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Notation and basic definitions

1.1 Notation

If not indicated otherwise, the symbols throughout the book are used in the following manner:

1.1.1 Symbols

The symbols const, c, c_i denote generic real constants. Their concrete value may change at different parts of the text.

The symbols $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of integers, and complex numbers, respectively. The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^N$ is the N-dimensional Euclidean space.

1.1.2 Euclidean space

The symbol $\Omega \subset \mathbb{R}^N$ stands for a *domain* - an open connected subset of the Euclidean space $\mathbb{R}^N$. The closure of a set $Q \subset \mathbb{R}^N$ is denoted by $\overline{Q}$, its boundary is ∂Q. By the symbol 1_Q we denote the characteristic function of the set Q. The outer normal vector to ∂Q is usually denoted by $\boldsymbol{n}$. The symbol $B(a;r)$ denotes an (open) ball in $\mathbb{R}^N$ of center $a \in R^N$ and radius $r > 0$.

We also write $B_X(a;r)$ to indicate a ball with respect to a more general norm X.

Vectors and vector valued functions ranging in an Euclidean space are denoted by symbols beginning by a boldface minuscule, for example $\boldsymbol{u}$, $\boldsymbol{v}$. Matrices (tensors) and matrix valued functions are represented by special Roman characters as $\mathbb{S}$, $\mathbb{T}$, in particular, the identity matrix is denoted by $\mathbb{I} = \{\delta_{i,j}\}_{i,j=1}^N$. The symbol $\mathbb{I}$ may also be used to denote the identity operator in a general setting.

The transpose of a square matrix $\mathbb{A} = \{a_{i,j}\}_{i,j=1}^N$ is $\mathbb{A}^T = \{a_{j,i}\}_{i,j=1}^N$. The trace of a square matrix $\mathbb{A} = \{a_{i,j}\}_{i,j=1}^N$ is $\mathrm{trace}[\mathbb{A}] = \sum_{i=1}^N a_{i,i}$.

The scalar product of vectors $\boldsymbol{a} = [a_1, \dots, a_N]$, $\boldsymbol{b} = [b_1, \dots, b_N]$ is denoted by

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i,$$

the scalar product of tensors $\mathbb{A} = \{A_{i,j}\}_{i,j=1}^N$, $\mathbb{B} = \{B_{i,j}\}_{i,j=1}^N$ reads

$$\mathbb{A} : \mathbb{B} = \sum_{i,j=1}^N A_{i,j} B_{j,i}.$$

The symbol $\boldsymbol{a} \otimes \boldsymbol{b}$ denotes the tensor product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$, specifically,

$$\boldsymbol{a} \otimes \boldsymbol{b} = \{\boldsymbol{a} \otimes \boldsymbol{b}\}_{i,j} = a_i b_j.$$

The vector product $\boldsymbol{a} \times \boldsymbol{b}$ is the antisymmetric part of $\boldsymbol{a} \otimes \boldsymbol{b}$. If $N = 3$, the vector product of vectors $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$ is identified with a vector

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

The product of a matrix $\mathbb{A}$ with a vector $\boldsymbol{b}$ is a vector $\mathbb{A}\boldsymbol{b}$ whose components are

$$[\mathbb{A}\boldsymbol{b}]_i = \sum_{j=1}^N A_{i,j} b_j \text{ for } i = 1, \dots, N,$$

while the product of a matrix $\mathbb{A} = \{A_{i,j}\}_{i,j=1}^{N,M}$ and a matrix $\mathbb{B} = \{B_{i,j}\}_{i,j=1}^{M,S}$ is a matrix $\mathbb{A}\mathbb{B}$ with components

$$[\mathbb{A}\mathbb{B}]_{i,j} = \sum_{k=1}^M A_{i,k} B_{k,j}.$$

1.1.3 Norms

The norm in a Banach space X is denoted by the symbol $\|\cdot\|_X$. The Euclidean norm of a vector $\boldsymbol{a} \in \mathbb{R}^N$ is denoted by

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{\sum_{i=1}^N a_i^2}.$$

The distance of a vector $\boldsymbol{a}$ to a set $K \subset \mathbb{R}^N$ is denoted as

$$\mathrm{dist}[\boldsymbol{a}, K] = \inf\{|\boldsymbol{a} - \boldsymbol{k}| \mid \boldsymbol{k} \in K\},$$

the semidistance of sets A, B is

$$\mathrm{dist}[A, B] = \sup\{\mathrm{dist}[x, B] \mid x \in A\}.$$

We occasionally use the symbol $\mathrm{dist}_X[A, B]$ to indicate the metric used.

The diameter of K is

$$\mathrm{diam}[K] = \sup_{(x,y)\in K^2} |x - y|.$$

The closure of K with respect to the topology X is denoted by $\mathrm{cl}_X(K)$, the Lebesgue measure of a set Q is $|Q|$ or $\mathrm{vol}\, Q$.

1.1.4 Differential operators

The symbol

$$\partial_{y_i} g(y) \equiv \frac{\partial g}{\partial y_i}(y),\ i = 1, \ldots, N,$$

denotes the partial derivative of a function $g = g(y)$, $y = [y_1, \ldots, y_N]$, with respect to the variable y_i evaluated at a point $y \in \mathbb{R}^N$. The same notation is used for the generalized (distributional) derivatives introduced below. In most cases, we consider functions $g = g(t, x)$ of the *time variable* $t \in (0, T)$ and the *spatial coordinate* $x = [x_1, x_2, x_3] \in \Omega \subset \mathbb{R}^3$. We use italics rather than boldface minuscules to denote the independent variables although they may be vectors as the case may be.

Occasionally, we write d/dt to indicate the derivative with respect to the variable t; it is also often used if $u = u(t, x)$ is considered as a Bochner function, namely a function $t \mapsto u(t, \cdot)$ with values in suitable function space, thus suppressing the variable x.

The *gradient* of a scalar function $g = g(t, x)$ with respect to the spatial variable x is a vector

$$\nabla_x g(t, x) = [\partial_{x_1} g(t, x), \partial_{x_2} g(t, x), \partial_{x_3} g(t, x)].$$

The *gradient* of a vector function $\boldsymbol{v} = [v_1(t, x), v_2(t, x), v_3(t, x)]$ with respect to the space variables x is the matrix

$$\nabla_x \boldsymbol{v}(t, x) = \{\partial_{x_j} v_i(t, x)\}_{i,j=1}^3;$$

The *divergence* of a vector valued function depending on the spatial and time variables $\boldsymbol{v} = [v_1(t,x), v_2(t,x), v_3(t,x)]$ with respect to the space variable x is a scalar

$$\mathrm{div}_x \boldsymbol{v}(t,x) = \sum_{i=1}^{3} \partial_{x_i} v_i(t,x).$$

The *divergence* of a tensor (matrix-valued) function $\mathbb{B} = \{B_{i,j}(t,x)\}_{i,j=1}^{3}$ with respect to the space variable x is a vector

$$[\mathrm{div}\mathbb{B}]_i = [\mathrm{div}_x \mathbb{B}(t,x)]_i = \sum_{j=1}^{3} \partial_{x_j} B_{i,j}(t,x), \; i = 1, \dots, 3.$$

The vorticity of a vectorial function $\boldsymbol{v} = [v_1(t,x), \dots, v_3(t,x)]$ is an antisymmetric matrix

$$\mathbf{curl}_x \boldsymbol{v} = \nabla_x \boldsymbol{v} - \nabla_x^T \boldsymbol{v} = \left\{ \partial_{x_j} v_i - \partial_{x_i} v_j \right\}_{i,j=1}^{3}.$$

The vorticity operator in $\mathbb{R}^3$ may be interpreted as a vector $\mathbf{curl}\, \boldsymbol{v} = \nabla_x \times \boldsymbol{v}$.

The symbol $\Delta = \Delta_x$ denotes the *Laplace operator*,

$$\Delta_x = \mathrm{div}_x \nabla_x.$$

1.2 Measures and distributions

If not otherwise stated, all function spaces considered in what follows are real. The duality pairing between an abstract vector space X and its dual X^* is denoted as $\langle \cdot ; \cdot \rangle_{X^*;X}$, or simply $\langle \cdot ; \cdot \rangle$. In particular, if X is a Hilbert space, the symbol $\langle \cdot ; \cdot \rangle$ denotes the scalar product in X.

1.2.1 Continuously differentiable functions

For $Q \subset \mathbb{R}^N$, the symbol $C(Q)$ denotes the set of all continuous functions on Q. In Q is bounded, the symbol $C(\overline{Q})$ denotes the Banach space of functions continuous on the closure $\overline{Q}$ endowed with norm

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} |g(y)|.$$

Similarly, $C(\overline{Q}; X)$ denotes the Banach space of vectorial functions continuous in $\overline{Q}$ and ranging in a Banach space X with norm

$$\|g\|_{C(\overline{Q})} = \sup_{y \in \overline{Q}} \|g(y)\|_X.$$

The symbol $C_{\rm weak}(\overline{Q}; X)$ denotes the space of all vector-valued functions on $\overline{Q}$ ranging in a Banach space X continuous with respect to the weak topology. More specifically, $g \in C_{\rm weak}(\overline{Q}; X)$ if the mapping $y \mapsto \|g(y)\|_X$ is bounded and

$$y \mapsto \langle f; g(y) \rangle_{X^*;X}$$

is continuous on $\overline{Q}$ for any linear form f belonging to the dual space X^*.

We say that $g_n \to g$ in $C_{\rm weak}(\overline{Q}; X)$ if

$$\langle f; g_n \rangle_{X^*;X} \to \langle f; g \rangle_{X^*;X} \text{ in } C(\overline{Q}) \text{ for all } f \in X^*.$$

The symbol $C^k(\overline{Q})$, $Q \subset \mathbb{R}^N$, where k is a non-negative integer, denotes the space of functions on $\overline{Q}$ that are restrictions of k-times continuously differentiable functions on $\mathbb{R}^N$. $C^{k,\nu}(\overline{Q})$, $\nu \in (0,1)$ is the subspace of $C^k(\overline{Q})$ of functions having their k-th derivatives ν-Hölder continuous in $\overline{Q}$. $C^{k,1}(\overline{Q})$ is a subspace of $C^k(\overline{Q})$ of functions whose k-th derivatives are Lipschitz on $\overline{Q}$. For a bounded domain Q, the spaces $C^k(\overline{Q})$ and $C^{k,\nu}(\overline{Q})$, $\nu \in (0,1]$ are Banach spaces with norms

$$\|u\|_{C^k(\overline{Q})} = \max_{|\alpha| \le k} \sup_{x \in Q} |\partial^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\nu}(\overline{Q})} = \|u\|_{C^k(\overline{Q})} + \max_{|\alpha|=k} \sup_{(x,y) \in Q^2,\, x \neq y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\nu},$$

where $\partial^\alpha u$ stands for the partial derivative $\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N} u$ of order $|\alpha| = \sum_{i=1}^N \alpha_i$. The spaces $C^{k,\nu}(\overline{Q}; \mathbb{R}^M)$ are defined in a similar way. Finally, we set $C^\infty = \cap_{k=0}^\infty C^k$.

1.2.2 Compactness of sets of continuous functions

Arzelà-Ascoli Theorem:

Theorem 1.1 *Let $Q \subset \mathbb{R}^M$ be a compact set and X a compact topological metric space endowed with a metric d_X. Let $\{v_n\}_{n=1}^\infty$ be a sequence of functions in $C(Q; X)$ that is equi-continuous, meaning, for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$d_X \Big[v_n(y), v_n(z) \Big] \le \varepsilon \text{ provided } |y - z| < \delta \text{ independently of } n = 1, 2, \ldots$$

Then $\{v_n\}_{n=1}^{\infty}$ is precompact in $C(Q;X)$, that is, there exists a subsequence (not relabeled) and a function $v \in C(Q;X)$ such that

$$\sup_{y\in Q} d_X \Big[v_n(y), v(y) \Big] \to 0 \text{ as } n \to \infty.$$

Proof: See Kelley [96, Chapter 7, Theorem 17].

□

1.2.3 Measures

The symbol $C_c(Q)$ denotes the space of continuous functions with compact support in a locally compact Hausdorff metric space Q. The symbol $\mathcal{M}(Q)$ stands for the space of signed Borel measures on Q. The symbol $\mathcal{M}^+(Q)$ denotes the cone of non-negative Borel measures on Q. A measure $\nu \in \mathcal{M}^+(Q)$ such that $\nu[Q] = 1$ is called *probability measure*. Lebesgue measure of a set $A \subset \mathbb{R}^N$ is denoted $|A|$ or $\text{vol}\, A$.

Riesz Representation Theorem:

Theorem 1.2 *Let Q be a locally compact Hausdorff metric space. Let f be a non-negative linear functional defined on the space $C_c(Q)$.*

Then there exist a $\sigma-$algebra of measurable sets containing all Borel sets and a unique non-negative measure on $\mu_f \in \mathcal{M}^+(Q)$ such that

$$\langle f; g \rangle = \int_Q g \, \mathrm{d}\mu_f \text{ for any } g \in C_c(Q). \tag{1.1}$$

Moreover, the measure μ_f enjoys the following properties:

- *$\mu_f[K] < \infty$ for any compact $K \subset Q$.*
- $$\mu_f[E] = \sup \{\mu_f[K] \mid K \subset E, \ K \text{ compact}\}$$
  for any open set $E \subset Q$.
- $$\mu_f[V] = \inf \{\mu(E) \mid V \subset E, \ E \text{ open}\}$$
  for any Borel set V.
- *If E is μ_f measurable, $\mu_f(E) = 0$, and $A \subset E$, then A is μ_f measurable.*

Proof: See Rudin [140, Chapter 2, Theorem 2.14].

□

Corollary 1.1 *Assume that* $f : C_c^\infty(Q) \to \mathbb{R}$ *is a linear and non-negative functional, where* Q *is a domain in* $\mathbb{R}^N$*.*

Then there exists a measure μ_f *enjoying the same properties as in Theorem 1.2 such that* f *is represented through (1.1).*

Proof: See Evans and Gariepy [61, Chapter 1.8, Corollary 1].

□

1.2.4 Distributions

The symbol $C_c^k(Q;\mathbb{R}^M)$, $k \in \{0,1,\dots,\infty\}$ denotes the vector space of functions belonging to $C^k(Q;\mathbb{R}^M)$ and having compact support in Q. The space $C_c^\infty(Q;\mathbb{R}^M)$ may be endowed with the topology induced by the convergence:

$$\varphi_n \to \varphi \in C_c^\infty(Q)$$

if

$$\begin{aligned} &\mathrm{supp}[\varphi_n] \subset K,\ K \subset Q \text{ a compact set,} \\ &\varphi_n \to \varphi \text{ in } C^k(K) \text{ for any } k = 0,1,\dots \end{aligned} \tag{1.2}$$

We write $C_c^\infty(Q)$ instead of $C_c^\infty(Q;\mathbb{R})$.

The dual space $[C_c^\infty(Q;\mathbb{R}^M)]^*$ is the space of *distributions* on Ω with values in $\mathbb{R}^M$. Continuity of a linear form belonging to this space is understood with respect to the convergence introduced in (1.2).

A differential operator ∂^α of order $|\alpha|$ can be identified with a distribution

$$\langle \partial^\alpha v; \varphi \rangle = (-1)^{|\alpha|} \int_Q v \partial^\alpha \varphi \ \mathrm{d}y$$

for any locally integrable function v.

1.3 Bochner and Sobolev spaces

1.3.1 Lebesgue (Bochner) spaces

The *Lebesgue spaces* $L^p(Q;X)$ are spaces of (Bochner) measurable functions v ranging in a Banach space X such that the norm

$$\|v\|^p_{L^p(Q;X)} = \int_Q \|v\|_X^p \ \mathrm{d}y \text{ is finite, } 1 \le p < \infty.$$

Similarly, $v \in L^\infty(Q;X)$ if v is (Bochner) measurable and

$$\|v\|_{L^\infty(Q;X)} = \operatorname*{ess\,sup}_{y \in Q} \|v(y)\|_X < \infty.$$

The symbol $L^p_{\text{loc}}(Q;X)$ denotes the vector space of locally L^p-integrable functions, meaning

$$v \in L^p_{\text{loc}}(Q;X) \text{ if } v \in L^p(K;X) \text{ for any compact set } K \text{ in } Q.$$

We write $L^p(Q)$ for $L^p(Q;\mathbb{R})$.

Let $f \in L^1_{loc}(Q)$ where Q is an open set. A *Lebesgue point* $a \in Q$ of f in Q is characterized by the property

$$\lim_{r \to 0+} \frac{1}{|B(a,r)|} \int_{B(a;r)} f(x)\mathrm{d}x = f(a). \tag{1.3}$$

For $f \in L^1(Q)$ the set of all Lebesgue points is of full measure, meaning its complement in Q is of zero Lebesgue measure. A similar statement holds for vector valued functions $f \in L^1(Q;X)$, where X is a Banach space (see Brezis [16]).

If $f \in C(Q)$, then identity (1.3) holds for all points a in Q.

Aubin-Lions lemma:

Theorem 1.3 *Let p, $q \in [1,\infty)$, let X, Y and Z be Banach spaces, such that $Y \subset X \subset Z$, the embedding $Y \subset Z$ being compact, and let moreover Y, Z be reflexive.*

Then the space

$$\left\{u \in L^p(0,T;Y) \mid \frac{\mathrm{d}}{\mathrm{d}t}u \in L^q(0,T;Z)\right\}$$

is compactly embedded into the space $L^p(0,T;X)$.

Proof: See Simon [149] for a general case; a constructive proof for certain special cases is also given in Section 9.5 of the Appendix.

□

Linear Functionals on $L^p(Q;X)$:

Theorem 1.4 *Let $Q \subset \mathbb{R}^N$ be a measurable set, X a Banach space that is reflexive and separable, $1 \le p < \infty$.*

Then any continuous linear form $\xi \in [L^p(Q;X)]^*$ admits a unique representation $w_\xi \in L^{p'}(Q;X^*)$,

$$\langle \xi; v \rangle_{L^{p'}(Q,X^*);L^p(Q;X)}$$

$$= \int_Q \langle w_\xi(y); v(y)\rangle_{X^*;X} \mathrm{d}y \textit{ for all } v \in L^p(Q;X),$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Moreover the norm on the dual space is given as

$$\|\xi\|_{[L^p(Q;X)]^*} = \|w_\xi\|_{L^{p'}(Q;X^*)}.$$

Accordingly, the spaces $L^p(Q;X)$ are reflexive for $1 < p < \infty$ as soon as X is reflexive and separable.

Proof: See Gajewski et al. [74, Chapter IV, Theorem 1.14, Remark 1.9]. □

Identifying ξ with w_ξ, we write

$$[L^p(Q;\mathbb{R}^N)]^* = L^{p'}(Q;\mathbb{R}^N),\ \|\xi\|_{[L^p(Q;\mathbb{R}^N)]^*} = \|\xi\|_{L^{p'}(Q;\mathbb{R}^N)},\ 1 \leq p < \infty.$$

This formula is known as *Riesz representation theorem.*

If the Banach space X in Theorem 1.4 is merely separable, we have

$$[L^p(Q;X)]^* = L^{p'}_{\text{weak}-(*)}(Q;X^*) \text{ for } 1 \leq p < \infty,$$

where

$$L^{p'}_{\text{weak}-(*)}(Q;X^*)$$
$$:= \Big\{ \xi : Q \to X^* \ \Big|\ y \in Q \mapsto \langle \xi(y); v\rangle_{X^*;X} \text{ measurable for any fixed } v \in X,$$
$$y \mapsto \|\xi(y)\|_{X^*} \in L^{p'}(Q) \Big\}$$

(see Edwards [56], Pedregal [128, Chapter 6, Theorem 6.14]).

Hölder's inequality reads

$$\|uv\|_{L^r(Q)} \leq \|u\|_{L^p(Q)}\|v\|_{L^q(Q)},\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

for any $u \in L^p(Q)$, $v \in L^q(Q)$, $Q \subset \mathbb{R}^N$ (see Adams and Fournier [1, Chapter 2]).

Interpolation inequality for L^p-spaces reads

$$\|v\|_{L^r(Q)} \leq \|v\|^{\lambda}_{L^p(Q)}\|v\|^{(1-\lambda)}_{L^q(Q)},\ \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q},\ p < r < q,\ \lambda \in (0,1)$$

for any $v \in L^p \cap L^q(Q)$, $Q \subset \mathbb{R}^N$ (see Adams and Fournier [1, Chapter 2]).

1.3.2 Sobolev spaces

The *Sobolev spaces* $W^{k,p}(Q;\mathbb{R}^M)$, $1 \leq p \leq \infty$, k a positive integer, are the spaces of functions having all distributional derivatives up to order k in $L^p(Q;\mathbb{R}^M)$. The norm in $W^{k,p}(Q;\mathbb{R}^M)$ is defined as

$$\|\boldsymbol{v}\|_{W^{k,p}(Q;\mathbb{R}^M)} = \left\{ \begin{array}{c} \left(\sum_{i=1}^{M} \sum_{|\alpha|\leq k} \|\partial^\alpha v_i\|^p_{L^p(Q)} \right)^{1/p} \text{ if } 1 \leq p < \infty \\ \\ \max_{1\leq i\leq M,\, |\alpha|\leq k} \{ \|\partial^\alpha v_i\|_{L^\infty(Q)} \} \text{ if } p = \infty \end{array} \right\},$$

where the symbol ∂^α stands for any partial derivative of order $|\alpha|$.

If Q is a bounded domain with boundary of class $C^{k-1,1}$, then there exists a continuous linear operator which maps $W^{k,p}(Q)$ to $W^{k,p}(\mathbb{R}^N)$; it is called *extension operator*. If, in addition, $1 \leq p < \infty$, then $W^{k,p}(Q)$ is separable and the space $C^k(\overline{Q})$ is its dense subspace.

By a regular domain, we will understand a bounded domain with boundary of class $C^{3,1}$; that is to say, the boundary can be covered by a finite number of graphs of C^3 functions.

The space $W^{1,\infty}(Q)$, where Q is a bounded domain, is isometrically isomorphic to the space $C^{0,1}(\overline{Q})$ of Lipschitz functions on $\overline{Q}$.

For basic properties of Sobolev functions, see Adams and Fournier [1, Chapter 2] or Ziemer [163].

1.3.3 Dual Sobolev spaces

The symbol $W^{k,p}_0(Q;\mathbb{R}^M)$ denotes the completion of $C^\infty_c(Q;\mathbb{R}^M)$ with respect to the norm $\|\cdot\|_{W^{k,p}(Q;\mathbb{R}^M)}$. In what follows, we identify $W^{0,p}(\Omega;R^N) = W^{0,p}_0(\Omega;\mathbb{R}^N)$ with $L^p(\Omega;\mathbb{R}^N)$.

We denote $\dot{L}^p(Q) = \{u \in L^p(Q) \mid \int_Q u \,\mathrm{d}y = 0\}$ and $\dot{W}^{1,p}(Q) = W^{1,p}(Q) \cap \dot{L}^p(Q)$. If $Q \subset \mathbb{R}^N$ is a bounded domain, then $\dot{L}^p(Q)$ and $\dot{W}^{1,p}(Q)$ can be viewed as closed subspaces of $L^p(Q)$ and $W^{1,p}(Q)$, respectively.

Dual Sobolev Spaces:

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $1 \leq p < \infty$. Then the dual space $[W^{k,p}_0(\Omega)]^*$ is a proper subspace of the space of distributions $\mathrm{d}'(\Omega)$. Moreover, any linear form $f \in [W^{k,p}_0(\Omega)]^*$ admits a representation*

$$\langle f; v\rangle_{[W^{k,p}_0(\Omega)]^*;W^{k,p}_0(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega (-1)^{|\alpha|} w_\alpha \, \partial^\alpha v \,\mathrm{d}x, \tag{1.4}$$

$$\text{where } w_\alpha \in L^{p'}(\Omega),\ \frac{1}{p} + \frac{1}{p'} = 1.$$

The norm of f in the dual space is given as

$$\|f\|_{[W_0^{k,p}(\Omega)]^*} = \begin{cases} \inf\left\{ \left(\sum_{|\alpha|\le k} \|w_\alpha\|^{p'}_{L^{p'}(\Omega)} \right)^{1/p'} \,\middle|\, w_\alpha \text{ satisfy (1.4)} \right\} \\ \text{for } 1 < p < \infty; \\ \inf\left\{ \max_{|\alpha|\le k}\{\|w_\alpha\|_{L^\infty(Q)}\} \,\middle|\, w_\alpha \text{ satisfy (1.4)} \right\} \\ \text{if } p = 1. \end{cases}$$

The infimum is attained in both cases.

Proof: See Adams and Fournier [1, Theorem 3.12], Maz'ya [121, Section 1.1.14].

□

The dual space to the Sobolev space $W_0^{k,p}(\Omega)$ is denoted as $W^{-k,p'}(\Omega)$. The dual to the Sobolev space $W^{k,p}(\Omega)$ admits formally the same representation formula as (1.4). However it cannot be identified as a space of distributions on Ω. A typical example is the linear form

$$\langle f; v\rangle = \int_\Omega \boldsymbol{w}_f \cdot \nabla_x v \, \mathrm{d}x, \text{ with } \mathrm{div}_x \boldsymbol{w}_f = 0$$

that vanishes on $\mathrm{d}(\Omega)$ but generates a non-zero linear form when applied to $v \in W^{1,p}(\Omega)$.

Rellich-Kondrachov Embedding Theorem:

Theorem 1.6 *Let $\Omega \subset R^N$ be a bounded Lipschitz domain.*

(i) Then, if $kp < N$ and $p \ge 1$, the space $W^{k,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any

$$1 \le q \le p^* = \frac{Np}{N - kp}.$$

Moreover, the embedding is compact if $k > 0$ and $q < p^*$.

(ii) If $kp = N$, the space $W^{k,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for any $q \in [1, \infty)$.

(iii) If $kp > N$ then $W^{k,p}(\Omega)$ is continuously embedded in $C^{k-[N/p]-1,\nu}(\overline{\Omega})$, where $[\,]$ denotes the integer part and

$$\nu = \begin{cases} [\frac{N}{p}] + 1 - \frac{N}{p} \ \textit{if} \ \frac{N}{p} \notin \mathbb{Z}, \\ \textit{arbitrary positive number in } (0,1) \ \textit{if} \ \frac{N}{p} \in \mathbb{Z}. \end{cases}$$

Moreover, the embedding is compact if $0 < \nu < [\frac{N}{p}] + 1 - \frac{N}{p}$.

Proof: See Ziemer [163, Theorem 2.5.1, Remark 2.5.2].

□

1.3.4 Embeddings for dual Sobolev spaces

The following result may be regarded as a direct consequence of Theorem 1.6.

Embedding Theorem for Dual Sobolev Spaces:

Theorem 1.7 *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain. Let* $k > 0$ *and* $q < \infty$ *satisfy*

$$q > \frac{p^*}{p^* - 1}, \ \textit{where} \ p^* = \frac{Np}{N - kp} \ \textit{if} \ kp < N,$$

$$q > 1 \ \textit{for} \ kp = N,$$

or

$$q \geq 1 \ \textit{if} \ kp > N.$$

Then the space $L^q(\Omega)$ *is compactly embedded into the space* $W^{-k,p'}(\Omega)$, $1/p + 1/p' = 1$.

1.3.5 Traces

The *Sobolev-Slobodeckii spaces* $W^{k+\beta,p}(Q)$, $1 \leq p < \infty$, $0 < \beta < 1$, $k = 0, 1, \ldots$, where Q is a domain in $\mathbb{R}^L$, L a positive integer, are Banach spaces of functions with finite norm

$$W^{k+\beta,p}(Q) = \left(\|v\|^p_{W^{k,p}(Q)} + \sum_{|\alpha|=k} \int_Q \int_Q \frac{|\partial^\alpha v(y) - \partial^\alpha v(z)|^p}{|y - z|^{L+\beta p}} \, \mathrm{d}y \, \mathrm{d}z \right)^{\frac{1}{p}},$$

see Nečas[126, Section 2.3.8] for example.

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Referring to the notation introduced in **(i)**, we say that $f \in W^{k+\beta,p}(\partial\Omega)$ if $(\varphi f) \circ (\mathbb{I}', \gamma) \in W^{k+\beta,p}(\mathbb{R}^{N-1})$ for any $\Gamma = \partial\Omega \cap B$ with B belonging to the covering $\mathcal{B}$ of $\partial\Omega$ and φ the corresponding term in the partition of unity $\mathcal{F}$. The space $W^{k+\beta,p}(\partial\Omega)$ is a Banach space endowed with a norm

$$\|v\|^p_{W^{k+\beta,p}(\partial\Omega)} = \sum_{i=1}^{M} \|(v\varphi_i)\circ(\mathbb{I}',\gamma)\|^p_{W^{k+\beta,p}(\mathbb{R}^{N-1})}.$$

In the above formulas $(\mathbb{I}',\gamma) : \mathbb{R}^{N-1} \to \mathbb{R}^N$ maps y' to $(y', \gamma(y'))$. For more details see for example Nečas [126, Section 3.8].

In the situation when $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, the Sobolev-Slobodeckii spaces admit similar embeddings as classical Sobolev spaces. The embeddings

$$W^{k+\beta,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ and } W^{k+\beta,p}(\Omega) \hookrightarrow C^s(\overline{\Omega})$$

are compact provided $0 < (k+\beta)p < N$, $1 \le q < \frac{Np}{N-(k+\beta)p}$, and $s = 0,1,\dots,k$, $(k-s+\beta)p > N$, respectively. The former embedding remains continuous (but not compact) at the border case $q = \frac{Np}{N-(k+\beta)p}$. See Adams and Fournier [1, Chapter 6].

Trace Theorem for Sobolev Spaces and Green's formula:

Theorem 1.8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain.*

Then there exists a linear operator γ_0 with the following properties:

$$[\gamma_0(v)](x) = v(x) \text{ for } x \in \partial\Omega \text{ provided } v \in C^\infty(\overline{\Omega}),$$

$$\|\gamma_0(v)\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le c\|v\|_{W^{1,p}(\Omega)} \text{ for all } v \in W^{1,p}(\Omega),$$

$$\ker[\gamma_0] = W_0^{1,p}(\Omega)$$

provided $1 < p < \infty$.

Conversely, there exists a continuous linear operator

$$\ell : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$$

such that

$$\gamma_0(\ell(v)) = v \text{ for all } v \in W^{1-\frac{1}{p},p}(\partial\Omega)$$

provided $1 < p < \infty$.

In addition, the following formula holds:

$$\int_\Omega \partial_{x_i} uv \,\mathrm{d}x + \int_\Omega u\partial_{x_i} v \,\mathrm{d}x = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v) n_i \,\mathrm{dS}_x, \; i = 1,\dots,N,$$

for any $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$, where $\boldsymbol{n}$ is the outer normal vector to the boundary $\partial\Omega$.

Proof: See Nečas [126, Theorems 5.5, 5.7].

□

The dual $[W^{1-\frac{1}{p},p}(\partial\Omega)]^*$ to the Sobolev-Slobodeckii space $W^{1-\frac{1}{p},p}(\partial\Omega)) = W^{\frac{1}{p'},p}(\partial\Omega)$ is denoted simply by $W^{-\frac{1}{p'},p'}(\partial\Omega)$.

1.3.6 Interpolation

If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then we have the interpolation inequality

$$\|v\|_{W^{\alpha,r}(\Omega)} \le c\|v\|^{\lambda}_{W^{\beta,p}(\Omega)}\|v\|^{1-\lambda}_{W^{\gamma,q}(\Omega)},\ 0 \le \lambda \le 1, \tag{1.5}$$

for

$$0 \le \alpha,\beta,\gamma \le 1,\ 1 < p,q,r < \infty,\ \alpha = \lambda\beta + (1-\lambda)\gamma,\ \frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$$

(see Sections 2.3.1, 2.4.1, 4.3.2 in Triebel [155]). We also refer to Appendix for proofs of several other interpolation, as for example Ladyzhenskaya inequality.

1.4 Fourier analysis

Let $v = v(x)$ be a complex valued function integrable on $\mathbb{R}^N$. The *Fourier transform* of v is a complex valued function $\mathcal{F}_{x\to\xi}[v]$ of the variable $\xi \in \mathbb{R}^N$ defined as

$$\mathcal{F}_{x\to\xi}[v](\xi) = \left(\frac{1}{2\pi}\right)^{N/2} \int_{\mathbb{R}^N} v(x)\exp(-\mathrm{i}\xi\cdot x)\ \mathrm{d}x. \tag{1.6}$$

Therefore, the Fourier transform $\mathcal{F}_{x\to\xi}$ can be viewed as a continuous linear mapping defined on $L^1(\mathbb{R}^N)$ with values in $L^\infty(\mathbb{R}^N)$.

For u, v complex valued square integrable functions on $\mathbb{R}^N$, we have *Parseval's identity*:

$$\int_{\mathbb{R}^N} u(x)\overline{v}(x)\ \mathrm{d}x = \int_{\mathbb{R}^N} \mathcal{F}_{x\to\xi}[u](\xi)\overline{\mathcal{F}_{x\to\xi}[v]}(\xi)\ \mathrm{d}\xi,$$

where bar denotes the complex conjugate. Parseval's identity implies that $\mathcal{F}_{x\to\xi}$ can be extended as a continuous linear mapping defined on $L^2(\mathbb{R}^N)$ with values in $L^2(\mathbb{R}^N)$. Its inverse reads

$$\mathcal{F}^{-1}_{\xi\to x}[f] = \left(\frac{1}{2\pi}\right)^{N/2} \int_{\mathbb{R}^N} f(\xi)\exp(ix\cdot\xi)\mathrm{d}\xi. \tag{1.7}$$

1.4.1 Fourier transform of tempered distributions

The symbol $\mathcal{S}(\mathbb{R}^N)$ denotes the space of smooth rapidly decreasing (complex valued) functions, specifically, $\mathcal{S}(\mathbb{R}^N)$ consists of functions u such that

$$\sup_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} (1+|x|^2)^s |\partial^\alpha u| < \infty$$

for all $s, m = 0, 1, \ldots$. We say that $u_n \to u$ in $\mathcal{S}(\mathbb{R}^N)$ if

$$\sup_{|\alpha|\leq m} \sup_{x\in\mathbb{R}^N} (1+|x|^2)^s |\partial^\alpha (u_n - u)| \to 0,\ s, m = 0, 1, \ldots. \tag{1.8}$$

The space of *tempered distributions* is identified as the dual $\mathcal{S}'(\mathbb{R}^N)$. Continuity of a linear form belonging to $\mathcal{S}'(\mathbb{R}^N)$ is understood with respect to convergence introduced in (1.8).

The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^N)$ is defined as

$$\langle \mathcal{F}_{x\to\xi}[f]; g\rangle = \langle f; \mathcal{F}_{x\to\xi}[g]\rangle \text{for any } g \in \mathcal{S}(\mathbb{R}^N). \tag{1.9}$$

It is a continuous linear operator defined on $\mathcal{S}'(\mathbb{R}^N)$ onto $\mathcal{S}'(\mathbb{R}^N)$ with the inverse $\mathcal{F}^{-1}_{\xi\to x}$,

$$\langle \mathcal{F}^{-1}_{\xi\to x}[f], g\rangle = \langle f, \mathcal{F}^{-1}_{\xi\to x}[g]\rangle, \quad f \in \mathcal{S}'(R^N), g \in \mathcal{S}(\mathbb{R}^N). \tag{1.10}$$

We recall formulas

$$\partial_{\xi_k} \mathcal{F}_{x\to\xi}[f] = \mathcal{F}_{x\to\xi}[-\mathrm{i}x_k f], \quad \mathcal{F}_{x\to\xi}[\partial_{x_k} f] = \mathrm{i}\xi_k \mathcal{F}_{x\to\xi}[f], \tag{1.11}$$

where $f \in \mathcal{S}'(\mathbb{R}^N)$, and

$$\mathcal{F}_{x\to\xi}[f * g] = \Big(\mathcal{F}_{x\to\xi}[f]\Big) \times \Big(\mathcal{F}_{x\to\xi}[g]\Big), \tag{1.12}$$

where $f \in \mathcal{S}(\mathbb{R}^N)$, $g \in \mathcal{S}'(\mathbb{R}^N)$ and $*$ denotes denotes *convolution.*

A differential operator D of order m,

$$D = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha,$$

can be defined by means of a *Fourier multiplier* in the form

$$\tilde{D} = \sum_{|\alpha|\leq m} a_\alpha (\mathrm{i}\xi)^\alpha,\ \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_N^{\alpha_N},$$

in the sense that

$$D[v](x) = \mathcal{F}^{-1}_{\xi\to x}\left[\sum_{|\alpha|\leq m} a_\alpha (\mathrm{i}\xi)^\alpha \mathcal{F}_{x\to\xi}[v](\xi)\right],\ v \in \mathcal{S}(\mathbb{R}^N).$$

The operators defined through the right-hand side of the above expression are called *pseudodifferential operators*.

Various *pseudodifferential operators* used in the book are identified through their Fourier symbols:

- *Riesz transform*:

$$\mathcal{R}_j \approx \frac{-i\xi_j}{|\xi|},\ j = 1, \ldots, N.$$

- *Inverse Laplacian*:

$$(-\Delta)^{-1} \approx \frac{1}{|\xi|^2}.$$

- *The "double" Riesz transform*:

$$\{\mathcal{R}\}_{i,j=1}^{N},\ \mathcal{R} = \Delta^{-1}\nabla_x \otimes \nabla_x,\ \mathcal{R}_{i,j} \approx \frac{\xi_i \xi_j}{|\xi|^2},\ i, j = 1, \ldots, N.$$

- *Inverse divergence*:

$$\mathcal{A}_j = \partial_{x_j}\Delta^{-1} \approx \frac{\mathrm{i}\xi_j}{|\xi|^2},\ j = 1, \ldots, N.$$

We denote

$$\mathbb{A} : \mathcal{R} \equiv \sum_{i,j=1}^{3} A_{i,j}\mathcal{R}_{i,j},\ \mathcal{R}[\boldsymbol{v}]_i \equiv \sum_{j=1}^{3} \mathcal{R}_{i,j}[v_j],\ i = 1, 2, 3.$$

1.4.2 Boundedness of Fourier multipliers

Hörmander-Mikhlin Theorem:

Theorem 1.9 *Consider an operator $\mathcal{L}$ defined by means of a Fourier multiplier $m = m(\xi)$,*

$$\mathcal{L}[v](x) = \mathcal{F}^{-1}_{\xi \to x}\left[m(\xi)\mathcal{F}_{x \to \xi}[v](\xi)\right],$$

where $m \in L^\infty(\mathbb{R}^N)$ has classical derivatives up to order $[N/2]+1$ in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$|\partial_\xi^\alpha m(\xi)| \leq c_\alpha |\xi|^{-|\alpha|},\ \xi \neq 0,$$

for any multiindex α such that $|\alpha| \leq [N/2] + 1$, where $[\,]$ denotes the integer part.

Then $\mathcal{L}$ is a bounded linear operator on $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.

Proof: See Stein [150, Chapter 4, Theorem 3].

□

1.5 Compactness of families of functions

Let X be a Banach space, B_X the (closed) unit ball in X, and B_{X^*} the (closed) unit ball in the dual space X^*.

Here are some known facts concerning *weak compactness*:

(1) B_X is weakly compact only if X is reflexive. This is stated in Kakutani's theorem, see Theorem III.6 in Brezis [17].
(2) B_{X^*} is weakly-(*) compact. This is Banach-Alaoglu-Bourbaki theorem, see Theorem III.15 in Brezis [17].
(3) If X is separable, then B_{X^*} is sequentially weakly-(*) compact, see Theorem III.25 in Brezis [17].
(4) A non-empty subset of a Banach space X is weakly relatively compact only if it is sequentially weakly relatively compact. This is stated in Eberlein-Shmuliyan-Grothendieck theorem, see Köthe [97], Paragraph 24, 3.(8); 7.

In view of these facts:

(1) Any bounded sequence in $L^p(Q)$, where $1 < p < \infty$ and $Q \subset \mathbb{R}^N$ is a domain, is weakly (relatively) compact.
(2) Any bounded sequence in $L^\infty(Q)$, where $Q \subset \mathbb{R}^N$ is a domain, is weakly-(*) (relatively) compact.

1.5.1 Weak compactness of integrable functions

Since L^1 is neither reflexive nor dual of a Banach space, the uniformly bounded sequences in L^1 are in general not weakly relatively compact in L^1. On the other hand, the property of weak compactness is equivalent to the property of sequential weak compactness.

Weak Compactness in the Space L^1:

Theorem 1.10 *Let $\mathcal{V} \subset L^1(Q)$, where $Q \subset \mathbb{R}^M$ is a bounded measurable set.*

Then the following statements are equivalent:

(i) any sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{V}$ contains a subsequence weakly converging in $L^1(Q)$;
(ii) for any $\varepsilon > 0$ there exists $k > 0$ such that

$$\int_{\{|v| \geq k\}} |v(y)| \, \mathrm{d}y \leq \varepsilon \text{ for all } v \in \mathcal{V};$$

(iii) for any $\varepsilon > 0$ there exists $\delta > 0$ such for all $v \in \mathcal{V}$

$$\int_M |v(y)| \mathrm{d}y < \varepsilon$$

for any measurable set $M \subset Q$ such that

$$|M| < \delta;$$

(iv) there exists a non-negative function $\Phi \in C([0,\infty))$,

$$\lim_{z\to\infty} \frac{\Phi(z)}{z} = \infty,$$

such that

$$\sup_{v\in\mathcal{V}} \int_Q \Phi(|v(y)|) \ \mathrm{d}y < \infty.$$

Proof: See Ekeland and Temam [58, Chapter 8, Theorem 1.3], or Pedregal [128, Lemma 6.4].

□

Condition (iii) is termed *equi-integrability* of a given set of integrable functions and the equivalence of (i) is *Dunford-Pettis theorem* (cf. Diestel [47, p.93]). Condition (iv) is called De la Vallé-Poussin criterion, see Pedregal [128, Lemma 6.4]. The statement "there exists a non-negative function $\Phi \in C([0,\infty))$" in condition (iv) can be replaced by "there exists a non-negative convex function on $[0,\infty)$".

1.5.2 Young measures

Let $Q \subset \mathbb{R}^N$ be a domain. We say that $\psi : Q \times \mathbb{R}^M$ is a *Carathéodory function* on $Q \times \mathbb{R}^M$ if

$$\left\{ \begin{array}{c} \text{for a. a. } x \in Q, \text{ the function } \lambda \mapsto \psi(x,\lambda) \text{ is continuous on } \mathbb{R}^M; \\ \\ \text{for all } \lambda \in \mathbb{R}^M, \text{ the function } x \mapsto \psi(x,\lambda) \text{ is measurable on } Q. \end{array} \right\} \quad (1.13)$$

We say that $\{\nu_x\}_{x\in Q}$ is a *family of parametrized measures* if ν_x is a probability measure for a.a. $x \in Q$, and if

$$\left\{ \begin{array}{c} \text{the function } x \to \int_{\mathbb{R}^M} \phi(\lambda)\,\mathrm{d}\nu_x(\lambda) := \langle \nu_x, \phi \rangle \text{ is measurable on } Q \\ \\ \text{for all } \phi : \mathbb{R}^M \to \mathbb{R},\ \phi \in C(\mathbb{R}^M) \cap L^\infty(\mathbb{R}^M)). \end{array} \right\} \quad (1.14)$$

Fundamental Theorem of the Theory of Parameterized (Young) Measures:

Theorem 1.11 *Let* $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$, $\boldsymbol{v}_n : Q \subset \mathbb{R}^N \to \mathbb{R}^M$ *be a sequence of functions bounded in* $L^1(Q;\mathbb{R}^M)$, *where* Q *is a domain in* $\mathbb{R}^N$.

Then there exist a subsequence (not relabeled) and a parameterized family $\{\nu_y\}_{y\in Q}$ *of probability measures on* $\mathbb{R}^M$ *depending measurably on* $y \in Q$ *with the following property:*

For any Carathéodory function $\Phi = \Phi(y,z)$, $y \in Q$, $z \in \mathbb{R}^M$ *such that*

$$\Phi(\cdot, \boldsymbol{v}_n) \to \overline{\Phi} \text{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{\mathbb{R}^M} \psi(y,z) \ \mathrm{d}\nu_y(z) \text{ for a.a. } y \in Q.$$

Proof: See Pedregal [128, Chapter 6, Theorem 6.2].

□

The family of measures $\{\nu_y\}_{y\in Q}$ associated to a sequence $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$,

$$\boldsymbol{v}_n \to \boldsymbol{v} \text{ weakly in } L^1(Q;\mathbb{R}^M),$$

is termed *Young measure*. We shall systematically denote by the symbol $\overline{\Phi(\cdot,\boldsymbol{v})}$ the weak limit associated to $\{\Phi(\cdot,\boldsymbol{v}_n)\}_{n=1}^{\infty}$ via the corresponding Young measure constructed in Theorem 1.11. Note that Young measure need not be unique for a given sequence.

1.6 Gronwall's lemma

Gronwall's lemma:

Theorem 1.12 *Let* η *be a nonnegative, continuous function, let* a *be a nonnegative integrable function, and let* $b \geq 0$ *be a real constant. Assume that*

$$\eta(t) \leq b + \int_0^t a(s)\eta(s) \ \mathrm{d}s \quad \text{for all } t \in [0,T].$$

Then

$$\eta(t) \leq b \exp\left(\int_0^t a(s) \ \mathrm{d}s\right) \quad \text{for all } t \in [0,T].$$

In particular, $\eta \equiv 0$ *on* $[0,T]$ *if* $b = 0$.

Proof: See for example Chicone [31, Chapter 2, Theorem 2.1].

□