzcoefficients\ M_\sigma, M_k\ and\ M_\mu\ respectively;$
$(\mathbf{H3})There\ exists\ b_0 \in [0,b), such\ that\ \sigma(a), k(c), \mu(a) \geq b_0,\ for\ all\ a,c>0.$

For $a, c > 0$, we denote

$$\begin{aligned}\varepsilon(a) =& \sigma(a)+\mu(a)+b+\delta_e,\ \eta(c)=k(c)+b, \rho_1(a)=e^{-\int_0^a \varepsilon(s)ds},\ \rho_2(c)=e^{-\int_0^c \eta(s)ds},\\ \theta_1 =& \int_0^{+\infty}\sigma(a)\rho_1(a)da,\ \theta_2=\int_0^{+\infty}k(c)\rho_2(c)dc, \theta_3=\int_0^{+\infty}\mu(a)\rho_1(a)da.\end{aligned}$$

According to Webb [30], by solving the PDE parts of (1) along the characteristic lines $t-a=const$ and $t-c=const$ respectively, we obtain

$$e(t,a)=\begin{cases} e(t-a,0)e^{-\int_0^a \varepsilon(s)ds},\ t>a\geq 0,\\ e_0(a-t)e^{-\int_{a-t}^a \varepsilon(s)ds},\ a\geq t\geq 0,\end{cases} \tag{4}$$

$$r(t,c)=\begin{cases} r(t-c,0)e^{-\int_0^c \eta(s)ds},\ t>c\geq 0,\\ r_0(c-t)e^{-\int_{c-t}^c \eta(s)ds},\ c\geq t\geq 0.\end{cases} \tag{5}$$

Define the space of functions $X$ as

$$X := R_+\times L^1_+(0,+\infty)\times R_+\times L^1_+(0,+\infty)$$

equipped with the norm

$$\parallel (x_1,x_2,x_3,x_4)\parallel_X = |x_1|+\int_0^{+\infty}|x_2(a)|da+|x_3|+\int_0^{+\infty}|x_4(c)|dc.$$

The norm has the biological interpretation of giving the total population size. The initial conditions (3) for system (1) can be rewritten as $x_0=(S_0, e(0,\cdot), I_0, r(0,\cdot))\in X$. Using standard methods we can verify the existence and uniqueness of solutions to model (1) with the boundary conditions (2) and initial conditions (3) (see Iannelli [12] and Webb [30]). Meanwhile, we can claim that any solution of system (1) with nonnegative initial conditions remains nonnegative. The nonnegativity of $e(t,a)$ and $r(t,c)$ follows from (4) and (5). Next, we shall show that $S(t)>0$ for $t\geq 0$ and $I(t)>0$ for $t\geq 0$. Otherwise, assume that $S(t)$ would lose its positivity for the first time at $t_1>0$, i.e., $S(t_1)=0$. However, from the first equation of (1) we obtain

$$S(t_1)=e^{-bt_1-\int_0^{t_1}\beta I(\tau)d\tau}\{S(0)+\int_0^{t_1}e^{bs+\int_0^s \beta I(\tau)d\tau}\Lambda ds\}>0.$$

Similarly, assume that $I(t)$ would lose its positivity for the first time at $t_2>0$, i.e., $I(t_2)=0$. However, from the third equation of (1) we obtain

$$\begin{aligned}I(t_2) =& e^{-(r_1+r_2+b+\delta_i)t_2}\int_0^{t_2}(\int_0^\infty \sigma(a)e(t,a)da+\int_0^\infty k(c)r(t,c)dc)e^{(r_1+r_2+b+\delta_i)s})ds\\ &+e^{-(r_1+r_2+b+\delta_i)t_2}I(0)>0.\end{aligned}$$

Thus $S(t)>0$ and $I(t)>0$ are true for $\forall t\geq 0$. This verifies our claim.

Let us consider a function $N(t)=S(t)+\int_0^{+\infty}e(t,a)da+I(t)+\int_0^{+\infty}r(t,c)dc$, which is the total population at time $t$. We can easily see that the time derivative of $N$ along solutions of model (1) is

$$\frac{d}{dt}N(t) = \frac{d}{dt}S(t) + \frac{d}{dt}\int_0^{+\infty} e(t,a)da + \frac{d}{dt}I(t) + \frac{d}{dt}\int_0^{+\infty} r(t,c)dc.$$

due to $\rho_1(0) = 1$, $\frac{d\rho_1(a)}{da} = -\varepsilon(a)\rho_1(a)$, we have

$$\begin{aligned}
\frac{d}{dt}\int_0^{+\infty} e(t,a)da =& \frac{d}{dt}\Big(\int_0^{t} e(t-a,0)\rho_1(a)da + \int_t^{+\infty} e_0(a-t)\frac{\rho_1(a)}{\rho_1(a-t)}da\Big)\\
=& \frac{d}{dt}\int_0^{t}(\beta S(t-a)I(t-a) + r_2 I(t-a))\rho_1(a)da\\
&+ \frac{d}{dt}\int_t^{+\infty} e_0(a-t)\frac{\rho_1(a)}{\rho_1(a-t)}da\\
=& \frac{d}{dt}\int_0^{t}(\beta S(\tau)I(\tau) + r_2 I(\tau))\rho_1(t-\tau)d\tau\\
&+ \frac{d}{dt}\int_0^{+\infty} e_0(\tau)\frac{\rho_1(t+\tau)}{\rho_1(\tau)}d\tau\\
=& \beta S(t)I(t) + r_2 I(t) - \int_0^{+\infty}\varepsilon(a)e(t,a)da.
\end{aligned}$$

Similarly, by using $\rho_2(0) = 1$, $\frac{d\rho_2(c)}{dc} = -\eta(c)\rho_2(c)$, we can get

$$\begin{aligned}
\frac{d}{dt}\int_0^{+\infty} r(t,c)dc =& \frac{d}{dt}\int_0^{t}\Big(r_1 I(t-c) + \int_0^{+\infty}\mu(a)e(t-c,a)da\Big)\rho_2(c)dc\\
&+ \frac{d}{dt}\int_t^{+\infty} r_0(c-t)\frac{\rho_2(c)}{\rho_2(c-t)}dc\\
=& r_1 I(t) + \int_0^{+\infty}\mu(a)e(t,a)da - \int_0^{+\infty}\eta(c)r(t,c)dc.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(S(t) + \int_0^{+\infty} e(t,a)da + I(t) + \int_0^{+\infty} r(t,c)dc\Big)\\
=& \Lambda - bS(t) - \beta S(t)I(t) + \beta S(t)I(t) + r_2 I(t) - \int_0^{+\infty}\varepsilon(a)e(t,a)da\\
&- (r_1 + r_2 + b + \delta_i)I(t) + \int_0^{\infty}\sigma(a)e(t,a)da + \int_0^{\infty}k(c)r(t,c)dc\\
&+ r_1 I(t) + \int_0^{+\infty}\mu(a)e(t,a)da - \int_0^{+\infty}\eta(c)r(t,c)dc\\
=& \Lambda - bS(t) - \int_0^{+\infty} br(t,c)dc - bI(t) - \int_0^{+\infty}(b+\delta_e)e(t,a)da - \delta_i I(t)\\
\le& \Lambda - bN(t).
\end{aligned}$$

It follows from the variation of constants formula that $N(t) \le \frac{\Lambda}{b}$, for any $t \ge 0$, which implies that

$$\Omega = \left\{(S(t), e(t,\cdot), I(t), r(t,\cdot)) \in R_+ \times L^1_+(0,+\infty) \times R_+ \times L^1_+(0,+\infty) : N(t) \le \frac{\Lambda}{b}\right\}$$

is positively invariant absorbing set for system (1).

**Proposition 1.** *If $x_0 \in X$ and $\|x_0\|_X \leq M$ for some constant $M \geq \frac{\Lambda}{b}$, then the following statements hold for $t \geq 0$,*

$$(i)\ 0 \leq S(t), \int_0^{+\infty} e(t,a)da, I(t), \int_0^{+\infty} r(t,c)dc \leq M,$$
$$(ii)\ e(t,0) \leq \beta M^2 + r_2 M, r(t,0) \leq r_1 M + \bar{\mu} M.$$

For convenience, we rewrite (1) as follows

$$\begin{cases} (\frac{\partial}{\partial t} + \frac{\partial}{\partial a})e(t,a) = -(b+\delta_e+\mu(a)+\sigma(a))e(t,a), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial c})r(t,c) = -(k(c)+b)r(t,c), \\ \frac{dV(t)}{dt} = G(e(t,a), r(t,c), V(t)) - CV(t), \\ e(t,0) = \beta S(t)I(t) + r_2 I(t), \\ e(0,a) = e_0(a), \\ r(t,0) = r_1 I(t) + \int_0^\infty \mu(a)e(t,a)da, \\ r(0,c) = r_0(c), \\ V(0) = V_0, \end{cases} \tag{6}$$

where

$$V(t) = \begin{pmatrix} S(t) \\ I(t) \end{pmatrix},$$
$$C = \begin{pmatrix} b & 0 \\ 0 & r_1+r_2+b+\delta_i \end{pmatrix},$$
$$G(e(t,a), r(t,c), V(t)) = \begin{pmatrix} \Lambda - \beta S(t)I(t) \\ \int_0^\infty \sigma(a)e(t,a)da + \int_0^\infty k(c)r(t,c)dc, \end{pmatrix}.$$

Set $Z = Y \times R^2$, where $Y = R \times L^1(R_+, R)$ , for any $\begin{pmatrix} \alpha \\ \phi \end{pmatrix} \in Y$, we have $\| \begin{pmatrix} \alpha \\ \phi \end{pmatrix} \| = |\alpha| + \|\phi\|_{L^1(R_+,R)}$. Furthermore, we define

$$Z_+ = Y_+ \times R_+^2, Z_0 = Y_0 \times R^2, Z_{0+} = Z_0 \bigcap Z_+,$$

where

$$Y_+ = R_+ \times L'_+(R_+, R), Y_0 = \{0\} \times L'(R_+, R).$$

We define $A_1 : Dom(A_1) \subset Y \to Y$, by

$$A_1 \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} -\phi_1(0) \\ -\phi_1' - (b+\delta_e+\mu(a)+\sigma(a))\phi_1 \end{pmatrix}$$

with $Dom(A_1) = \{0\} \times W^{1,1}(R_+, R)$. If $\lambda$ is a complex number with $Re\lambda > -(b+\delta_e)$, then $\lambda \in \rho(A_1)$ which is the resolvent set of $A_1$. Moreover, if $\lambda \in \rho(A_1)$ and

$$(\lambda I - A_1)^{-1} \begin{pmatrix} \theta_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix},$$

then we can get

$$\phi_1(a) = e^{-(\lambda+b+\delta_e)a}\theta_1 + \int_0^a e^{-\int_s^a(\mu(l)+\sigma(l))dl} e^{-(\lambda+b+\delta_e)(a-s)}\psi_1(s)ds.$$

Similarly, we define $A_2 : Dom(A_2) \subset Y \to Y$, by

$$A_2 \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_2(0) \\ -\phi_2' - (b + k(c))\phi_2 \end{pmatrix}$$

with $Dom(A_2) = \{0\} \times W^{1,1}(R_+, R)$, we can obtain

$$\phi_1(a) = e^{-(\lambda+b)c}\theta_2 + \int_0^c e^{-\int_s^c (k(l))dl} e^{-(\lambda+b)(c-s)}\psi_2(s)ds.$$

Thus (6) can be rewritten as

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix} = A_1 \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix} + F_1\left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(t) \right), \\ \frac{d}{dt}\begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix} = A_2 \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix} + F_2\left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(t) \right), \\ \frac{dV(t)}{dt} = -CV(t) + F_3\left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(t) \right), \\ e(0,a) = e_0(a), \\ r(0,c) = r_0(c), \\ V(0) = V_0, \end{cases} \tag{7}$$

where

$$F_1\left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(t) \right) = \begin{pmatrix} \beta S(t)I(t) + r_2 I(t) \\ 0 \end{pmatrix},$$

$$F_2\left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(t) \right) = \begin{pmatrix} r_1 I(t) + \int_0^\infty \mu(a)e(t,a)da \\ 0 \end{pmatrix},$$

$$F_3\left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(t) \right) = \begin{pmatrix} \Lambda - \beta S(t)I(t) \\ \int_0^\infty \sigma(a)e(t,a)da + \int_0^\infty k(c)r(t,c)dc \end{pmatrix}.$$

Let $L : D(L) \subset X \to X$ be the linear operator defined by

$$L(u(t)) = \left( A_1 \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, A_2 \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, -CV(t) \right),$$

where $D(L) = P \times R^2$ with $P = \{0\} \times W^{1,1}(R_+, R) \times \{0\} \times W^{1,1}(R_+, R)$. It follows that $X_0 = \overline{D(L)}$ and $X_{0+} = \overline{D(L)} \bigcap X_+$. So $\overline{D(L)} = X_0$ is not dense in $X$. We consider the nonlinear map $F : \overline{D(L)} \to X$ defined by

$$F(u(t)) = \begin{pmatrix} F_1(u(t)) \\ F_2(u(t)) \\ F_3(u(t)) \end{pmatrix}.$$

Therefore (7) can be rewritten as an abstract Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = L(u(t)) + F(u(t)), \\ u(0) = \left( \begin{pmatrix} 0 \\ e_0(\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r_0(\cdot) \end{pmatrix}, V_0 \right). \end{cases} \tag{8}$$

By using the results in Magal [19] and Chen et al. [3], there exists a uniquely determined semiflow $\{U(t)\}_{t\geq 0}$ on $X_{0+}$ such that, for each $u = \left( \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}, \begin{pmatrix} 0 \\ r(t,\cdot) \end{pmatrix}, V(0) \right)$, there exists a unique continuous map $U \in C(R_+, X_{0+})$, which is an integrated solution of the Cauchy problem (8), that is, for all $t \geq 0$, $\int_0^t U(s)uds \in D(L)$ and $U(t)u = u + L\int_0^t U(s)uds + \int_0^t F(U(s)u)ds$. And $\Omega$ is the positively invariant absorbing set under the semi-flow $U$ can be verified, that is, $U(t)\Omega \subseteq \Omega$ and for each $x \in X_{0+}$, $d(U(t),\Omega) := inf_{y\in\Omega} \parallel U(t)x - y \parallel \to 0$, as $t \to \infty$ which means that the semi-flow $\{U(t)\}_{t\geq 0}$ is bound dissipative on $X_{0+}$. A semi-flow $U(t,x_0) : R_+ \times X \longrightarrow X$ is called asymptotically smooth if each forward invariant bounded closed set is attracted by a nonempty compact set [8], [22]. In order to obtain global properties of the dynamics of the semi-flow $U(t)$, it is important to prove the asymptotically smooth of semi-flow $U(t)$. First we give the following useful lemma.

**Lemma 2.1.** *([1]) Let $D \subseteq R$. For $j = 1, 2$, suppose $f_j : D \to R$ is a bounded Lipschitz continuous function with bound $K_j$ and Lipschitz coefficient $M_j$. Then the product function $f_1 f_2$ is Lipschitz with coefficient $K_1M_2 + K_2M_1$.*

**Lemma 2.2.** *([1]) If the following two conditions hold, then the semi-flow $U(t,x_0) = \phi(t,x_0) + \varphi(t,x_0) : R_+ \times X \longrightarrow X$ is asymptotically smooth in $X$:*
*(i) There exists a continuous function $v : R_+\times R_+ \longrightarrow R_+$ such that $\lim_{t\to+\infty} v(t,h) = 0$ and $\|\phi(t,x_0)\|_X \leq v(t,h)$ if $\|x_0\|_X \leq h$;*
*(ii) For $t \geq 0, \varphi(t,x_0)$ is completely continuous.*

In order to prove Lemma 2.2, we first decompose $U : R_+ \times X \longrightarrow X$ into the following two operators $\phi(t,x_0), \varphi(t,x_0) : R_+ \times X \longrightarrow X$, $\phi(t,x_0) = (0, y_2(t,\cdot), 0, y_4(t,\cdot))$, $\varphi(t,x_0) = (S(t), \tilde{y}_2(t,\cdot), I(t), \tilde{y}_4(t,\cdot))$, where

$$
\begin{aligned}
y_2(t,a) &= \begin{cases} 0, & t > a \geq 0, \\ e_0(a-t)\frac{\rho_1(a)}{\rho_1(a-t)}, & a \geq t \geq 0, \end{cases} \\
y_4(t,c) &= \begin{cases} 0, & t > c \geq 0, \\ r_0(c-t)\frac{\rho_2(c)}{\rho_2(c-t)}, & c \geq t \geq 0. \end{cases}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\tilde{y}_2(t,a) &= \begin{cases} e(t-a,0)\rho_1(a), & t > a \geq 0, \\ 0, & a \geq t \geq 0, \end{cases} \\
\tilde{y}_4(t,c) &= \begin{cases} r(t-c,0)\rho_2(c), & t > c \geq 0, \\ 0, & c \geq t \geq 0. \end{cases}
\end{aligned} \tag{10}
$$

In order to verify condition (i) of Lemma 2.2, we need to prove the following proposition.

**Proposition 2.** *For $h > 0$, let $v(t,h) = he^{-(b+2b_0+\delta_e)t}$. Then $v(t,h) \longrightarrow 0$ as $t \longrightarrow +\infty$ and $\|\phi(t,x_0)\|_X \leq v(t,h)$ if $\|x_0\|_X \leq h$.*

*Proof.* It is obvious that $v(t,h) \longrightarrow 0$ as $t \longrightarrow +\infty$, with the help of (9) and **(H3)**, we have

$$
\begin{aligned}
\|\phi(t,x_0)\|_X =&|0| + \int_0^{+\infty} |y_2(t,a)|da + |0| + \int_0^{+\infty} |y_4(t,c)|dc \\
=&\int_t^{+\infty} |e_0(a-t)\frac{\rho_1(a)}{\rho_1(a-t)}|da + \int_t^{+\infty} |r_0(c-t)\frac{\rho_2(c)}{\rho_2(c-t)}|dc \\
=&\int_0^{+\infty} |e_0(\tau)e^{-\int_\tau^{\tau+t}\varepsilon(s)ds}|d\tau + \int_0^{+\infty} |r_0(\tau)e^{-\int_\tau^{\tau+t}\eta(s)ds})|d\tau \\
\leq& e^{-(b+2b_0+\delta_e)t}(|0| + \int_0^{+\infty} |e_0(\tau)|d\tau + |0| + \int_0^{+\infty} |r_0(\tau)|d\tau) \\
=&e^{-(b+2b_0+\delta_e)t}\|x_0\|_X,
\end{aligned}
$$

by the known condition $\|x_0\|_X \leq h, \forall x_0 \in \Omega$, we have $\|\phi(t,x_0)\|_X \leq he^{-(b+2b_0+\delta_e)t} \equiv v(t,h)$. ∎

**Lemma 2.3.** *([1]) Let $K \subset L^p(0,+\infty)$ be closed and bounded where $p \geq 1$. Then $K$ is compact if the following conditions hold*

$$(i) \lim_{h\to 0}\int_0^{+\infty} |f(z+h)-f(z)|^p dz = 0 \ uniformly\ for\ f\in K,$$

$$(ii) \lim_{h\to +\infty}\int_h^{+\infty} |f(z)|^p dz = 0 \ uniformly\ for\ f\in K.$$

**Proposition 3.** *For $t \geq 0, \phi(t, x_0)$ is completely continuous.*

*Proof.* According to Proposition 1(i), $S(t)$ and $I(t)$ remain in the compact set $[0, \Lambda/b] \subset [0, M]$, where $M \geq \Lambda/b$. Thus, it needs only to show that $\tilde{y}_2(t,a)$ and $\tilde{y}_4(t,c)$ remain in a precompact subset of $L^1_+(0,+\infty)$, which is independent of $x_0 \in \Omega$. It suffices to verify that (i) and (ii) in Lemma 2.3 hold. Now, from Proposition 1(ii) and (10) we have

$$\tilde{y}_2(t,a) \leq (\beta M^2 + r_2 M)e^{-(b+2b_0+\delta_e)a}. \tag{11}$$

$$\begin{aligned}
&\int_0^{+\infty} |\tilde{y}_2(t,a+h)-\tilde{y}_2(t,a)|da\\
=&\int_0^{t-h} |e(t-a-h,0)\rho_1(a+h)-e(t-a,0)\rho_1(a)|da\\
&+\int_{t-h}^{t} |0-e(t-a,0)\rho_1(a)|da\\
\leq&\int_0^{t-h} e(t-a-h,0)|\rho_1(a+h)-\rho_1(a)|da\\
&+\int_0^{t-h} \rho_1(a)|e(t-a-h,0)-e(t-a,0)|da\\
&+\int_{t-h}^{t} |e(t-a,0)\rho_1(a)|da.
\end{aligned} \tag{12}$$

Recall that $\rho_1(a) = e^{-\int_0^a(\sigma(s)+\mu(s)+b+\delta_e)ds} \leq e^{-(2b_0+b+\delta_e)a} \leq 1$, then $\rho_1(a)$ is a non-increasing function with respect to $a$, we have

$$\begin{aligned}
&\int_0^{t-h} |\rho_1(a+h)-\rho_1(a)|da = \int_0^{t-h}\rho_1(a)da - \int_h^{t}\rho_1(a)da\\
=&\int_0^{t-h}\rho_1(a)da - \int_h^{t-h}\rho_1(a)da - \int_{t-h}^{t}\rho_1(a)da\\
=&\int_0^{h}\rho_1(a)da - \int_{t-h}^{t}\rho_1(a)da \leq h.
\end{aligned} \tag{13}$$

From Proposition 1 and (**H1**), we find that $|dS(t)/dt|$ is bounded by $M_S = \Lambda + bM + \beta M^2$ and $|dI(t)/dt|$ is bounded by $M_I = \bar{\sigma}M + \bar{k}M + (b+\delta_i+r_1+r_2)M$. Therefore, $S(\cdot)$ and $I(\cdot)$ are Lipschitz on $[0,+\infty)$ with coefficients $M_S$ and $M_I$. By Lemma 2.1, $S(\cdot)I(\cdot)$ is Lipschitz on $[0,+\infty)$ with coefficient $M_{SI} = M(M_S+M_I)$. Thus

$$\int_0^{t-h} \rho_1(a)|e(t-a-h,0)-e(t-a,0)|da$$

$$\leq \int_0^{t-h} \rho_1(a)(|\beta S(t-a-h)I(t-a-h) - \beta S(t-a)I(t-a)|$$
$$+ |r_2 I(t-a-h) - r_2 I(t-a)|)da$$
$$\leq \int_0^{t-h} (\beta M_{SI} + r_2 M_I)(-h)e^{-(b+2b_0+\delta_e)a}da \tag{14}$$
$$=(\beta M_{SI} + r_2 M_I)\frac{h}{b+2b_0+\delta_e}(1 - e^{-(b+2b_0+\delta_e)(t-h)})$$
$$\leq(\beta M_{SI} + r_2 M_I)\frac{h}{b+2b_0+\delta_e}.$$

From (12)-(14), we have

$$\int_0^{+\infty} |\tilde{y}_2(t, a+h) - \tilde{y}_2(t, a)|da \leq (\frac{\beta M_{SI} + r_2 M_I}{b+2b_0+\delta_e} + 2(\beta M^2 + r_2 M))h$$

which converges uniformly to 0 as $h \to 0$. The condition (i) in Lemma 2.3 is proved for $\tilde{y}_2(t, a)$.

From (11) we have

$$\lim_{h\to+\infty} \int_h^{+\infty} |\tilde{y}_2(t, a)|da \leq \lim_{h\to+\infty} \int_h^{+\infty} (\beta M^2 + r_2 M))e^{-(b+2b_0+\delta_e)a}da$$
$$= \lim_{h\to+\infty} \frac{\beta M^2 + r_2 M}{b+2b_0+\delta_e}e^{-(b+2b_0+\delta_e)h} = 0$$

which meet the condition (ii) in Lemma 2.3. Similarly, $y_4(t, c)$ also satisfies the conditions of Lemma 2.3. □

**Theorem 2.4.** *The semi-flow $\{U(t)\}_{t\geq 0}$ generated by system (1) is asymptotically smooth, and has a global attractor $A$ contained in $X$, which attracts the bounded sets of $X$.*

## 3. Existence of the equilibria.

Now we consider the existence of equilibria of system (1) . The steady state $(S^*, e^*(\cdot), I^*, r^*(\cdot))$ of system (1) satisfies the equalities

$$\begin{cases} 0 = \Lambda - bS^* - \beta S^* I^*, \\ \frac{d}{da}e^*(a) = -\varepsilon(a)e^*(a), \\ 0 = -(r_1 + r_2 + b + \delta_i)I^* + \int_0^\infty \sigma(a)e^*(a)da + \int_0^\infty k(c)r^*(c)dc, \\ \frac{d}{dc}r^*(c) = -\eta(c)r^*(c), \end{cases} \tag{15}$$

with boundary conditions

$$\begin{cases} e^*(0) = \beta S^* I^* + r_2 I^*, \\ r^*(0) = r_1 I^* + \int_0^\infty \mu(a)e^*(a)da. \end{cases} \tag{16}$$

From the second equation of (15) and the first equation of (16), we obtain

$$e^*(a) = (\beta S^* I^* + r_2 I^*) e^{-\int_0^a \varepsilon(s)ds}. \tag{17}$$

Similarly, by using the fourth equation of (15) and the second equation of (16), we get

$$r^*(c) = (r_1 I^* + \int_0^{+\infty} \mu(a) e^*(a) da) e^{-\int_0^c \eta(s)ds}. \tag{18}$$

If $I^* = 0$, then we have $e^*(a) = 0$ and $r^*(c) = 0$ respectively from (17) and (18). Furthermore, it is easy to know that $S^0 = \frac{\Lambda}{b}$ from the first equation of (15). Thus, system (15) has a disease-free equilibrium $E_0$, and

$$E_0 = (S^0, 0, 0, 0), \ where \ S^0 = \frac{\Lambda}{b}.$$

In order to find any endemic equilibrium, we introduce the basic reproduction number $R_0$, which is the average number of new infections generated by a single newly infectious individual during the full infectious period [7]. It is given by the following expression

$$R_0 = \frac{\beta S^0(\theta_1 + \theta_2\theta_3)}{r_1 + r_2 + b + \delta_i - r_1\theta_2 - r_2(\theta_1 + \theta_2\theta_3)}.$$

Now, if $I^* \neq 0$, substituting (17) and (18) into the third equation of (15), we have

$$\begin{aligned} 0 =& -(r_1 + r_2 + b + \delta_i) I^* + \int_0^{+\infty} \sigma(a)(\beta S^* I^* + r_2 I^*) e^{-\int_0^a \varepsilon(s)ds} da \\ &+ \int_0^{+\infty} k(c)(r_1 I^* + \int_0^{+\infty} \mu(a)(\beta S^* I^* + r_2 I^*) e^{-\int_0^a \varepsilon(s)ds} da) e^{-\int_0^c \eta(s)ds} \\ =& -(r_1 + r_2 + b + \delta_i) I^* + (\beta S^* I^* + r_2 I^*)(\theta_1 + \theta_2\theta_3) + r_1 I^* \theta_2. \end{aligned} \tag{19}$$

Thus, combining the first equation of (15) and the equation (19), we get

$$S^* = \frac{\Lambda}{bR_0}, I^* = \frac{b}{\beta}(R_0 - 1). \tag{20}$$

If $R_0 > 1$, we get $e^*(a) > 0$ and $r^*(c) > 0$ from (17) and (18). Therefore, system (15) has a unique positive endemic equilibrium $E^*$, and

$$E^* = (S^*, e^*(\cdot), I^*, r^*(\cdot)),$$

where

$$\begin{aligned} S^* &= \frac{\Lambda}{bR_0}, \ e^*(a) = (\beta S^* I^* + r_2 I^*)\rho_1(a), \\ I^* &= \frac{b}{\beta}(R_0 - 1), \ r^*(c) = ((\beta S^* I^* + r_2 I^*)\theta_3 + r_1 I^*)\rho_2(c). \end{aligned}$$

4. **Local asymptotic stability of the equilibria.** In this section, sufficient conditions for the local asymptotic stability of the equilibria will be derived.

**Theorem 4.1.** *The disease-free equilibrium $E_0$ is locally asymptotically stable in the positive variant set $\Omega$ if $R_0 \leq 1$ and unstable if $R_0 > 1$.*

*Proof.* First, we introduce the change of variables as follows

$$x_1(t) = S(t) - S^0,\ x_2(t,a) = e(t,a),\ x_3(t) = I(t),\ x_4(t,c) = r(t,c).$$

Linearizing the system (1) about disease-free equilibrium $E_0$, we obtain the following system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = -bx_1(t) - \beta\dfrac{\Lambda}{b}x_3(t), \\ (\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a})x_2(t,a) = -(b+\delta_e+\mu(a)+\sigma(a))x_2(t,a), \\ \dfrac{dx_3(t)}{dt} = -(r_1+r_2+b+\delta_i)x_3(t) + \displaystyle\int_0^{+\infty}\sigma(a)x_2(t,a)da \\ + \displaystyle\int_0^{+\infty}k(c)x_4(t,c)dc, \\ (\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c})x_4(t,c) = -(k(c)+b)x_4(t,c), \\ x_2(t,0) = (\dfrac{\beta\Lambda}{b} + r_2)x_3(t), \\ x_4(t,0) = r_1x_3(t) + \displaystyle\int_0^{+\infty}\mu(a)x_2(t,a)da. \end{cases} \tag{21}$$

Set

$$x_1(t) = x_1^0e^{\lambda t},\ x_2(t,a) = x_2^0(a)e^{\lambda t},\ x_3(t) = x_3^0e^{\lambda t},\ x_4(t,c) = x_4^0(c)e^{\lambda t}, \tag{22}$$

where $x_1^0, x_2^0(a), x_3^0, x_4^0(c)$ will be determined. Substituting (22) into (21), we get

$$\lambda x_1^0 = -bx_1^0 - \frac{\beta\Lambda}{b}x_3^0, \tag{23}$$

$$\begin{cases} \lambda x_2^0(a) + \dfrac{dx_2^0(a)}{da} = -(b+\delta_e+\mu(a)+\sigma(a))x_2^0(a), \\ x_2^0(0) = (\beta\dfrac{\Lambda}{b} + r_2)x_3^0, \end{cases} \tag{24}$$

$$\lambda x_3^0 = -(r_1+r_2+b+\delta_i)x_3^0 + \int_0^{+\infty}\sigma(a)x_2^0(a)da + \int_0^{+\infty}k(c)x_4^0(c)dc, \tag{25}$$

$$\begin{cases} \lambda x_4^0(c) + \dfrac{dx_4^0(c)}{dc} = -(k(c)+b)x_4^0(c), \\ x_4^0(0) = r_1x_3^0 + \displaystyle\int_0^{+\infty}\mu(a)x_2^0(a)da. \end{cases} \tag{26}$$

Integrating the first equation of (24) from 0 to $a$ yields

$$x_2^0(a) = (\beta\frac{\Lambda}{b} + r_2)x_3^0e^{-(\lambda+b+\delta_e)a-\int_0^a(\sigma(s)+\mu(s))ds}. \tag{27}$$

Similarly, we have from (26) that

$$x_4^0(c) = (r_1x_3^0 + \int_0^{+\infty}\mu(a)x_2^0(a)da)e^{-(\lambda+b)c-\int_0^c k(s)ds}. \tag{28}$$

Substituting (27) and (28) into (25) and solving (25) gives

$$\begin{aligned}\lambda =& -(r_1+r_2+b+\delta_i)+\int_0^{+\infty} k(c) r_1 e^{-(\lambda+b)c-\int_0^c k(s)ds} dc \\ &+\int_0^{+\infty} \sigma(a)(\beta\frac{\Lambda}{b}+r_2) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s)ds} da \\ &+\int_0^{+\infty} k(c)\int_0^{+\infty} \mu(a)(\beta\frac{\Lambda}{b}+r_2) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s))ds} da \cdot \\ & e^{-(\lambda+b)c-\int_0^c k(s)ds} dc\end{aligned} \tag{29}$$

which is the characteristic equation. Let

$$\begin{aligned}F(\lambda)=&\int_0^{+\infty} k(c)\int_0^{+\infty} \mu(a)(\beta\frac{\Lambda}{b}+r_2) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s))ds} da \cdot \\ & e^{-(\lambda+b)c-\int_0^c k(s)ds} dc-\lambda-(r_1+r_2+b+\delta_i) \\ &+\int_0^{+\infty} \sigma(a)(\beta\frac{\Lambda}{b}+r_2) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s)ds} da \\ &+\int_0^{+\infty} k(c) r_1 e^{-(\lambda+b)c-\int_0^c k(s)ds} dc.\end{aligned}$$

Obviously, $F(\lambda)$ is a continuously differential function and satisfies

$$\begin{aligned}F'(\lambda)=&-(\beta\frac{\Lambda}{b}+r_2)\int_0^{+\infty} a\sigma(a) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s))ds} da \\ &-a\int_0^{+\infty} k(c) e^{-(\lambda+b)c-\int_0^c k(s)ds} dc \cdot \\ &\int_0^{+\infty} \mu(a)(\beta\frac{\Lambda}{b}+r_2) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s))ds} da \\ &-c\int_0^{+\infty} k(c) e^{-(\lambda+b)c-\int_0^c k(s)ds} dc \cdot \\ &\int_0^{+\infty} \mu(a)(\beta\frac{\Lambda}{b}+r_2) e^{-(\lambda+b+\delta_e)a-\int_0^a (\sigma(s)+\mu(s))ds} da \\ &-r_1\int_0^{+\infty} ck(c) e^{-(\lambda+b)c-\int_0^c k(s)ds} dc-1<0,\end{aligned}$$

and

$$\lim_{\lambda\to+\infty} F(\lambda)=-\infty,\quad \lim_{\lambda\to-\infty} F(\lambda)=+\infty.$$

Thus, we know (29) has a unique real root $\lambda^*$. Obviously,

$$F(0)=[(r_1+r_2+b+\delta_i)-r_1\theta_2-r_2(\theta_1+\theta_2\theta_3)](R_0-1),$$

we have $\lambda^*<0, if\ R_0<1$, and $\lambda^*>0, if\ R_0>1$. Let $\lambda=x+yi$ be an arbitrary complex root to (29), then

$$0=F(\lambda)=F(x+yi)\le F(x)$$

which means that $\lambda^*>x$. Thus, all the roots of (29) have negative real part if and only if $R_0\le 1$ and have at least one eigenvalue with positive real part if $R_0>1$. Therefore we have that the disease-free equilibrium $E_0$ is locally asymptotically stable if $R_0\le 1$ and unstable if $R_0>1$. □

**Theorem 4.2.** *The unique endemic equilibrium $E^*$ is locally asymptotically stable if $R_0 > 1$.*

*Proof.* First, we introduce the perturbation variables as follows

$y_1(t) = S(t) - S^*,\ y_2(t,a) = e(t,a) - e^*(a),\ y_3(t) = I(t) - I^*,\ y_4(t,c) = r(t,c) - r^*(c).$

Linearizing system (1) at the endemic equilibrium $E^*$ yields the following system

$$\begin{cases} \frac{dy_1(t)}{dt} = -by_1(t) - \beta I^* y_1(t) - \beta S^* y_3(t), \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial a}) y_2(t,a) = -(b + \delta_e + \mu(a) + \sigma(a)) y_2(t,a), \\ \frac{dy_3(t)}{dt} = -(r_1 + r_2 + b + \delta_i) y_3(t) + \int_0^{+\infty} \sigma(a) y_2(t,a) da \\ + \int_0^{+\infty} k(c) y_4(t,c) dc, \\ (\frac{\partial}{\partial t} + \frac{\partial}{\partial c}) y_4(t,c) = -(k(c) + b) y_4(t,c), \\ y_2(t,0) = \beta y_1(t) I^* + \beta S^* y_3(t) + r_2 y_3(t), \\ y_4(t,0) = r_1 y_3(t) + \int_0^{+\infty} \mu(a) y_2(t,a) da. \end{cases} \tag{30}$$

Set

$$y_1(t) = y_1^0 e^{\lambda t},\ y_2(t,a) = y_2^0(a) e^{\lambda t},\ y_3(t) = y_3^0 e^{\lambda t},\ y_4(t,c) = y_4^0(c) e^{\lambda t}, \tag{31}$$

Substituting (31) into (30) gives

$$\lambda y_1^0 = -by_1^0 - \beta I^* y_1^0 - \beta S^* y_3^0, \tag{32}$$

$$\begin{cases} \frac{dy_2^0(a)}{da} = -(\lambda + b + \delta_e + \mu(a) + \sigma(a)) y_2^0(a), \\ y_2^0(0) = \beta I^* y_1^0 + \beta S^* y_3^0 + r_2 y_3^0, \end{cases} \tag{33}$$

$$\lambda y_3^0 = -(r_1 + r_2 + b + \delta_i) y_3^0 + \int_0^{+\infty} \sigma(a) y_2^0(a) da + \int_0^{+\infty} k(c) y_4^0(c) dc, \tag{34}$$

$$\begin{cases} \frac{dy_4^0(c)}{dc} = -(\lambda + k(c) + b) y_4^0(c), \\ y_4^0(0) = r_1 y_3^0 + \int_0^{+\infty} \mu(a) y_2^0(a) da. \end{cases} \tag{35}$$

Integrating the first equation of (33) and (35) from 0 to $a$ and from 0 to $c$ respectively, together with the boundary conditions, yields

$$\begin{aligned} y_2^0(a) =& (\beta I^* y_1^0 + \beta S^* y_3^0 + r_2 y_3^0) e^{-(\lambda + b + \delta_e)a - \int_0^a (\sigma(s) + \mu(a)) ds}, \\ y_4^0(c) =& (r_1 y_3^0 + \int_0^{+\infty} \mu(a) y_2^0(a) da) e^{-(\lambda + b)c - \int_0^c k(s) ds}. \end{aligned} \tag{36}$$

substituting the above two equations into (34) and solving (34) we get

$$\begin{aligned} \lambda y_3^0 =& (\beta I^* y_1^0 + \beta S^* y_3^0 + r_2 y_3^0)(K_1(\lambda) + K_2(\lambda) K_3(\lambda)) \\ &+ r_1 y_3^0 K_2(\lambda) - (r_1 + r_2 + b + \delta_i) y_3^0, \end{aligned} \tag{37}$$

where

$$K_1(\lambda)=\int_0^{+\infty}\sigma(a)e^{-(\lambda+b+\delta_e)a-\int_0^a(\sigma(s)+\mu(s))ds}da,$$
$$K_2(\lambda)=\int_0^{+\infty}k(c)e^{-(\lambda+b)c-\int_0^c k(s)ds}dc,$$
$$K_3(\lambda)=\int_0^{+\infty}\mu(a)e^{-(\lambda+b+\delta_e)a-\int_0^a(\sigma(s)+\mu(a))ds}da,$$

By combining (37) and (32) we obtain the characteristic equation

$$det\begin{pmatrix}\lambda+b+\beta I^* & \beta S^*\\ \beta I^*(K_1(\lambda)+K_2(\lambda)K_3(\lambda)) & M\end{pmatrix}=0.$$

where $M=(\beta S^*+r_2)(K_1(\lambda)+K_2(\lambda)K_3(\lambda))+r_1K_2(\lambda)-(\lambda+r_1+r_2+b+\delta_i)$, that is

$$M=\frac{\beta^2S^*I^*}{\lambda+b+\beta I^*}(K_1(\lambda)+K_2(\lambda)K_3(\lambda)).\tag{38}$$

It follows from (20) that (38) can also be rewritten as

$$\begin{aligned}&(\frac{\beta S^0}{R_0}+r_2)(K_1(\lambda)+K_2(\lambda)K_3(\lambda))+r_1K_2(\lambda)\\ &=\frac{\beta bS^0(R_0-1)}{(\lambda+bR_0)R_0}(K_1(\lambda)+K_2(\lambda)K_3(\lambda))+\lambda+r_1+r_2+b+\delta_i.\end{aligned}\tag{39}$$

Note that $K_i'(\lambda)<0, i=1,2,3$. Thus, $K_i(\lambda), i=1,2,3$ is decreasing. Further, $K_i(0)=\theta_i, i=1,2,3$. Assume that $Re\lambda\geq 0$, then $K_1(\lambda)\leq\theta_1$, $K_2(\lambda)\leq\theta_2$ and $K_3(\lambda)\leq\theta_3$ hold. Hence, the modulus of the left-hand side of (39) satisfies

$$\begin{aligned}&(\frac{\beta S^0}{R_0}+r_2)\mid(K_1(\lambda)+K_2(\lambda)K_3(\lambda))\mid+r_1\mid K_2(\lambda)\mid\\ &\leq(\frac{\beta S^0}{R_0}+r_2)(\theta_1+\theta_2\theta_3)+r_1\theta_2=r_1+r_2+b+\delta_i\end{aligned}$$

which, together with (39), leads to

$$\mid\frac{\beta bS^0(R_0-1)}{(\lambda+bR_0)R_0}(K_1(\lambda)+K_2(\lambda)K_3(\lambda))+\lambda+r_1+r_2+b+\delta_i\mid\leq r_1+r_2+b+\delta_i.\tag{40}$$

Since $R_0>1$, hence

$$\frac{\beta bS^0(R_0-1)}{(\lambda+bR_0)R_0}(K_1(\lambda)+K_2(\lambda)K_3(\lambda))+\lambda\leq 0.\tag{41}$$

that is $Re\lambda\leq 0$. There is a contradiction. This means that all roots of (39) have negative real parts. Consequently, the endemic equilibrium $E^*$ of (1) is locally asymptotically stable if $R_0>1$. □

5. **Uniform persistence.** In this section, we study the uniform persistence of system (1). Define $M_0=\{(S,I,0,0,e,r)^T\in X_{0+}: I+\int_0^\infty e(a)da+\int_0^\infty r(c)dc>0\}$, and $\partial M_0=X_{0+}\backslash M_0$.

**Theorem 5.1.** *$M_0$ and $\partial M_0$ are both positively invariant under the semiform $\{U(t)\}_{t\geq 0}$ generated by system (1) on $X_{0+}$. Moreover, the infection-free equilibrium $E_0=(S^0,0,0,0,0_{L^1},0_{L^1})$ is globally asymptotically stable for the semiflow $\{U(t)\}_{t\geq 0}$ restricted to $\partial M_0$.*

*Proof.* Let $(S^0, I_0, 0, 0, e_0, r_0) \in M_0, T(t) = I(t) + \int_0^\infty e(t,a)da + \int_0^\infty r(t,c)dc$. It follows that

$$T'(t) \geq -max\{(r_1 + r_2 + b + \delta_i), (b + \delta_e + \mu_{max})\}T(t),$$

where $\mu_{max} = esssup_{a\in(0,\infty)}\mu(a)$. Then,

$$T(t) \geq e^{-max\{(r_1+r_2+b+\delta_i),(b+\delta_e+\mu_{max})\}t}T(0).$$

This completes the fact that $U(t)M_0 \subset M_0$. Now let $(S^0, I_0, 0, 0, e_0, r_0) \in \partial M_0$, using (4) and (5), we easily find that $I(t) = 0$, for $t \geq 0$, and

$$\begin{aligned}\int_0^\infty e(t,a)da &= \int_0^t e(t-a,0)e^{-\int_0^a \varepsilon(s)ds}da + \int_t^\infty e_0(a-t)e^{-\int_{a-t}^a \varepsilon(s)ds}da \\ &= \int_0^t [\beta S(t-a)I(t-a) + r_2 I(t-a)]e^{-\int_0^a \varepsilon(s)ds}da \\ &\quad + \int_t^\infty e_0(a-t)e^{-\int_{a-t}^a \varepsilon(s)ds}da \\ &\leq e^{-\varepsilon_{min}t}\|e_0\|_{L_1} \to 0,\ t\to\infty.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_0^\infty r(t,c)dc &= \int_0^t r(t-c,0)e^{-\int_0^c \eta(s)ds}dc + \int_t^\infty r_0(c-t)e^{-\int_{c-t}^c \eta(s)ds}dc \\ &\leq e^{-\eta_{min}t}\|r_0\|_{L_1} \to 0,\ t\to\infty.\end{aligned}$$

Thus $U(t)\partial M_0 \subset \partial M_0$. Let $(S^0, I_0, 0, 0, e_0, r_0) \in \partial M_0$, we obtain

$$\begin{cases} \dfrac{dI}{dt} = -(r_1 + r_2 + b + \delta_i)I(t) + \displaystyle\int_0^\infty \sigma(a)e(t,a)da + \int_0^\infty k(c)r(t,c)dc, \\ (\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a})e(t,a) = -(b + \delta_e + \mu(a) + \sigma(a))e(t,a), \\ e(t,0) = \beta SI + r_2 I, \\ (\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c})r(t,c) = -(k(c) + b)r(t,c), \\ r(t,0) = r_1 I(t) + \displaystyle\int_0^\infty \mu e(t,a)da, \\ I(0) = 0, e(0,a) = e_0(a), r(0,c) = r_0(c). \end{cases}$$

Since $S(t) \leq S^0$ as $t$ is large enough, we get $I(t) \leq \tilde{I}(t), e(t,a) \leq \tilde{e}(t,a)$ and $r(t,c) \leq \tilde{r}(t,c)$, where

$$\begin{cases} \dfrac{d\tilde{I}}{dt} = -(r_1 + r_2 + b + \delta_i)\tilde{I} + \displaystyle\int_0^\infty \sigma(a)\tilde{e}(t,a)da + \int_0^\infty k(c)\tilde{r}(t,c)dc, \\ (\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a})\tilde{e}(t,a) = -(b + \delta_e + \mu(a) + \sigma(a))\tilde{e}(t,a), \\ \tilde{e}(t,0) = \beta S^0\tilde{I} + r_2\tilde{I}, \\ (\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial c})\tilde{r}(t,c) = -(k(c) + b)\tilde{r}(t,c), \\ \tilde{r}(t,0) = r_1\tilde{I} + \displaystyle\int_0^\infty \mu(a)\tilde{e}(t,a)da, \\ \tilde{I}(0) = 0, \tilde{e}(0,a) = e_0(a), \tilde{r}(0,c) = r_0(c). \end{cases} \tag{42}$$

By the formulations (4), (5), we have

$$\tilde{e}(t,a) = \begin{cases} \tilde{e}(t-a,0)e^{-\int_0^a \varepsilon(s)ds}, & t > a \geq 0, \\ e_0(a-t)e^{-\int_{a-t}^a \varepsilon(s)ds}, & a \geq t \geq 0. \end{cases} \tag{43}$$

$$\tilde{r}(t,c) = \begin{cases} \tilde{r}(t-c,0)e^{-\int_0^c \eta(s)ds}, & t > c \geq 0, \\ r_0(c-t)e^{-\int_{c-t}^c \eta(s)ds}, & c \geq t \geq 0. \end{cases} \tag{44}$$

Substituting (43) and (44) into the first equation of (42), with the help of the third and the fifth equations of (42), we obtain

$$\begin{cases} \dfrac{d\tilde{I}(t)}{dt} = (H_1 + H_2 + H_3 + H_4)\tilde{I}(t) + F_e(t) + F_r(t) + F_{er}(t), \\ \tilde{I}(0) = 0, \end{cases} \tag{45}$$

where

$$\begin{aligned} H_1 &= -(r_1 + r_2 + b + \delta_i), \\ H_2 &= \int_0^t \sigma(a)(\beta S^0 + r_2)e^{-\int_0^a \varepsilon(s)ds}da, \\ H_3 &= \int_0^t k(c)r_1 e^{-\int_0^c \eta(s)ds}dc, \\ H_4 &= \int_0^t k(c)\int_0^t \mu(a)(\beta S^0 + r_2)e^{-\int_0^a \varepsilon(s)ds}da e^{-\int_0^c \eta(s)ds}dc, \\ F_e(t) &= \int_t^\infty \sigma(a)e_0(a-t)e^{-\int_{a-t}^a \varepsilon(s)ds}da, \\ F_r(t) &= \int_t^\infty k(c)r_0(c-t)e^{-\int_{c-t}^c \eta(s)ds}dc, \\ F_{er}(t) &= \int_0^t k(c)\int_t^\infty \mu(a)e_0(a-t)e^{-\int_{a-t}^a \varepsilon(s)ds}da e^{-\int_0^c \eta(s)ds}dc. \end{aligned}$$

It's simple to know that, for each $t \to \infty$,

$$\begin{aligned} F_e(t) &\leq e^{-\varepsilon_{min}t}\int_t^\infty \sigma(a)e_0(a-t)da = 0, \\ F_r(t) &\leq e^{-\eta_{min}t}\int_t^\infty k(c)r_0(c-t)dc = 0, \\ F_{er}(t) &\leq \int_0^t k(c)e^{-\varepsilon_{min}t}\int_t^\infty \mu(a)e_0(a-t)da e^{-\int_0^c \eta(s)ds}dc = 0. \end{aligned}$$

Then, we know that equation (45) has a unique solution $\tilde{I}(t) = 0$ and we obtain $\tilde{e}(t,0) = 0$, $\tilde{r}(t,0) = 0$ from the third and fifth equations of (42). If $0 \leq a < t$, according to (43), we have $\tilde{e}(t,a) = 0$. Similarly, If $0 \leq c < t$, according to (44), we have $\tilde{r}(t,c) = 0$. If $a \geq t, \| \tilde{e}(t,a) \|_{L^1} \leq e^{-\varepsilon_{min}t} \| e_0 \|_{L^1}$, if $c \geq t, \| \tilde{r}(t,c) \|_{L^1} \leq e^{-\eta_{min}t} \| r_0 \|_{L^1}$, which yields that $\tilde{e}(t,a) \to 0$ as $t \to \infty$, and $\tilde{r}(t,c) \to 0$ as $t \to \infty$. By using $I(t) \leq \tilde{I}(t), e(t,a) \leq \tilde{e}(t,a)$ and $r(t,c) \leq \tilde{r}(t,c)$ , we have $I(t) \to 0, e(t,a) \to 0$ and $r(t,c) \to 0$. □

**Theorem 5.2.** *Assume $R_0 > 1$, the semiflow $\{U(t)\}_{t\geq 0}$ generated by system (1) is uniformly persistent with respect to the pair $(\partial M_0, M_0)$, that is there exists $\varepsilon > 0$,* *such that for each* $y \in M_0$,

$$\lim_{t\to+\infty} inf d(U(t)y, \partial M_0) \geq \varepsilon.$$

*Furthermore, there exists a compact subset* $A_0 \subset M_0$ *which is a global attractor for* $\{U(t)\}_{t\geq 0}$ *in* $M_0$.

*Proof.* Since the infection-free equilibrium $E_0 = (S^0, 0, 0, 0, 0_{L^1}, 0_{L^1})$ is globally asymptotically stable in $\partial M_0$, applying Theorem 4.2 in Hale and Waltman [10], we only need to investigate the behavior of the solutions starting in $M_0$ in some neighborhood of $E_0$. Then, we will show that $W^s(\{E_0\}) \bigcap M_0 = \emptyset$, where $W^s(\{E_0\}) = \{y \in X_{0+} : \lim_{t\to+\infty} U(t)y = E_0\}$. Assume there exists $y \in W^s(\{E_0\}) \bigcap M_0$, it follows that there exists $t_0 > 0$ such that $I(t_0) + \int_0^\infty e(t_0, a)da + \int_0^\infty r(t_0, c)dc > 0$. Using the same argument in the proof of Lemma 3.6(i) in Demasse and Ducrot [6], we have that $I(t) > 0$ for $t \geq 0$, and $e(t, a) > 0$ for $(t, a) \in [0, \infty) \times [0, \infty)$, $r(t, c) > 0$ for$(t, c) \in [0, \infty) \times [0, \infty)$. By means of the method of Brauer et al. [2], we define the following functions

$$\begin{aligned} A(a) &= \int_a^\infty (\sigma(\theta) + B(0)\mu(\theta))e^{-\int_a^\theta \varepsilon(s)ds}d\theta, \\ B(c) &= \int_c^\infty k(\theta)e^{-\int_c^\theta \eta(s)ds}d\theta. \end{aligned} \tag{46}$$

For, $\forall a, c > 0, A(a), B(c) \geq 0$, and $A(0) = \theta_1 + \theta_2\theta_3, B(0) = \theta_2$. Furthermore, for $\forall a \geq 0, c \geq 0$,

$$\begin{aligned} A'(a) &= -\sigma(a) + \varepsilon(a)A(a) - \theta_2\mu(a), \\ B'(c) &= -k(c) + \eta(c)B(c). \end{aligned} \tag{47}$$

Consider the function

$$\Phi(t) = I(t) + \int_0^\infty A(a)e(t, a)da + \int_0^\infty B(c)r(t, c)dc,$$

which satisfies

$$\frac{d\Phi(t)}{dt} = \beta I(\theta_1 + \theta_2\theta_3)(S - \frac{S^0}{R_0}).$$

Since $y \in W^s(\{E_0\})$, we have $S(t) \to S^0, I(t) \to 0$, as $t \to \infty$. When $R_0 > 1$, we know that the function $\Phi(t)$ is not decreasing for t large enough. Thus there exists $t_0 > 0$ such that $\Phi(t) \geq \Phi(t_0)$ for all $t \geq t_0$. Since $\Phi(t_0) > 0$, this prevents that the function $(I(t), e(t, a), r(t, c))$ converges to $(0, 0_{L^1}, 0_{L^1})$ as $t \to \infty$. A contradiction with $S(t) \to S^0$. □

## 6. **Global asymptotic stability of the endemic equilibrium.** Let

$$g(x) = x - 1 - \ln x,$$

denote $g'(x) = 1 - \frac{1}{x}$. Thus, $g : R^+ \to R^+$ is concave up. Also, the function $g$ has only one extremum which is a global minimum at 1, satisfying $g(1) = 0$ and $\forall x, y \in R,\ g(xy) \geq g(x) + g(y)$.

**Theorem 6.1.** *The unique endemic equilibrium* $E^*$ *is globally asymptotically stable if* $R_0 > 1$ .

*Proof.* Constructing the Lyapunov functional as follows

$$V_* = W_s + W_e + W_i + W_r,$$

where

$$\begin{aligned} W_s =&(\theta_1+\theta_2\theta_3)S^* g(\frac{S}{S^*}),\ W_i = I^* g(\frac{I}{I^*}),\\ W_e =& \int_0^{+\infty} A(a)e^*(a)g(\frac{e(t,a)}{e^*(a)})da,\ W_r = \int_0^{+\infty} B(c)r^*(c)g(\frac{r(t,c)}{r^*(c)})dc. \end{aligned}$$

Since $\Lambda = bS^* + \beta S^* I^*$, then the derivative of $W_s$ along with the solutions of (1) is

$$\begin{aligned} \frac{dW_s}{dt} =&(\theta_1+\theta_2\theta_3)bS^*(2-\frac{S}{S^*}-\frac{S^*}{S})\\ &+(\theta_1+\theta_2\theta_3)\beta S^* I^*(1-\frac{SI}{S^*I^*}-\frac{S^*}{S}+\frac{I}{I^*}). \end{aligned}$$

Calculating the derivative of $W_e$ along with the solutions of system (1) yields

$$\begin{aligned} \frac{dW_e}{dt} =& \int_0^{+\infty} A(a)e^*(a)\frac{\partial}{\partial t}g(\frac{e(t,a)}{e^*(a)})da\\ =& \int_0^{+\infty} A(a)e^*(a)\frac{\partial}{\partial t}(\frac{e(t,a)}{e^*(a)}-1-\ln\frac{e(t,a)}{e^*(a)})da\\ =& \int_0^{+\infty} A(a)e^*(a)(\frac{1}{e^*(a)}-\frac{1}{e(t,a)})\frac{\partial}{\partial t}e(t,a)da\\ =& \int_0^{+\infty} A(a)e^*(a)(\frac{1}{e^*(a)}-\frac{1}{e(t,a)})(-\frac{\partial}{\partial a}e(t,a)-\varepsilon(a)e(t,a))da\\ =& -\int_0^{+\infty} A(a)e^*(a)(\frac{e(t,a)}{e^*(a)}-1)(\frac{e_a(t,a)}{e(t,a)}+\varepsilon(a))da. \end{aligned}$$

Note that

$$\begin{aligned} \frac{\partial}{\partial a}g(\frac{e(t,a)}{e^*(a)}) =&\frac{e_a(t,a)+e(t,a)\varepsilon(a)}{e^*(a)}-\frac{e_a(t,a)}{e(t,a)}+\frac{e^*(a)(-\varepsilon(a))}{e^*(a)}\\ =&(\frac{e(t,a)}{e^*(a)}-1)(\frac{e_a(t,a)}{e(t,a)}+\varepsilon(a)). \end{aligned}$$

And

$$\begin{aligned} \frac{dA(a)}{da} =&A(a)\varepsilon(a)-\sigma(a)-\mu(a)B(0),\\ \frac{de^*(a)}{da} =&-\varepsilon(a)e^*(a). \end{aligned}$$

Hence, using integration by parts, we have

$$\begin{aligned} \frac{dW_e}{dt} =& -\int_0^{+\infty} A(a)e^*(a)\frac{\partial}{\partial a}g(\frac{e(t,a)}{e^*(a)})da\\ =& -A(a)e^*(a)g(\frac{e(t,a)}{e^*(a)})\ |_0^{+\infty} + \int_0^{+\infty}(\frac{d}{da}A(a))e^*(a)g(\frac{e(t,a)}{e^*(a)})da \end{aligned}$$

$$
\begin{aligned}
&+\int_0^{+\infty} A(a)(\frac{d}{da}e^*(a))g(\frac{e(t,a)}{e^*(a)})da \\
=&-A(a)e^*(a)g(\frac{e(t,a)}{e^*(a)})\mid_{+\infty} +A(0)e^*(0)g(\frac{e(t,0)}{e^*(0)}) \\
&-\int_0^{+\infty} e^*(a)(\sigma(a)+\mu(a)B(0))g(\frac{e(t,a)}{e^*(a)})da.
\end{aligned}
$$

Note $A(0)=\theta_1+\theta_2\theta_3, B(0)=\theta_2, e^*(0)=\beta S^*I^*+r_2I^*, e(t,0)=\beta S(t)I(t)+r_2I(t)$, thus

$$
\begin{aligned}
\frac{dW_e}{dt}=&-A(a)e^*(a)g(\frac{e(t,a)}{e^*(a)})\mid_{+\infty} +(\theta_1+\theta_2\theta_3)(\beta S^*I^*+r_2I^*)g(\frac{e(t,0)}{e^*(0)}) \\
&-\int_0^{+\infty} e^*(a)(\sigma(a)+\mu(a)\theta_2)g(\frac{e(t,a)}{e^*(a)})da.
\end{aligned}
$$

Further, it follows from $(r_1+r_2+b+\delta_i)I^*=\int_0^{+\infty}\sigma(a)e^*(a)da+\int_0^{+\infty}k(c)r^*(c)dc$, that the derivative of $W_i$ along with the solutions of system (1) gives

$$
\begin{aligned}
\frac{dW_i}{dt}=&I^*(\frac{I_t}{I^*}-\frac{I_t}{I}) \\
=&I^*(\frac{1}{I^*}-\frac{1}{I})[-(r_1+r_2+b+\delta_i)I \\
&+\int_0^{+\infty}\sigma(a)e(t,a)da+\int_0^{+\infty}k(c)r(t,c)dc] \\
=&\int_0^{+\infty}\sigma(a)e^*(a)(1-\frac{I}{I^*}-\frac{I^*e(t,a)}{Ie^*(a)}+\frac{e(t,a)}{e^*(a)})da \\
&+\int_0^{+\infty}k(c)r^*(c)(1-\frac{I}{I^*}-\frac{I^*r(t,c)}{Ir^*(c)}+\frac{r(t,c)}{r^*(c)})dc.
\end{aligned}
$$

Similar to $W_e$, by using $B(0)=\theta_2,\ r^*(0)=r_1I^*+\int_0^{+\infty}\mu(a)e^*(a)da$, and $r(t,0)=r_1I(t)+\int_0^{+\infty}\mu(a)e(t,a)da$, the derivative of $W_r$ along with the solutions of system (1) reads

$$
\begin{aligned}
\frac{dW_r}{dt}=&\int_0^{+\infty}B(c)r^*(c)\frac{\partial}{\partial t}g(\frac{r(t,c)}{r^*(c)})dc \\
=&\int_0^{+\infty}B(c)r^*(c)\frac{\partial}{\partial t}[\frac{r(t,c)}{r^*(c)}-1-\ln\frac{r(t,c)}{r^*(c)}]dc \\
=&\int_0^{+\infty}B(c)r^*(c)[(\frac{1}{r^*(c)}-\frac{1}{r(t,c)})\frac{\partial}{\partial t}r(t,c)]dc \\
=&\int_0^{+\infty}B(c)r^*(c)[(\frac{1}{r^*(c)}-\frac{1}{r(t,c)})(-\frac{\partial}{\partial c}r(t,c)-\eta(c)r(t.c))]dc \\
=&-\int_0^{+\infty}B(c)r^*(c)(\frac{r(t,c)}{r^*(c)}-1)(\frac{r_c(t,c)}{r(t,c)}+\eta(c))dc.
\end{aligned}
$$

Note

$$
\frac{\partial}{\partial c}g(\frac{r(t,c)}{r^*(c)})=(\frac{r(t,c)}{r^*(c)}-1)(\frac{r_c(t,c)}{r(t,c)}+\eta(c)),
$$

and

$$
\frac{dB(c)}{dc}=B(c)\eta(c)-k(c),\ \frac{dr^*(c)}{dc}=-\eta(c)r^*(c).
$$

Hence, using integration by parts, we have

$$\begin{aligned}\frac{dW_r}{dt} =& - \int_0^{+\infty} B(c)r^*(c)\frac{\partial}{\partial c}g(\frac{r(t,c)}{r^*(c)})dc \\ =& - B(c)r^*(c)g(\frac{r(t,c)}{r^*(c)}) \mid_{+\infty} + B(0)r^*(0)g(\frac{r(t,0)}{r^*(0)}) \\ &- \int_0^{+\infty} r^*(c)k(c)g(\frac{r(t,c)}{r^*(c)})dc \\ =& - B(c)r^*(c)g(\frac{r(t,c)}{r^*(c)}) \mid_{+\infty} - \int_0^{+\infty} r^*(c)k(c)g(\frac{r(t,c)}{r^*(c)})dc \\ &+ \theta_2(r_1 I^* + \int_0^{+\infty} \mu(a)e^*(a)da)g(\frac{r(t,0)}{r^*(0)}).\end{aligned}$$

Note

$$\begin{aligned}&\int_0^{+\infty} (\sigma(a) + \mu(a)\theta_2)e^*(a)da = (\theta_1 + \theta_2\theta_3)(\beta S^* I^* + r_2 I^*), \\ &\int_0^{+\infty} k(c)r^*(c)dc = \theta_2 r_1 I^* + \theta_2\theta_3(\beta S^* I^* + r_2 I^*).\end{aligned}$$

We derive

$$\begin{aligned}\frac{dV_*}{dt} =&(\theta_1 + \theta_2\theta_3)bS^*(2 - \frac{S}{S^*} - \frac{S^*}{S}) - A(a)e^*(a)g(\frac{e(t,a)}{e^*(a)}) \mid_{+\infty} \\ &- B(c)r^*(c)g(\frac{r(t,c)}{r^*(c)}) \mid_{+\infty} + \int_0^{+\infty} \sigma(a)e^*(a)dag(\frac{e(t,0)}{e^*(0)}) \\ &- \int_0^{+\infty} \mu(a)\theta_2 e^*(a)(g(\frac{e(t,a)}{e^*(a)}) - g(\frac{e(t,0)}{e^*(0)}))da \\ &+ \int_0^{+\infty} k(c)r^*(c)dcg(\frac{r(t,0)}{r^*(0)}) + H_1 + H_2 + H_3\end{aligned}$$

where

$$\begin{aligned}H_1 =&(\theta_1 + \theta_2\theta_3)\beta S^* I^*[-g(\frac{SI}{S^* I^*}) - g(\frac{S^*}{S}) + g(\frac{I}{I^*})] \\ \leq&(\theta_1 + \theta_2\theta_3)(\beta S^* I^* + r_2 I^*)[-g(\frac{SI}{S^* I^*}) - g(\frac{S^*}{S}) + g(\frac{I}{I^*})] \\ =&\int_0^{+\infty} (\sigma(a) + \mu(a)\theta_2)e^*(a)[-g(\frac{SI}{S^* I^*}) - g(\frac{S^*}{S}) + g(\frac{I}{I^*})]da, \\ H_2 =& \int_0^{+\infty} \sigma(a)e^*(a)(1 - \frac{I}{I^*} - \frac{I^* e(t,a)}{I e^*(a)} + \frac{e(t,a)}{e^*(a)} - \frac{e(t,a)}{e^*(a)} + 1 + \ln\frac{e(t,a)}{e^*(a)})da \\ =& - \int_0^{+\infty} \sigma(a)e^*(a)(g(\frac{I}{I^*}) + g(\frac{I^* e(t,a)}{I e^*(a)})da \\ \leq& - \int_0^{+\infty} \sigma(a)e^*(a)(g(\frac{I}{I^*}) + g(\frac{e(t,a)}{e^*(a)})da, \\ H_3 =& \int_0^{+\infty} k(c)r^*(c)(1 - \frac{I}{I^*} - \frac{I^* r(t,c)}{I r^*(c)} + \frac{r(t,c)}{r^*(c)} - \frac{r(t,c)}{r^*(c)} + 1 + \ln\frac{r(t,c)}{r^*(c)})dc \\ =& - \int_0^{+\infty} k(c)r^*(c)(g(\frac{I}{I^*}) + g(\frac{I^* r(t,c)}{I r^*(c)}))dc\end{aligned}$$

$$\le -\int_0^{+\infty} k(c)r^*(c)(g(\frac{I}{I^*}) + g(\frac{r(t,c)}{r^*(c)}))dc.$$

Hence, $dV_*/dt \le 0$ holds true. Furthermore, the strict equality holds only if $S = S^*, I = I^*, e(t,a) = e^*(a), r(t,c) = r^*(c)$. Consequently, the endemic equilibrium $E^*$ of (1) is globally asymptotically stable if $R_0 > 1$. □

7. **Numerical simulations.** In the following, we provide some numerical simulations to illustrate the global stability of the disease-free equilibrium and the endemic equilibrium for system (1). We choose parameters $\Lambda = 3; b = 0.065; n = 0.02; \alpha_1 = 0.01; \alpha_2 = 0.03; r_1 = 0.1; r_2 = 0.2$; and

$$\sigma(a) = \begin{cases} 0.3 & a \ge \tau \\ 0 & \tau \ge a \ge 0 \end{cases}, k(c) = \begin{cases} 0.1 & c \ge \tau \\ 0 & \tau \ge c \ge 0 \end{cases}, \mu(a) = \begin{cases} 0.25 & a \ge \tau \\ 0 & \tau \ge a \ge 0 \end{cases}.$$

Under the initial values

$$S(0) = 30, e(0,a) = 6e^{-0.3a}, I(0) = 10, r(0,c) = 6e^{-0.3c}.$$

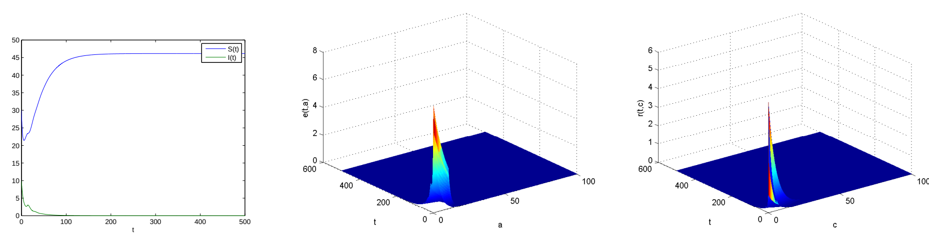


FIGURE 2. The time series of $S(t)$ and $I(t)$, and the age distributions of $e(t,a)$ and $r(t,c)$ when $\tau = 12$.

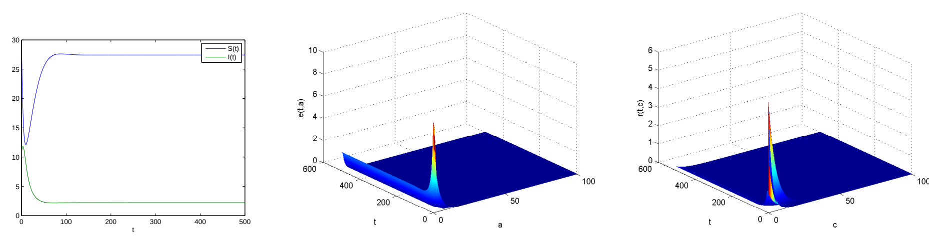


FIGURE 3. The time series of $S(t)$ and $I(t)$, and the age distributions of $e(t,a)$ and $r(t,c)$ when $\tau = 1$.

In Figure 2, we choose $\tau = 12$ , then $R_0 < 1$, while in Figure 3, we choose $\tau = 1$, then $R_0 > 1$. The figures show the series of $S(t)$ and $I(t)$ which converge to their equilibrium values, and the age distribution and time series of $e(t,a)$ and $r(t,c)$, respectively.

8. **Discussion.** In this section, we briefly summarize our results. First, a PDE tuberculosis model (1) is proposed here to incorporate the latent-stage progression age of latent individuals and the relapse age of removed individuals. In addition, we assumed that infectious individuals might come into the latent class $E$ due to incomplete treatment, and the relapse in the removed class. Under our assumptions, the expression of the basic reproduction number $R_0$ is given, and we proved that if $R_0 < 1$ the disease-free equilibrium $E_0$ is globally asymptotically stable, while if

$R_0 > 1$ the unique endemic equilibrium $E^*$ is globally asymptotically stable. Figure 2 and Figure 3 further verify our results.

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