OPTIMAL CONTROL OF INTEGRODIFFERENCE EQUATIONS IN A PEST-PATHOGEN SYSTEM

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ABSTRACT. We develop the theory of optimal control for a system of integrodifference equations modelling a pest-pathogen system. Integrodifference equations incorporate continuous space into a system of discrete time equations. We design an objective functional to minimize the damaged cost generated by an invasive species and the cost of controlling the population with a pathogen. Existence, characterization, and uniqueness results for the optimal control and corresponding states have been completed. We use a forward-backward sweep numerical method to implement our optimization which produces spatio-temporal control strategies for the gypsy moth case study.

1. Introduction. Invasive species are one of the world’s largest immediate environmental threats [51, 2]. They can have a strong negative impact on ecosystem function and services, if they are not properly removed in a reasonable time frame [70]. Many land-based invasions are either from plant species or from insects [4, 53], both of which commonly exhibit seasonally structured population or generational dynamics. Population dynamics in this case are well described using systems of nonlinear difference equations in discrete time [47, 39].

Spatial spread is a key element of invasion processes. There is a long history of modeling movement of biological organisms, starting with Skellam [61] who explored the role of random motion to describe biological movement using a linear diffusion partial differential equation. Many extensions to this structure have been proposed; see [41] for a recent review and [42, 7, 48] for an overview of partial differential equation (PDE) approaches to modelling biological movement and dynamics.

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An alternative to the PDE approach to modeling invasive spatial spread uses integrodifference equations in which a dispersal kernel is used to describe the redistribution of species [33, 64, 32, 22]. Examples of populations that have been modeled with integrodifference equations include house finches in the eastern North America, cane toads in Australia, blowflies in South Africa, and Platte thistle [65, 50, 40, 13]. Integrodifference equations have also been used to understand the following: migration of trees, plant invasions, evolution of flowering strategies, persistence of populations in fragmented habitats and river environments, invasion of structured populations, Allee effects, movement of individuals in patchy landscapes, and habitat shifting as a result of climate change [8, 38, 56, 44, 67, 25, 43, 71]. For a recent review of ecological applications see [31].

Some classical approaches to the study of PDEs have been applied to integrodifference model systems within the ecological literature to include calculation of traveling wave speeds [66] and determination of the conditions required for spatial pattern formation [32, 46]; theoretical work has also derived conditions by which coupled map lattice models can be analyzed using systems of integrodifference equations [68]. In the case where the underlying dynamics are generational or best described in discrete time, for a population $N_k(x)$ at location $x$ in the spatial domain $\Omega$ at time $k$ subject to local population demographic dynamics $f(N)$ and with spatial redistribution kernel $k_1(x, y)$, the model system takes the form:

$$N_{k+1}(x) = \int_{\Omega} k_1(x, y) f(N_t(y)) dy.$$
host-parasitoid and predator-prey [24, 45]. Following the analysis of the existence and the characterization of the optimal control, we apply our results to the control of gypsy moth populations which have spread west from an initial release point in north-eastern USA [62]. This pest population can be managed by the nucleopolyhedrosis virus biocontrol product, Gypchek [52]. We choose growth functions appropriate for gypsy moth and virus population.

In section 2, we formulate a two species coupled integrodifference system and its associated objective functional for an invasive pest species and its biological control agent. We focus on this structure as the simplest system for control in which any meaningful interaction between the two species exists (in the case of a non-biological control, the problem can be collapsed into a single species invasion problem; using biological control will create a feedback mechanism between the two species). In section 3, we present the existence and characterization of the optimal control. In section 4, we prove uniqueness of the optimal control. Finally in section 5, we apply the theory to the problem of gypsy moth spread and we conclude with a discussion section with the results presented in a broader context.

2. The integrodifference model. In a spatial domain Ω, the model comprises a pair of coupled nonlinear integrodifference equations, in which the first state variable $N(x) = (N_0(x), N_1(x), ..., N_T(x))$, represents the density of the invasive species at location $x$ and time step $k$, and the second state variable, $Z(x) = (Z_0(x), Z_1(x), ..., Z_T(x))$, denotes the density of the biological control agent. We consider the problem in a bounded one dimensional spatial domain although the analytical and numerical techniques employed here can be extended in a straightforward manner to higher dimensions. Assuming that the invasive species has dispersal kernel $k_1$, the biological control agent has dispersal kernel $k_2$ and the applied control agent has dispersal kernel $k_3$, we obtain the spatio-temporal model system:

For $x \in \Omega$, at time $k$,

$$N_{k+1}(x) = \int_{\Omega} k_1(x, y) F(N_k(y), Z_k(y)) dy$$

$$Z_{k+1}(x) = \int_{\Omega} k_2(x, y) G(N_k(y), Z_k(y)) dy + \int_{\Omega} k_3(x, y) u_k(y) dy$$

for $k = 0, ..., T - 1$

where $u(x) = (u_0(x), u_1(x), ..., u_{T-1}(x))$, is the density of biocontrol added to the system. Initial distributions for the invasive species and the biocontrol are given as non-negative bounded functions $N_0(x)$ and $Z_0(x)$ respectively. We are interested in minimizing costs associated with the invasive species and with the controls subject to the underlying spatio-temporal dynamics described by (1).

The function $F(N_k(y), Z_k(y))$ describes the local population growth of the host at location $y$, and we are assuming that the pathogen population causes the growth of the gypsy moth population to decline. The function $G(N_k(y), Z_k(y))$ describes the local population growth of the pathogen. The dispersal kernel $k_i(x, y)$ represents the probability of moving from point $x$ to point $y$. The component $u_k(y)$ of the control is the density of biocontrol added to the system at position $y$ at time $k$.

Our goal is to minimize the costs associated with damage from the invasive population and the implementation of the control. We assume that there is a cost $A_k$ associated with the host density $N_k$, connected to the damages caused directly or indirectly by the presence of the host species. The cost of applying the
biocontrol is a function of the control level and we assume a quadratic cost of the form, $B_k u_k + C_k u_k^2$. The objective functional to be minimized is defined as

$$J(u) = \int_{\Omega} A_T N_T(x) dx + \sum_{k=0}^{T-1} \int_{\Omega} \left[ A_k N_k(x) + B_k u_k(x) + C_k u_k^2(x) \right] dx,$$

with the set of bounded controls given by

$$U = \left\{ u = (u_0(x),...,u_{T-1}(x)) \in (L^\infty(\Omega))^T \mid 0 \leq u_k(x) \leq u_{max}, k = 0, 1, ..., T - 1 \right\}.$$

(2)

3. Existence and characterization of an optimal control. To set our model in an appropriate mathematical setting corresponding to an underlying pest-pathogen scenario, we make the following assumptions:

**Assumptions on $F$ and $G$.** Functions $F(N,Z)$ and $G(N,Z)$ are twice differentiable in $N$ and $Z$. We also assume that the partial derivatives, $\frac{\partial F(N,Z)}{\partial N}$, $\frac{\partial F(N,Z)}{\partial Z}$, $\frac{\partial^2 F(N,Z)}{\partial N^2}$, $\frac{\partial^2 F(N,Z)}{\partial N \partial Z}$, $\frac{\partial^2 F(N,Z)}{\partial Z^2}$, $\frac{\partial^2 G(N,Z)}{\partial N^2}$, $\frac{\partial^2 G(N,Z)}{\partial N \partial Z}$, $\frac{\partial^2 G(N,Z)}{\partial Z^2}$ are $L^\infty$ bounded for any $N,Z \in L^\infty(\Omega)$.

**Assumptions on the kernels.** The kernels are bounded and Lebesgue measurable such that

$$0 < \int_{\Omega} k_i(x,y) dy \leq 1 \text{ for all } x \in \Omega, \text{ where } i = 1, 2, 3 \quad 0 \leq k_i(x,y) \leq K \text{ for } (x,y) \in \Omega.$$

One can easily show the following boundedness and non-negativity lemma, by showing the bounds on $N_1$, $Z_1$ from the initial conditions and then iterating to get the bounds for the other $k$ values [27].

**Lemma 3.1.** There exists a constant $D$ such that $0 \leq N_k(x) \leq D$, $0 \leq Z_k(x) \leq D$ for all $x \in \Omega$, $k = 0, 1, ..., T$.

We proceed to prove the existence, characterization and uniqueness of the optimal control.

**Theorem 3.2.** Under above assumptions, an optimal control $u^* \in U$ exists such that the objective functional $J(u)$ is minimized.

**Proof.** The objective functional is bounded below by zero. Thus let $\{u^i\}$ be a minimizing sequence for the objective functional $J$ and $\{N^i\}$ and $\{Z^i\}$ be the corresponding state sequences. As the sequences are bounded from Lemma 3.1 and definition of $U$, there exists $u^* \in U$ and $N^*, Z^* \in (L^\infty(\Omega))^{T+1}$ such that on a subsequence

$$u^i_k(x) \to u^*_k(x) \quad N^i_k(x) \to N^*_k(x) \quad Z^i_k(x) \to Z^*_k(x)$$

as $i \to \infty$, weakly in $L^2(\Omega)$ for each $k$. We now show that the states $N^*$ and $Z^*$ correspond to the control $u^*$. Using the structure of the model, we show that the state sequences converge pointwise. Due to the fixed initial conditions.

$$N^*_1(x) = \int_{\Omega} k_1(x,y) F(N_0(y), Z_0(y)) dy$$
\[ Z_t^i(x) = \int_{\Omega} k_2(x,y) G(N_0(y), Z_0(y)) dy + \int_{\Omega} k_3(x,y) u_0^i(y) dy. \]

Hence the sequence \( \{N_t^i\} \) is constant with value \( N_1 \). The weak convergence of the sequence \( \{u_0^i\} \) gives \( Z_t^1(x) \to Z_t^i(x) \) pointwise for each \( x \) as \( i \to \infty \). For \( k = 2 \), this pointwise convergence of the sequences \( \{N_t^i\} \) and \( \{Z_t^i\} \) and the continuity of \( F,G \) gives convergence pointwise of \( \{F(N_t^i, Z_t^i)\} \), \( \{G(N_t^i, Z_t^i)\} \) and then gives convergence of the integral terms by Lebesgue’s Dominated Convergence Theorem,

\[ N_t^2(x) = \int_{\Omega} k_1(x,y) F(N_1(y), Z_t^i(y)) dy \]
\[ Z_t^2(x) = \int_{\Omega} k_2(x,y) G(N_1(y), Z_t^i(y)) dy + \int_{\Omega} k_3(x,y) u_1^i(y) dy. \]

Thus we obtain pointwise convergence of \( N_t^2, Z_t^2 \) and by iteration, the pointwise convergence of \( N_t^k, Z_t^k \) as \( i \to \infty, k = 0,1,\ldots,T \). Hence \( N^* = N(u^*) \) and \( Z^* = Z(u^*) \). Now we verify that \( u^* \) is an optimal control. For the quadratic term of the control in the objective functional, the lower semi-continuity of the \( L^2 \) norm with respect to weak convergence of control sequences gives:

\[
\begin{align*}
\min_{u \in U} J(u) &= \lim_{i \to \infty} \int_{\Omega} \sum_{k=0}^{T-1} [A_k N_k^i(x) + B_k u_k^i(x) + C_k(u_k^i(x))^2] dx + \int_{\Omega} A_T N_T^i(x) dx \\
&\geq \lim_{i \to \infty} \int_{\Omega} \sum_{k=0}^{T-1} [A_k N_k^i(x) + B_k u_k^i(x)] dx + \liminf_{i \to \infty} \int_{\Omega} \sum_{k=0}^{T-1} C_k(u_k^i(x))^2 dx \\
&\quad + \lim_{i \to \infty} \int_{\Omega} A_T N_T^i(x) dx \\
&\geq \int_{\Omega} \sum_{k=0}^{T-1} [A_k N_k^i(x) + B_k u_k^i(x) + C_k(u_k^i(x))^2] dx + \int_{\Omega} A_T N_T^i(x) dx.
\end{align*}
\]

Consequently an optimal control \( u^* \) exists with corresponding states \( N^* \) and \( Z^* \).

To characterize an optimal control, we must differentiate the map \( u \to J(u) \), which requires first the differentiation of the control-to-solution maps \( u \to N = N(u), Z = Z(u) \). The two components of the direction derivative \( \psi \) of this solution map are called the sensitivities of the states with respect to the control.

**Theorem 3.3.** Under above assumptions, the mapping \( u \to N, Z \) is differentiable in the following sense: For any \( u \in U \) and \( l \in (L^\infty(\Omega))^T \), such that \( (u + \varepsilon l) \in U \) for \( \varepsilon \) small, there exists sensitivities \( \psi^N(x), \psi^Z(x) \in (L^\infty(\Omega))^{T+1} \) such that for \( k = 0,1,\ldots,T \):

\[ \frac{N_k^i(x) - N_k(x)}{\varepsilon} \to \psi^N_k(x), \]
\[ \frac{Z_k^i(x) - Z_k(x)}{\varepsilon} \to \psi^Z_k(x), \]

pointwise on \( \Omega \) as \( \varepsilon \to 0^+ \), for \( 0 < \varepsilon < 1 \) such that \( (u + \varepsilon l) \in U \). Furthermore the sensitivities satisfy the following system for \( k = 0,1,\ldots,T - 1 \):
Thus the quotients
\[ \frac{N(x)}{\varepsilon}, \frac{Z(x)}{\varepsilon} \]
converge pointwise to the desired sensitivities \( \psi^N(x), \psi^Z(x) \). For the second component of sensitivities, consider

\begin{align*}
N_2(x) - N_1(x) &= \int_\Omega k_1(x,y) \left[ F(N_1(y), Z_1(y)) - F(N_2(y), Z_2(y)) \right] dy \\
Z_2(x) - Z_1(x) &= \int_\Omega k_2(x,y) \left[ G(N_1(y), Z_1(y)) - G(N_2(y), Z_2(y)) \right] dy \\
&\quad + \int_\Omega k_3(x,y) (\varepsilon l_1(y)) dy.
\end{align*}

For simplicity we will use

\[ F(N_1(y), Z_1(y)) - F(N_1(y), Z_1(y)) = F(N_1, Z_1) - F(N_1, Z_1), \]
and we continue to analyze the difference quotients:

\[
\frac{N_{2}^{2}(x) - N_{2}(x)}{\varepsilon} = \int_{\Omega} k_{1}(x, y) \left[ \frac{F(N_{1}^{2}, Z_{1}^{2}) - F(N_{1}, Z_{1})}{\varepsilon} \right] dy
\]

\[
= \int_{\Omega} k_{1}(x, y) \left[ \frac{F(N_{1}^{2}, Z_{1}^{2}) - F(N_{1}, Z_{1})}{\varepsilon} + \frac{F(N_{1}, Z_{1}^{2}) - F(N_{1}, Z_{1})}{\varepsilon} \right] dy
\]

\[
= \int_{\Omega} k_{1}(x, y) \left[ \left( \frac{F(N_{1}^{2}, Z_{1}^{2}) - F(N_{1}, Z_{1})}{N_{1}^{2} - N_{1}} \right) \left( \frac{N_{1}^{2} - N_{1}}{\varepsilon} \right) \right] dy
\]

\[
+ \left( \frac{F(N_{1}, Z_{1}^{2}) - F(N_{1}, Z_{1})}{Z_{1}^{2} - Z_{1}} \right) \left( \frac{Z_{1}^{2} - Z_{1}}{\varepsilon} \right) \int_{\Omega} dy.
\]

From the uniform convergence of \( Z_{1}^{x}(x) \) to \( Z_{1}(x) \) and the differentiability of \( F \) and \( G \), we can pass the limit in the right hand side of (4) and get the pointwise convergence of the quotient \( \left\{ \frac{N_{2}^{x}(x) - N_{2}(x)}{\varepsilon} \right\} \). Doing a similar process for \( Z_{2}^{x} - Z_{2} \), we obtain the existence of \( \psi_{k}^{N}(x), \psi_{k}^{Z}(x) \). Continuing iteratively it follows that \( \left\{ \frac{N_{k}(x) - N_{k}(x)}{\varepsilon} \right\}, \left\{ \frac{Z_{k}(x) - Z_{k}(x)}{\varepsilon} \right\} \), converge pointwise to the desired sensitivities \( \psi_{k}^{N}(x), \psi_{k}^{Z}(x) \), which satisfy system (3).

To obtain the adjoint system, we will rewrite the system (3), leaving all the sensitivity terms on the left hand side and all other terms on the right hand side. By differentiating \( J(u) \) with respect to \( u \) and introducing the adjoint system, we can characterize the optimal control.

**Theorem 3.4.** Under above assumptions, given an optimal control \( u^{*} \) with corresponding solutions \( N^{*}, Z^{*} \), then there exists adjoint variables \( \lambda^{N}, \lambda^{Z} \), satisfying this system for \( k = 0, 1, \ldots, T - 1 \)

\[
\lambda_{k}^{N}(x) = \frac{\partial F(N_{k}^{*}(x), Z_{k}^{*}(x))}{\partial N} \int_{\Omega} k_{1}(y, x) \lambda_{k+1}^{N}(y) dy + \frac{\partial G(N_{k}^{*}(x), Z_{k}^{*}(x))}{\partial N} \int_{\Omega} k_{2}(y, x) \lambda_{k+1}^{N}(y) dy + A_{k}
\]

\[
\lambda_{k}^{Z}(x) = \frac{\partial F(N_{k}^{*}(x), Z_{k}^{*}(x))}{\partial Z} \int_{\Omega} k_{1}(y, x) \lambda_{k+1}^{Z}(y) dy + \frac{\partial G(N_{k}^{*}(x), Z_{k}^{*}(x))}{\partial Z} \int_{\Omega} k_{2}(y, x) \lambda_{k+1}^{Z}(y) dy
\]

with the transversality conditions

\[
\lambda_{T}^{N}(x) = A_{T}, \lambda_{T}^{Z}(x) = 0, \forall x \in \Omega.
\]

Furthermore \( u_{k}^{*}(x) \) is given by

\[
u_{k}^{*}(x) = \max \left( 0, \min \left( -\int_{\Omega} k_{3}(x, y) \lambda_{k+1}^{Z}(y) dy + B_{k} \frac{u_{max}}{2C_{k}}, \right) \right) (6)
\]

For \( k = 0, 1, \ldots, T - 1 \).
Proof. We start by taking the directional derivative of the objective functional. Since \( J(u^*) \) is the minimum value, for \( 0 < \varepsilon \) and directional vector \( l \), for \( u^* + \varepsilon l \in U \), we have

\[
0 \leq \lim_{\varepsilon \to 0^+} \frac{J(u^* + \varepsilon l) - J(u^*)}{\varepsilon}
\]

(7)

\[
= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[ \sum_{k=0}^{T-1} \int_{\Omega} (A_k N_k^* x) + B_k (u_k^* + \varepsilon l_k)(x) + C_k (u_k^* + \varepsilon l_k)^2(x) \right] dx
+ \int_{\Omega} A_T N_T^* x) dx - \sum_{k=0}^{T-1} \int_{\Omega} (A_k N_k^* x) + B_k u_k^* (x) + C_k (u_k^*)^2(x) \right] dx
- \int_{\Omega} A_T N_T^* x) dx \right]
= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[ \sum_{k=0}^{T-1} \int_{\Omega} \left[ A_k \left( \frac{N_k^* x) - N_k^* x)}{\varepsilon} \right) + B_k \left( \frac{(u_k^* + \varepsilon l_k)(x) - (u_k^*)(x)}{\varepsilon} \right)
+ C_k \left( \frac{(u_k^* + \varepsilon l_k)^2(x) - (u_k^*)^2(x)}{\varepsilon} \right) \right] dx + \lim_{\varepsilon \to 0^+} \int_{\Omega} \frac{A_T N_T^* x) - N_T^* x)}{\varepsilon} dx
= \sum_{k=0}^{T-1} \int_{\Omega} [A_k \psi_k^N(x) + B_k l_k(x) + C_k 2u_k^* l_k(x)] dx + \int_{\Omega} A_T \psi_T^N(x) dx.
\]

We focus just on the terms: \( \sum_{k=0}^{T-1} \int_{\Omega} A_k \psi_k^N(x) dx + \int_{\Omega} A_T \psi_T^N(x) \). First we add the term \( 0\psi_k^Z(x) \) and then substituting from the adjoint system (5), we obtain

\[
= \sum_{k=0}^{T-1} \int_{\Omega} \left[ \lambda_k^Z(x) - \frac{\partial F(N_k^* x, Z_k^* x)}{\partial N} \int_{\Omega} k_1(y, x) \lambda_{k+1}^N(x) dy \right] dx
- \frac{\partial G(N_k^* x, Z_k^* x)}{\partial N} \int_{\Omega} k_2(y, x) \lambda_{k+1}^N(x) dy \right] \psi_k^N(x)
+ \left( \lambda_k^Z(x) - \partial F(N_k^* x, Z_k^* x) \int_{\Omega} k_1(y, x) \lambda_{k+1}^N(x) dy \right) \lambda_k^Z(x) \psi_k^Z(x)
- \frac{\partial G(N_k^* x, Z_k^* x)}{\partial Z} \int_{\Omega} k_2(y, x) \lambda_{k+1}^Z(x) dy \right] \psi_k^Z(x) \right] dx + \int_{\Omega} A_T \psi_T^N(x) dx
= \sum_{k=0}^{T-1} \int_{\Omega} \left[ \lambda_k^N(x) \psi_k^N(x) + \lambda_k^Z(x) \psi_k^Z(x) \right] dx
+ \sum_{k=0}^{T-1} \left[ \int_{\Omega} k_1(y, x) \lambda_{k+1}^N(x) dy \left( - \frac{\partial F(N_k^* x, Z_k^* x)}{\partial N} \right) \psi_k^N(x) \right] dx.
\]
We introduce the term \( \int_\Omega 0 \psi_k^2(x) dx \) in the first sum and the last term and use the transversality conditions, and include these new terms in the first sum. Since the initial conditions for the sensitivities are all zero, we exclude those two terms and we reindex the first sum to match the other two sums. Finally working just in the second and third sum, we first change the order of integration, then interchange variables \( x \) and \( y \)

\[
\begin{align*}
&\sum_{k=0}^{T-1} \int_\Omega \left[ \lambda^N_{k+1}(x) \psi^N_{k+1}(x) + \lambda^Z_{k+1}(x) \psi^Z_{k+1}(x) \right] dx \\
&+ \sum_{k=0}^{T-1} \int_\Omega \left[ \lambda^N_{k+1}(x) \int_\Omega k_1(x,y) \left( -\frac{\partial F(N^*(y), Z^*(y))}{\partial N} \psi^N_k(y) \\ - \frac{\partial F(N^*(y), Z^*(y))}{\partial Z} \psi^Z_k(y) \right) dy \right] dx \\
&+ \sum_{k=0}^{T-1} \int_\Omega \left[ \lambda^Z_{k+1}(x) \int_\Omega k_2(x,y) \left( -\frac{\partial G(N^*(y), Z^*(y))}{\partial N} \psi^N_k(y) \\ - \frac{\partial G(N^*(y), Z^*(y))}{\partial Z} \psi^Z_k(y) \right) dy \right] dx \\
&= \sum_{k=0}^{T-1} \int_\Omega \lambda^N_{k+1}(x) \left[ \int_\Omega k_1(x,y) \left( -\frac{\partial F(N^*(y), Z^*(y))}{\partial N} \psi^N_k(y) \\ + \frac{\partial F(N^*(y), Z^*(y))}{\partial Z} \psi^Z_k(y) \right) dy \right] dx \\
&+ \sum_{k=0}^{T-1} \int_\Omega \lambda^Z_{k+1}(x) \left[ \int_\Omega k_2(x,y) \left( -\frac{\partial G(N^*(y), Z^*(y))}{\partial N} \psi^N_k(y) \\ + \frac{\partial G(N^*(y), Z^*(y))}{\partial Z} \psi^Z_k(y) \right) dy \right] dx \\
&= \sum_{k=0}^{T-1} \int_\Omega \lambda^N_{k+1}(x) [0] dx + \sum_{k=0}^{T-1} \int_\Omega \lambda^Z_{k+1}(x) \left[ \int_\Omega k_3(x,y) l_k(y) dy \right] dx.
\end{align*}
\]

Therefore we can say

\[
\begin{align*}
&\sum_{k=0}^{T-1} \int_\Omega A_k \psi^N_k(x) dx + \int_\Omega A_T \psi^N_T(x) dx \\
&= \sum_{k=0}^{T-1} \int_\Omega \lambda^Z_{k+1}(x) \left[ \int_\Omega k_3(x,y) l_k(y) dy \right] dx
\end{align*}
\]
Using the last expression in inequality (7),
\[
0 \leq \sum_{k=0}^{T-1} \int_{\Omega} \left[ A_k \psi_k^N(x) + B_k l_k(x) + C_k 2u_k^* l_k(x) \right] dx + \int_{\Omega} A_T \psi_T^N(x) dx
= \sum_{k=0}^{T-1} \int_{\Omega} \left[ \lambda_{k+1}^2(x) \int k_3(x,y) l_k(y) dy + B_k l_k(x) + C_k 2u_k^* l_k(x) \right] dx
= \sum_{k=0}^{T-1} \int_{\Omega} \left[ l_k(y) \int k_3(x,y) \lambda_{k+1}^2(x) dx \right] dy + \int_{\Omega} [B_k l_k(x) + C_k 2u_k^* l_k(x)] dx
= \sum_{k=0}^{T-1} \int_{\Omega} \left[ l_k(y) \int k_3(x,y) \lambda_{k+1}^2(x) dx \right] dy + \int_{\Omega} [B_k l_k(x) + C_k 2u_k^* l_k(y)] dy
= \sum_{k=0}^{T-1} \int_{\Omega} l_k(y) \int k_3(x,y) \lambda_{k+1}^2(x) dx + B_k + C_k 2u_k^* dy.
\]

Since a minimum occurs, at \( \varepsilon = 0 \), considering the sign of the variation \( l_k \) on subsets of \( \Omega \) where \( u_k^* = 0 \), \( u_k^* = u_{\max}^* \), or \( 0 < u_k^* < u_{\max}^* \) we obtain
\[
u_k^*(x) = \begin{cases} 
0 & \text{if } -f_{\Omega} k_3(y,x) \lambda_{k+1}^2(y) dy + B_k \leq 0 \\
\frac{f_{\Omega} k_3(y,x) \lambda_{k+1}^2(y) dy + B_k}{2C_k} & \text{if } 0 \leq -f_{\Omega} k_3(y,x) \lambda_{k+1}^2(y) dy + B_k \leq u_{\max} \\
u_{\max} & \text{if } u_{\max} \leq -f_{\Omega} k_3(y,x) \lambda_{k+1}^2(y) dy + B_k,
\end{cases}
\]
which gives the control characterization in (6).

\[ \square \]

4. Uniqueness result for the optimal control. We show uniqueness of the optimal control under a largeness assumption on the coefficients of the quadratic cost term.

**Theorem 4.1.** If \( C_k, k = 0, 1, ..., T - 1 \) are sufficiently large, then the optimal control is unique

**Proof.** We show uniqueness by showing strict concavity of the map:

\[ u \in U \rightarrow J(u). \]

The concavity follows from showing for all \( u, l \in U \), \( 0 < \varepsilon < 1 \)

\[ g''(\varepsilon) > 0 \text{ where } g(\varepsilon) = J(\varepsilon l + (1 - \varepsilon)u) = J(u + \varepsilon(l - u)). \]

For simplicity we use the notation:

\[ N^\varepsilon(x) = N(u + \varepsilon(l - u)) \quad N^{\varepsilon+\tau}(x) = N(u + (\varepsilon + \tau)(l - u)) \]
First we calculate $g'(\varepsilon)$:

$$g'(\varepsilon) = \lim_{\tau \to 0} \left( \frac{J(u_k + (\varepsilon + \tau)(l_k - u_k)) - J(u_k + \varepsilon(l_k - u_k))}{\tau} \right)$$

$$= \lim_{\tau \to 0} \frac{T-1}{\tau} \sum_{k=0}^{T-1} \int_{\Omega} \left\{ A_k \psi^{N^r}_k(x) + B_k \psi^{Z^r}_k(x) + C_k \right\} dx$$

The results in Theorem 2 were used in the limits of these quotients in $g'(\varepsilon)$:

$$\frac{N^r_k(x) - N^r_k(x)}{\tau} \to \psi^{N^r}_k(x)$$

$$\frac{Z^r_k(x) - Z^r_k(x)}{\tau} \to \psi^{Z^r}_k(x)$$

We want to estimate $\psi^{N^r}_k(x)$ in terms of the components of $l - u$. Using our assumptions and initial conditions, we can show that

$$\left| \psi^{N^r}_1(x) \right| = 0$$

$$\left| \psi^{Z^r}_1(x) \right| \leq E_1 \int_{\Omega} |(l_0 - u_0)(y)| dy.$$
\[ \int \psi_k(x,y)(l_1 - u_1(y))dy \leq \int k_2(x,y) \frac{\partial G(N_1(y), Z_1(y))}{\partial Z} \psi_1^T(y)dy + \int k_3(x,y)(l_1 - u_1(y))dy \]

\[ \leq E_2 \left[ \int |l_0 - u_0(y)|dy + \int |l_1 - u_1(y)|dy \right]. \]

If we continue this iterative process we will find

\[ |\psi_{k+1}^{N^r}(x)| \leq D_{k+1} \sum_{j=0}^{k} \int |l_j - u_j(y)|dy \]

\[ |\psi_{k+1}^{Z^r}(x)| \leq E_{k+1} \sum_{j=0}^{k} \int |l_j - u_j(y)|dy \]

with \( k = 0, 1, ..., T - 1 \)

where the constants \( D_k \) and \( E_k \) do not depend on \( \varepsilon \).

To find \( q''(\varepsilon) \), we calculate the difference quotient for directional derivative of \( \psi_{k}^{N^r} \) and \( \psi_{k}^{Z^r} \) with respect \( (u + \varepsilon(l - u)) \) in the direction \( l - u \):

\[
\frac{\psi_{k+1}^{N^{r+\tau}}(x) - \psi_{k+1}^{N^r}(x)}{\tau} = \frac{1}{\tau} \left[ \int \kappa_1(x,y) \left( \frac{\partial F(N_k^{r+\tau}(y), Z_k^{r+\tau}(y))}{\partial N} \psi_{k}^{N^{r+\tau}}(y) 
+ \frac{\partial F(N_k^{r+\tau}(y), Z_k^{r+\tau}(y))}{\partial Z} \psi_{k}^{Z^{r+\tau}}(y) \right) dy 
- \int \kappa_1(x,y) \left( \frac{\partial F(N_k^{r}(y), Z_k^{r}(y))}{\partial N} \psi_{k}^{N^r}(y) + \frac{\partial F(N_k^{r}(y), Z_k^{r}(y))}{\partial Z} \psi_{k}^{Z^r}(y) \right) dy \right] 
= \frac{1}{\tau} \int \kappa_1(x,y) \left\{ \left[ \frac{\partial F(N_k^{r+\tau}(y), Z_k^{r+\tau}(y))}{\partial N} - \frac{\partial F(N_k^{r}(y), Z_k^{r}(y))}{\partial N} \right] \psi_{k}^{N^{r+\tau}}(y) 
+ \frac{\partial F(N_k^{r}(y), Z_k^{r}(y))}{\partial N} \left( \psi_{k}^{N^{r+\tau}}(y) - \psi_{k}^{N^r}(y) \right) \right\} dy 
+ \left[ \left( \frac{\partial F(N_k^{r+\tau}(y), Z_k^{r+\tau}(y))}{\partial Z} - \frac{\partial F(N_k^{r}(y), Z_k^{r}(y))}{\partial Z} \right) \psi_{k}^{Z^{r+\tau}}(y) 
+ \frac{\partial F(N_k^{r}(y), Z_k^{r}(y))}{\partial Z} \left( \psi_{k}^{Z^{r+\tau}}(y) - \psi_{k}^{Z^r}(y) \right) \right\} dy. \]

We will use \( \sigma_{k}^{N^r} \) and \( \sigma_{k}^{Z^r} \) to represent the limit of the difference quotients for directional derivative of \( \psi_{k}^{N^r} \) and \( \psi_{k}^{Z^r} \) with respect \( (u_k + \varepsilon(l_k - u_k)) \) in the direction \( l_k - u_k \). From estimates on these difference quotients and using similar ideas as used in Theorem 2, we have the existence of \( \sigma_{k}^{N^r}, \sigma_{k}^{Z^r} \in (L^\infty(\Omega))^{T+1} \) such that for \( k = 0, 1, ..., T \):
If we take the limit in both sides of (8) as $\tau \to 0$, we obtain

$$
\lim_{\tau \to 0} \frac{\psi^{N^\ast \tau}_{k+1}(x) - \psi^{N^\ast}_{k+1}(x)}{\tau} = \int_{\Omega} k_1(x, y) \left[ \frac{\partial^2 F(N_k^\ast(y), Z_k^\ast(y))}{\partial N^2} \left( \psi^{N^\ast}_{k}(y) \right)^2 + 2 \frac{\partial^2 F(N_k^\ast(y), Z_k^\ast(y))}{\partial N \partial Z} \psi^{Z^\ast}_{k}(y) \psi^{N^\ast}_{k}(y) + \frac{\partial F(N_k^\ast(y), Z_k^\ast(y))}{\partial N} \sigma^{N^\ast}_{k}(y) \right] dy.
$$

Following the same ideas for $\frac{\psi^{Z^\ast \tau}_{k+1}(x) - \psi^{Z^\ast}_{k+1}(x)}{\tau}$, we obtain:

$$
\sigma^{N^\ast}_{k}(x) = \int_{\Omega} k_1(x, y) \left[ \frac{\partial^2 G(N_k^\ast(y), Z_k^\ast(y))}{\partial N^2} \left( \psi^{N^\ast}_{k}(y) \right)^2 + 2 \frac{\partial^2 G(N_k^\ast(y), Z_k^\ast(y))}{\partial N \partial Z} \psi^{N^\ast}_{k}(y) \psi^{N^\ast}_{k}(y) + \frac{\partial G(N_k^\ast(y), Z_k^\ast(y))}{\partial N} \sigma^{N^\ast}_{k}(y) \right] dy
$$

$$
\sigma^{Z^\ast}_{k}(x) = \int_{\Omega} k_2(x, y) \left[ \frac{\partial^2 G(N_k^\ast(y), Z_k^\ast(y))}{\partial N^2} \left( \psi^{N^\ast}_{k}(y) \right)^2 + 2 \frac{\partial^2 G(N_k^\ast(y), Z_k^\ast(y))}{\partial N \partial Z} \psi^{N^\ast}_{k}(y) \psi^{N^\ast}_{k}(y) + \frac{\partial G(N_k^\ast(y), Z_k^\ast(y))}{\partial N} \sigma^{N^\ast}_{k}(y) \right] dy,
$$

with $\sigma^{N^\ast}_{0} = \sigma^{Z^\ast}_{0} = 0$.

We now use an iterative method to estimate $\sigma^{N^\ast}_{k}$ and $\sigma^{Z^\ast}_{k}$ in terms of the components of $(I - u)^{2}$.

Using $\psi^{N^\ast}_{0} = \psi^{N^\ast}_{1} = \sigma^{N^\ast}_{0} = \sigma^{N^\ast}_{1} = 0$, we can show that $\sigma^{N^\ast}_{1}(x) = \sigma^{N^\ast}_{1}(x) = 0$.

Using these results and $\psi^{N^\ast}_{1} = 0$ we can obtain:

$$
|\sigma^{N^\ast}_{2}(x)| \leq \int_{\Omega} k_1(x, y) \left[ \frac{\partial^2 F(N_k^\ast(y), Z_k^\ast(y))}{\partial N^2} \left( \psi^{N^\ast}_{1}(y) \right)^2 + 2 \frac{\partial^2 F(N_k^\ast(y), Z_k^\ast(y))}{\partial N \partial Z} \psi^{N^\ast}_{1}(y) \psi^{N^\ast}_{1}(y) + \frac{\partial F(N_k^\ast(y), Z_k^\ast(y))}{\partial N} \sigma^{N^\ast}_{1}(y) \right] dy.
$$
\[
\begin{align*}
&+ \frac{\partial^2 F(N_1^c(y), Z_1^c(y))}{\partial Z^2} \left( \psi_1^{z^*}(y) \right)^2 + \frac{\partial F(N_1^c(y), Z_1^c(y))}{\partial Z} \sigma_1^{z^*}(y) \right) dy \\
\leq & \left| \int_{\Omega} k_1(x, y) \left( \frac{\partial^2 F(N_1^c(y), Z_1^c(y))}{\partial Z^2} \right) \left( \psi_1^{z^*}(y) \right)^2 \right| dy \\
\leq & L_1 \int_{\Omega} (l_0 - u_0)^2(y) dy \\
\left| \sigma_2^{z^*}(x) \right| = & \left| \int_{\Omega} k_2(x, y) \left[ \frac{\partial^2 G(N_1^c(y), Z_1^c(y))}{\partial N^2} \right] \left( \psi_1^{z^*}(y) \right)^2 \\
& + 2 \frac{\partial^2 G(N_1^c(y), Z_1^c(y))}{\partial N \partial Z} \psi_1^{z^*}(y) \psi_1^{z^*_c}(y) + \frac{\partial G(N_1^c(y), Z_1^c(y))}{\partial N} \sigma_1^{z^*_c}(y) \\
& + \frac{\partial^2 G(N_1^c(y), Z_1^c(y))}{\partial Z^2} \left( \psi_1^{z^*_c}(y) \right)^2 + \frac{\partial G(N_1^c(y), Z_1^c(y))}{\partial Z} \sigma_1^{z^*_c}(y) \right) dy \\
\leq & \left| \int_{\Omega} k_2(x, y) \left( \frac{\partial^2 G(N_1^c(y), Z_1^c(y))}{\partial Z^2} \right) \left( \psi_1^{z^*_c}(y) \right)^2 \right| dy \\
\leq & M_1 \int_{\Omega} (l_0 - u_0)^2(y) dy,
\end{align*}
\]

Next, we focus on \( \sigma_3^{z^*_c}(x) \):
\[
\begin{align*}
\left| \sigma_3^{z^*_c}(x) \right| \\
\leq & \left| \int_{\Omega} k_1(x, y) \frac{\partial^2 F(N_2^c(y), Z_2^c(y))}{\partial N^2} \left( \psi_2^{z^*_c}(y) \right)^2 \right| dy \\
& + 2 \left| \int_{\Omega} k_1(x, y) \frac{\partial^2 F(N_2^c(y), Z_2^c(y))}{\partial N \partial Z} \psi_2^{z^*_c}(y) \psi_2^{z^*_c}(y) \right| dy \\
& + \left| \int_{\Omega} k_1(x, y) \frac{\partial F(N_2^c(y), Z_2^c(y))}{\partial N} \sigma_2^{z^*_c}(y) \right| dy \\
& + \left| \int_{\Omega} k_1(x, y) \frac{\partial^2 F(N_2^c(y), Z_2^c(y))}{\partial Z^2} \left( \psi_2^{z^*_c}(y) \right)^2 \right| dy \\
& + \left| \int_{\Omega} k_1(x, y) \frac{\partial F(N_2^c(y), Z_2^c(y))}{\partial Z} \sigma_2^{z^*_c}(y) \right| dy \\
\leq & L_{21} \left\{ \int_{\Omega} (l_0 - u_0)^2(y) dy \\
& + \left[ \int_{\Omega} (l_0 - u_0)(y) dy + \int_{\Omega} (l_1 - u_1)(y) dy \right] \int_{\Omega} (l_0 - u_0)(y) dy \\
& + \left[ \int_{\Omega} (l_0 - u_0)(y) dy + \int_{\Omega} (l_1 - u_1)(y) dy \right]^2 + \int_{\Omega} (l_1 - u_1)^2(y) dy \right\} \\
\leq & L_2 \left[ \int_{\Omega} (l_0 - u_0)^2(y) dy + \int_{\Omega} (l_1 - u_1)^2(y) dy \right],
\end{align*}
\]
where the last step is justified by the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$. Using the same ideas for $\sigma_z^e$, we have:

$$\left| \sigma_z^e(x) \right| \leq M_2 \left[ \int_{\Omega} (l_0 - u_0)^2(y)dy + \int_{\Omega} (l_1 - u_1)^2(y)dy \right]$$

Then by iteration, we obtain

$$\left| \sigma_{k+1}^N(x) \right| \leq L_k \sum_{j=0}^{k} \int_{\Omega} (l_j - u_j)^2(y)dy$$

$$\left| \sigma_{k+1}^Z(x) \right| \leq M_k \sum_{j=0}^{k} \int_{\Omega} (l_j - u_j)^2(y)dy$$

where the constants, $L_k$ and $M_k$, do not depend on $\varepsilon$ or $\tau$. Therefore

$$\sigma_{k+1}^N(x) \geq - L_k \sum_{j=0}^{k} \int_{\Omega} (l_j - u_j)^2(y)dy$$

$$\sigma_{k+1}^Z(x) \geq - M_k \sum_{j=0}^{k} \int_{\Omega} (l_j - u_j)^2(y)dy.$$
which gives the desire concavity for $C_k$'s sufficiently large. Thus the uniqueness of the optimal control has been shown.

5. **Application to biocontrol of a gypsy moth population.** The gypsy moth, *Lymantria dispar*, is perhaps the most destructive forest defoliator in North America [59, 63]. Since its introduction to the USA near Boston in the 1860’s, it has spread as far as Virginia and North Carolina to the south, Indiana and Illinois to the west, and Wisconsin to the Midwest [58, 59]. Like many insect species, gypsy moth population dynamics are characterized by rapid population increases maintained for a couple of years which decline less rapidly to low population levels which are then maintained for several years [63, 15]. The reasons for such outbreak behavior are still unclear although there is evidence in the case of the gypsy moth that pathogens may play some role in reducing population levels from outbreak. In outbreaks, populations achieve very high levels but eventually decrease after one to three years, generally as a consequence of disease epizootics. In North America, of the pathogens affecting gypsy moth, *Lymantria dispar* nucleopolyhedrosis virus (*Ld*MNPV) is the most common, and it only infects gypsy moths [63, 9].

Despite large variation in population densities, the dynamics of the moth population remain highly synchronized with an annual generational structure. Adult moths produce eggs in mid-summer; the eggs remain dormant over winter after which larvae emerge during the spring. The larval stage lasts between 4-6 weeks and after a short pupal phase (around 2 weeks), the adult moth emerges. It is the larvae which defoliate trees and create the environmental and economic problems; they are highly polyphagous insects, feeding on over 300 tree species [63].

Since the female gypsy moth cannot fly, most eggs are found within a couple of meters of where the adult emerged. Dispersal of early stage larvae can occur via ballooning and atmospheric transport but again this is rather short range, with a range up to a few hundred meters. Long range dispersal also occurs through human transportation, particularly of over-wintering egg masses [63, 37]. Combining these elements, overall gypsy moth spread can be described by stratified dispersal. Gypsy moth outbreaks are commonly synchronized over regions covering several hundred kilometers [59, 63].

During the 1950’s, the USDA Forest Service decided to exploit the naturally occurring nucleopolyhedrosis virus (*Ld*MNPV) as a biocontrol agent for the gypsy moth [52, 55]. The reasons for this were that (i) the *Ld*MNPV virus has an extremely narrow host range [3]; (ii) it does not have deleterious effects on beneficial insects or vertebrates, [21, 54, 55]; and (iii) it is not known to be related to any human pathogen [1]. In 1978 the *Ld*MNPV virus product, Gypchek, was registered by the U.S. Environmental Protection Agency as an insecticide for the gypsy moth [52] and in this work we assume biocontrol by Gypchek.

For previous work on discrete time models for gypsy moth population without spatial features, see [12, 10, 11, 5, 23].

In model (1) we let $N_k(x)$ and $Z_k(x)$ represent the density of gypsy moth and virus populations respectively. To exploit the results from earlier sections, we must choose growth functions and spatial kernels for (1) which satisfy the assumptions given in Section 3. We use a simple modification of the Nicholson-Bailey model.

In the absence of the virus, gypsy moth dynamics are density dependent and can
be represented by the Ricker equation, where $\gamma$ denotes the average per capita number of moths produced and $e^{-rN_k}$ is the density dependent probability that each new moth will survive until the next generation. The virus naturally occurs, and our control will add more. The virus adds to the constraints on between generation survival for the gypsy moths; we assume that in the presence of virus the probability that a moth does not become infected at generation $k$ is $e^{-bZ_k}$. For the virus, we assume a fraction $f$ of virus in the environment can survive between moth generations (over winter survival). Following virus infection, gypsy moths will die which results in $\rho$ viral spores, on average, being released into the environment. Thus our corresponding growth functions are:

$$ F(N_k, Z_k) = \gamma N_k \exp (-rN_k - bZ_k) $$

$$ G(N_k, Z_k) = fZ_k + \rho N_k \left( 1 - \exp(-bZ_k) \right), $$

with initial conditions $N_0$ and $Z_0$ given.

We assume that the dispersal kernels for the gypsy moth, the virus and the control are the same, but this assumption can be easily changed. We choose Laplace kernels to represent stratified dispersal since they involve fat tails, which can represent the long distance dispersal,

$$ k_i(x) = \frac{\beta_i}{2} e^{-\beta_i |x|} \quad \beta_i > 0, \ i = 1, 2, 3 $$

Combining all of the modeling assumptions described above we are able to present the mathematical formulation for our control problem. The objective functional is defined as

$$ J(u) = \int_{\Omega} A_T N_T(x) dx + \sum_{k=0}^{T-1} \int_{\Omega} \left[ A_k N_k(x) + B_k u_k(x) + C_k u_k^2(x) \right] dx $$

and this is subject to the constrains of the population dynamics

$$ N_{k+1}(x) = \int_{\Omega} k_1(x, y) F(N_k(y), Z_k(y)) dy $$

$$ Z_{k+1}(x) = \int_{\Omega} k_2(x, y) G(N_k(y), Z_k(y)) dy + \int_{\Omega} k_3(x, y) u_k(y) dy. $$

The control $u$ is the amount of spray of the biocontrol agent Gypchek used at location $x$ for each generation, with control set $U$ in (2). We seek an optimal control $u^* \in U$ such that $J(u^*) = \min_{u \in U} J(u)$. From work in sections 2 - 4, we have the existence, characterization and uniqueness of an optimal control $u^*$.

We use the Forward-Backward sweep numerical method to approximate optimal control solutions and corresponding state solutions [35]. To find the optimal control we use an iterative process, using initial conditions for the states and final time conditions for the adjoints. Starting with the initial conditions of the state and a guess for the control, in our case half of the maximum control, we solve the state equations forward in time. Using these new values of the states, we solve the adjoint equations backwards. The adjoint values are used in the control characterization to calculate a new control value. The control is updated by taking a convex combination of the old value and the value calculated from the characterization (since the convex combination speeds up convergence). The new estimate of the states and the optimal control are compared with those from the previous iteration. We
use a tolerance of 0.1% and when relative errors in the states and control are below
the tolerance, the iteration stops [35]. The trapezoidal rule is used to get integral
approximations. We found that our numerical simulations always converged in less
than 30 iterations. We found no indications of any non-uniqueness in the optimal
control calculations.

We will display the optimal control results in the following way. The left part
(a) will present the results without control and the part right (b) the ones with
control. For both (a) and (b), we present in the top the density of the control,
followed by the density of gypsy moth and the density of the virus. To estimate the
scales in our figures, one gypsy moth unit represents $10^4$ larvae/hectare and one
virus unit represents $10^{10}$ particles/hectare. We visualize our spatial length going
along one side of area measured in hectares, and thus 1/10 unit of space represents
100 meter (one side of hectare). We start with constant initial conditions for both
gypsy moth and the virus. Some of the parameters ($\gamma$, $\rho$, $f$) are taken from previous
work [57, 19, 69], while others ($r$, $b$) are adjusted so that model solutions exhibit the
characteristic oscillatory behavior of gypsy moth populations. In Figure 1 we can
observe two interesting results when the initial spatial distributions for both moth
and virus are constant. Firstly, the second peak in gypsy moth density in Figure 1(b)
is considerable smaller than the equivalent peak in Figure 1(a). Secondly, the timing
and levels of applied control indicate a temporal intervention that may seem counter
intuitive. Investigating the top graph of Figure 1(b) we observe the control should
be applied at high levels initially and should have a second peak at around year 6
when both the virus and gypsy moth are at low densities. Controlling gypsy moth
at those times, does not eliminate the outbreak but the densities at the outbreak
are considerable lower.

Next we explore the impact of initial spatial distributions for both the moth
and virus on the optimal control of gypsy moths. The alternative distributions are
presented in Figure 2. Output from the numerical simulations also include part (c)
which is simply a rotation of the control graph presented in part (b). In the absence
of external control, spatially varying initial conditions result in spatially varying
moth-pathogen interactions such that moth populations are regulated in regions
where the pathogen can be found but outside of those regions gypsy moth levels
increase (see (a) in each of the figures). This gives rise to a bimodal distribution of
both pathogen and moth. Inclusion of control follows the pattern described above
for spatially homogeneous initial conditions. In particular, inclusion of the control
acts to reduce the moth population; moreover the optimal control is to apply large
amounts of control initially and then in advance of the second outbreak peak.

In both cases where the initial distribution of moth and virus is in the center of
the spatial domain (Figure 2(a) and (b)) we obtain these outcomes. Although moth
distributions become more homogeneous over time, this is the case both with and
without external control strategies. In particular, for the parameters used here, the
effects of the initial conditions are transient and after about 10 years, the densities
of gypsy moth and the virus become less heterogeneous through the space (see
Figures 3(a); 4(a)). When the species are aggregated in the border of the space, as
in Figure 2(c), the control strategy and its impact on gypsy moth levels remain as
before. However, the effects of initial conditions last longer as shown in Figure 5(a),
we still observe some aggregation effects around year 14.
6. Conclusion and future work. Optimal control theory for integrodifference equations is a relatively new research area. Several applications of optimal control in a single species model have been developed. In this paper, we present a new application of optimal control theory for a system of integrodifference equations. We formulated a two species coupled integrodifference system and its associated objective functional for the management of a pest population and its pathogen control agent.

Using appropriate analysis and control techniques, we prove the existence of the optimal control for our system of integrodifference equations. We also developed the characterization of the optimal control and proved the existence of the sensitivity system and adjoint system. Finally we showed uniqueness of the optimal control, under a concavity assumption on the objective functional. We apply the theory developed to the invasive pest gypsy moth. Our results could be used to develop...
a spatio-temporal control strategy. Most notable, the optimal strategy suggests that the biocontrol agent should be applied while gypsy moth densities were pre-outbreak.

Applying the control does not eliminate subsequent outbreaks but it does reduce the density of gypsy moth at outbreaks. This result is consistent over several different spatial initial conditions. Implementation of such a programme is likely to be challenging since it requires investment in resources at a time when there appears to be no real problem with the moth population. This result highlights the importance of collaboration between the different groups interested in pest control to convince policy makers of the importance of preventative interactions.

The theory developed can be used for implementing optimal control problems for many integrodifference systems. Although we focus on two integrodifference
equations, the theory is valid and easily extended for several equations. Our results could assist in developing optimal control for integrodifference equations for a variety of applications ranging from dispersal of invasive species to dispersal of species as response to climate change. In the particular case of the invasion of gypsy moth, future work could explore the impact of population dynamics on the control strategy, particularly those which consider external impact such as predation or intra-specific nonlinearities such as Allee effects.

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Figure 5. Effect of the optimal control over a period of 15 years with aggregate spatial initial conditions in Figure 2(c). a. Dynamics without control. b. Dynamics with control. c. Different view of the control display in b

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