Semi-Hyperbolicity and Bi-Shadowing

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This book is dedicated to the memory of our coauthor and friend Alexei Pokrovskii, who passed away, much too soon, shortly after the manuscript was completed. Alexei was the principal source of many of the ideas in this book.

Phil Diamond  
Peter Kloeden  
Victor Kozyakin
Hyperbolicity is a very important concept in dynamical systems theory. It has been extensively investigated over the past half century along with its associated concepts of robustness of dynamical behavior \[11,13,18,19,24,26,37,61,62,87,92,94,102\]. Many deep results have been obtained and there are now numerous monographs and textbooks on the subject \[1,14,20,109,114\].

Most of the work on hyperbolicity, however, concerns abstract differentiable dynamical systems and it is often very difficult to show that the results apply to systems generated by specific differential equations such as the Lorenz equations. In addition, in recent years, there have been many interesting and useful generalizations and extensions of dynamical systems ideas to what were previously ‘non-mainstream’ applications arising, for example, in nonsmooth systems, numerically generated systems and systems with hysteresis, to name just a few. The existing theory of hyperbolicity does not apply directly to these situations, but many of the associated results on robustness and shadowing are nevertheless very important and useful for them too.

Since we were not able to not find any suitable ‘working’ tools in the literature to handle such real, numerical or physical applications, during the 1990s, we developed a practical approach, first for noninvertible differentiable mappings and later for Lipschitz mappings, to investigate ‘robustness’ issues for ‘real-world’ systems. We called this concept semi-hyperbolicity. It arose indirectly in the context of our research on the effects of spatial discretization on the behavior of a dynamical system, in particular by using finite machine arithmetic in computer representations of dynamical systems.

Subsequently, this idea rapidly broadened into a series of papers in which differing aspects and applications of the concept were explored. These and some recent papers form the basis of this monograph, the aim of which is to present a more complete and systematic development of the concept of semi-hyperbolicity, as well as to illustrate its usefulness.

The concept of bi-shadowing was also developed in the above papers. Shadowing is a well known consequence of hyperbolicity, and also holds in weaker situations of semi-hyperbolicity. Essentially it says that there is always a true
solution near a pseudo-solution, which could be arbitrary or, more interestingly, a solution of some approximating system. Bi-shadowing includes the converse effect and is definitely nontrivial when the pseudo-solutions that are near a given true solution come from a specified class or approximating solution.

We would like to stress that semi-hyperbolicity has, essentially, very little in common with ‘classical’ hyperbolicity — no invariant splitting, no ‘real’ smoothness of a system, no invertibility, and so on. Of course, when all of these properties are present in a system, then semi-hyperbolicity is often a sufficient condition for hyperbolicity of the system. For this reason, in this monograph we cite only very general facts about hyperbolic systems that we need here, rather than going into very deep problems of hyperbolicity that are usually considered in monographs on the subject. We reiterate that while the connection with the theory of hyperbolic systems is important and cannot be ignored, much of our motivation comes from our interest and background in applications of dynamical systems and this has naturally influenced the types of questions asked and investigated here.

The theory of semi-hyperbolicity and bi-shadowing is developed systematically in this monograph in nine chapters. There are also two appendices, one by Marcin Mazur, Jacek Tabor and Piotr Kościelniak on the relationship between hyperbolicity and semi-hyperbolicity in linear systems and one by Janosch Rieger on semi-hyperbolicity and bi-shadowing in set-valued systems.

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1992–2012

Phil Diamond
Peter Kloeden
Victor Kozyakin
Alexei Pokrovskii
List of Symbols

$A^T$ transpose of the matrix $A$

$\mathcal{A}, \mathcal{D}$ linear operators associated with the sequence of matrices $\{A_n\}$, p. 57

$A^T$ transpose of the matrix $A$

$\mathcal{A}, \mathcal{D}$ linear operators associated with the sequence of matrices $\{A_n\}$, p. 57

$B(r)$ the closed ball of the radius $r$ centered at 0

$B(r,x)$ the closed ball of the radius $r$ centered at $x$

$\mathcal{C}(\mathbb{R}^d)$ the collection of all nonempty compact subsets of $\mathbb{R}^d$, p. 193

$C(X,\mathbb{R}^d)$ the Banach space of continuous functions $f : X \to \mathbb{R}^d$, p. 26

$C_\varepsilon(S)$ $\varepsilon$-capacity of the compact set $S$, p. 16

$\text{CR}(f, X)$ the set of chain recurrent points of the mapping $f$, p. 15

$\text{CR}(f, \varepsilon, X)$ the set of $\varepsilon$-chain recurrent points of the mapping $f$, p. 108

$\text{Cyc}(f, K, \gamma)$ the set of $\gamma$-pseudo-cycles of the mapping $f$, p. 103

$\text{Cyc}(f, K, 0)$, $\text{Cyc}(f, K)$ the set of cycles of the mapping $f$, p. 103

$Df$ differential of the mapping $f$

$\mathcal{D}(K)$ the space of diffeomorphisms $f : K \to K$

$\dim E$ dimension of the linear space $E$

$\partial X$ boundary of the set $X$

$E$ linear space or Banach space

$E_c$ complexification of the linear space $E$, p. 53

$E^s_x, E^u_x$ subspaces forming a splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$, p. 21
List of Symbols

\( f, \phi, \tilde{f}, \tilde{\phi}, \ldots \) mappings

\( f|_X \) restriction of the mapping \( f \) to the set \( X \)

\( f \circ g \) composition of mappings: \( (f \circ g)(x) := f(g(x)) \)

\( H(K) \) the space of homeomorphisms \( f : K \to K \)

\( h(f, K) \) topological entropy of the mapping \( f \), p. 17

\( h_\varepsilon(f, K) \) topological \( \varepsilon \)-entropy of the mapping \( f \), p. 17

\( \text{Int } X \) interior of the set \( X \)

\( \text{Lip}(X, \mathbb{R}^d) \) the Banach space of Lipschitz functions \( f : X \to \mathbb{R}^d \), p. 26

\( \ell^\infty(I, \mathbb{R}^d) \) the space of bounded sequences of vectors \( x_n \in \mathbb{R}^d \) with indices \( n \) taking values in an interval \( I \), p. 35

\( M(s) \) split matrix, p. 35

\( \mathcal{O}_\varepsilon(X) \) \( \varepsilon \)-neighborhood of the set \( X \)

\( \text{Per}(f, X) \) the set of periodic points of the mapping \( f \).

\( P^s_x, P^u_x \) projectors corresponding to a splitting \( \mathbb{R}^d = E^s_x \oplus E^u_x \), p. 21

\( \mathbb{R}^d = E^s_x \oplus E^u_x \) splitting of \( \mathbb{R}^d \), p. 21

\( s, s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) split, p. 22

\( x, y, z, \tilde{x}, \tilde{y}, \tilde{z} \ldots \) vectors from the space \( \mathbb{R}^d \)

\( TM \) tangent bundle

\( T_x M \) tangent space

\( T_f x \) tangent mapping

\( \mathbb{T}^d \) torus of dimension \( d \), p. 30

\( \text{Tr}(f, K, \gamma) \) the set of \( \gamma \)-pseudo-trajectories of the mapping \( f \), p. 93

\( \text{Tr}(f, K, 0), \text{Tr}(f, K) \) the set of trajectories of the mapping \( f \), p. 93

\( \text{Tr}_{\pm k}(f, K) \) the set of trajectories of the mapping \( f \) defined over the interval \( [-k, k] \), p. 76

\( W^s_\varepsilon(x), W^u_\varepsilon(x), W^s(x), W^u(x) \) stable and unstable sets of the mapping \( f \), p. 12

\( X, Y, Z \ldots \) subsets of \( \mathbb{R}^d \)

\( \mathbb{Z} \) the set of integers

\( \overline{X} \) closure of the set \( X \)

\( x = (x_1, x_2, \ldots, x_d) \) coordinate representation of the vector \( x \in \mathbb{R}^d \)
\{x_n\}, \quad x = \{x_n\}, \\
x = \{x_0, x_1, \ldots\}, \\
x = x_0, x_1, \ldots

sequence of elements \(x_0, x_1, \ldots\), trajectory, pseudo-trajectory, cycle, pseudo-cycle

\(x^T\)
transpose of the vector \(x\)

\(\alpha(s, h)\)
bi-shadowing parameter, p. 94

\(\beta(s, h, \delta)\)
bi-shadowing parameter, p. 94

\(\gamma(s)\)
split parameter, p. 36

\(\delta(E_1, E_2)\)
separation between linear subspaces \(E_1, E_2\), p. 9

\(\nu(s)\)
split parameter, p. 22

\(\Omega(f, X)\)
the set of non-wandering points of the mapping \(f\), p. 15

\(\rho_N(x, y)\)
metric or semi-metric on the space of sequences, p. 16

\(\Sigma\)
the set of bi-infinite binary sequences \(x = \{x_n\}\), p. 15

\(\Sigma(X)\)
the set of bi-infinite sequences \(x = \{x_n\} \in X\), p. 107

\(\sigma: \Sigma \to \Sigma\)
the shift mapping, p. 76

\(\sigma(A)\)
spectral radius of the matrix or linear operator \(A\)

\(\sigma(s)\)
spectral radius of the split matrix \(M(s)\), p. 36

\(#(\cdot)\)
cardinality of a set, p. 12

\(|\cdot|\)
absolute value of a number

\(|t - s|_{\text{mod } k}\)
distance between real \(t\) and \(s\) modulo \(k\), p. 30

\(\|\cdot\|\)
a norm in \(\mathbb{R}^d\), a norm of a matrix or a linear operator

\(\|\cdot\|_x\)
Riemannian norm on a manifold

\(\|\cdot\|_*\)
weighted ‘cubic’ norm in \(\mathbb{R}^2\), p. 39

\(\|\cdot\|_\infty\)
supremum norm in \(\ell^\infty(\mathbb{I}, \mathbb{R}^d)\), p. 35

\(\|\cdot\|_{\infty}^*\)
‘cubic’ supremum norm in \(\ell^\infty(\mathbb{I}, \mathbb{R}^d)\), p. 40

\(\|\cdot\|_{\alpha\infty}\)
weighted ‘cubic’ supremum norm in \(\ell^\infty(\mathbb{I}, \mathbb{R}^d)\), p. 39

\(\|f - \varphi\|_\infty\)
supremum distance between mappings \(f\) and \(\varphi\), p. 87

\(\|\cdot\|_C\)
norm in the Banach space of continuous functions \(C(X, \mathbb{R}^d)\), p. 26

\(\|\cdot\|_{\text{Lip}}\)
norm in the Banach space of Lipschitz functions \(\text{Lip}(X, \mathbb{R}^d)\), p. 26

\((X, \rho)\)
metric space \(X\) with the norm \(\rho\)
List of Symbols
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In this chapter the appeal of introducing ideas of semi-hyperbolicity and bi-shadowing is explained from the point of view of mathematical modeling of real processes.

1.1 Modeling Dynamical Processes

Real world dynamical processes can be extremely complicated, yet very many quite simple idealized mathematical models often appear to capture the essence of what is happening and allow useful and testable predictions to be made. This fact, which is to some extent justifiable mathematically, has been central to the resounding success of rationalist scientific thinking on the physical world over the past four hundred years. Moreover, a large amount of mathematical analysis owes its origin to the development of concepts and techniques needed to formulate and investigate such idealized mathematical models, which, though relatively simple to state, are by no means trivial to analyze and develop.

Traditionally mathematical models have involved differential equations, both ordinary and partial, and thus represent continuous time dynamical systems on appropriately chosen state spaces. In contrast, much of the modern theory of dynamical systems has focussed on discrete time dynamical systems generated by iteration of a mapping $f$ of a given state space $X$ into itself, that is on difference equations

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots . \quad (1.1)$$

Though seemingly less complicated, the dynamical behavior can often be richer without the restricting constraint of continuous time, but in any case many results can often be transferred to and from the differential context by means of suspensions, time-1 maps or Poincaré return maps. Indeed, motivated by the properties of solutions of smooth differential equations, the
mapping $f$ in (1.1) is often assumed to be a diffeomorphism and the state space $X$ to be a compact manifold.

Investigations of long term or asymptotic behavior have dominated the development of the theory of dynamical systems. Not only has the behavior of individual systems been of interest, especially the existence of attractors, but also the classification of classes of systems that share common behavioral characteristics or retain them under transformations and perturbations. Stability is the crucial concept here, both within a given system and across a class of systems, and respectively described as dynamical stability or structural stability. Both types of stability here have connotations of robustness of dynamical behavior.

An idealized mathematical model $f$ is typically taken as given and as the starting point of a mathematical investigation, for which it remains a fixed reference point. Approximation methods may be required, but an approximating system $f_h$, for example the difference equation (1.1) implemented in the finite arithmetic field of a computer, and its behavior is then usually referred back to that of the original mathematical model.

In practice, a mathematical model $f$ is precisely that, a conveniently selected idealization of some nearby, but only imprecisely known system $\tilde{f}$. In applications it is thus desirable that the properties of an approximation $f_h$ to a modeled system $f$ also relate to those of possible underlying systems $\tilde{f}$ on which the model is based. Hence, robustness of dynamical behavior is a particularly important issue in mathematical modeling.

1.1.1 Hyperbolicity

The concept of hyperbolicity in dynamical systems provides an elegant means of addressing one the fundamental problems raised in mathematical modeling, namely that all systems close to an idealized model should share its essential dynamical properties. In a sense it combines and subsumes the idea of total stability, where all systems close to one with an asymptotically stable steady state have a small attracting set about this steady state, along with the analogous idea of total instability. It is based on the elementary observation that a linear mapping with a saddle point, hence with both contracting stable and expanding unstable directions, is robust under small parameter changes of the linear mapping itself as well as under small nonlinear perturbations. This extends easily to a hyperbolic cycle with the proviso that the corresponding stable and unstable linear manifolds are successively mapped onto their counterparts under the linearized mapping. The deep insight of Anosov [6-8] was to generalize this idea to an arbitrary compact invariant subset, in his case a manifold, for each point of which there is a splitting of the tangent space into stable and unstable linear manifolds.

Hyperbolicity and its implications have been intensively investigated since the 1960s led by Smale and his coworkers, with interest being heightened by the presence of chaos generating mechanisms within noncyclic hyperbolic
sets (see, e.g. \cite{60,65,113} and bibliography therein). Identifying and classifying such sets have been fundamental tasks, as also has been establishing their structural stability, where all systems in some neighborhood of a system with such a hyperbolic set are homeomorphic to it. This last generalizes the robustness proved in the Hartman–Grobman theorem \cite{65,124,125} for a system with a simple saddle point. Symbolic dynamics \cite{65,132} has been a key tool here, in showing the existence of a homeomorphism between a system restricted to a hyperbolic set and the shift operator on a space of symbol sequences. This conjugacy of these systems allows the more transparent complicated behavior of the latter to be transferred back to the original system. Shadowing \cite{16,29–32,35,82,103,104,110–112,134}, which asserts the existence of a true trajectory close to any approximate, pseudo-trajectory, is another useful property, suggesting that computer simulations do indeed reflect the dynamics of chaotic hyperbolic systems.

The importance of hyperbolicity in the theory of dynamical systems cannot be understated, yet there remains a very wide gap between the deep theoretical understanding that it provides and applying these mathematical results to specific, practical dynamical systems. There are two reasons for this, one involving the conditions placed upon the systems, and the other concerning the nature of the mathematical theorems themselves and their manner of proof. Much of the theory of hyperbolic systems deals with diffeomorphisms on compact manifolds, yet many interesting systems lack the invertibility, as do many difference equations in population modeling, or the smoothness of diffeomorphisms, for example in hysteresis or control switching. Of course, there have been attempts to generalize hyperbolicity to homeomorphisms or to noninvertible maps, as in Ruelle’s pre-hyperbolicity, but then the second issue of practical application becomes paramount. Hyperbolicity is an extremely difficult property to verify for specific systems generating diffeomorphisms and even more so for its generalizations to homeomorphisms and noninvertible maps. Indeed, nearly all mathematically confirmed examples of hyperbolic systems are artificial constructs. The nature of many theorems on hyperbolic systems and their proofs is another obstacle to their applicability to specific systems, particularly as many proofs are nonconstructive or lack clear, tight estimates. Moreover, many assertions in theorems hold only generically, that is for systems belonging to some residual subset of systems. Although typical in such a sense, it says little about any specific model to hand.

1.1.2 Semi-Hyperbolicity

Although in part restrictive, as argued above, hyperbolicity is nevertheless an ubiquitous and important property of dynamical systems, at least in the sense that many dynamical systems satisfy some if not all of its defining properties and enjoy many of the dynamical consequences. In particular, smoothness and invertibility of the system are not essential, nor are continuity and equivariance of the tangent space splitting at each point of a hyperbolic set, nor
indeed is the invariance of the hyperbolic set itself. As mentioned above, various generalizations of the definition of hyperbolicity have been proposed, but their actual applicability to specific systems is perhaps questionable. What is urgently required is a pragmatic reformulation of the concept of hyperbolicity that is both widely and effectively applicable and at the same time directly addresses both the practical and theoretical issues raised by the use and analysis of mathematical models.

The authors have previously introduced such a reformulation, first for differentiable mappings and then for Lipschitz mappings, called semi-hyperbolicity. This development arose indirectly in the context of research on the effect of spatial discretization on the behavior of a dynamical system, in particular that resulting from computer representations, in finite machine arithmetic, of a dynamical system, and appeared as a series of papers in which differing aspects and applications of the concept were explored. These papers \[2, 5, 41, 51, 74, 75, 80\] form the basis of this monograph, the aim of which is to present a more coherent and systematic development of the concept of semi-hyperbolicity, and to illustrate its practicality in applications. While the relationship with the theory of hyperbolic systems is important and will be discussed, much of the motivation of this work stems from the authors’ interests and background in applications of dynamical systems. This has naturally influenced the types of questions asked and investigated.

1.2 Outline of Book

The book consists of nine Chapters, two Appendices, and a List of Symbols, Index and References. In Chap. 1 the appeal of introducing ideas of semi-hyperbolicity and bi-shadowing is explained from the point of view of mathematical modeling of real processes.

Chapter 2 briefly reviews background material on dynamical systems, particularly that involving differentiable hyperbolic mappings, and introduces terminology and results that will be required in later chapters.

In Chap. 3 the concept of semi-hyperbolicity is introduced and some examples are considered. Here, modified definitions of semi-hyperbolicity are given to apply hyperbolicity to other contexts, such as differentiable mappings which need not be invertible, differentiable mappings on a general compact subset of \(\mathbb{R}^d\) and for Lipschitz mappings. In the last two cases the subset on which the mapping is considered is not necessarily invariant. Further generalizations of the concept of semi-hyperbolicity to mappings in Banach spaces are given in Chaps. 8 and 9, and Appendix A.

The usefulness of first Liapunov approximation methods to investigate the properties of semi-hyperbolic mappings and their trajectories naturally leads to considering linear operators in spaces of sequences generated in one way or another by derivatives of semi-hyperbolic mappings. Some elementary
properties of such linear operators that will be needed throughout the book are considered in Chap. [4].

Chapter [4] also shows that a semi-hyperbolic sequence of matrices is hyperbolic, that is in the linear case semi-hyperbolicity implies hyperbolicity. This is generally not true for nonlinear mappings, as an example in Sect. [5.1] demonstrates that a semi-hyperbolic mapping may not possess an invariant splitting and so cannot be a hyperbolic. Nevertheless, Sect. [5.2] shows that a semi-hyperbolic mapping which is smooth and invertible in a neighborhood of a compact invariant set is hyperbolic on that set. The proofs depend substantially on background material and results in Chap. [4].

In Sect. [2.2] it was mentioned that hyperbolic systems possess some rather strong and useful properties such as expansivity (see Definition [2.2.1] and Theorem [2.2.2]) and shadowing (see Theorem [2.2.3]). Chap. [6] shows that these and further properties are also true for semi-hyperbolic systems. Moreover, explicit values or sharp estimates of relevant parameters and parameter regions in which semi-hyperbolicity holds are found.

Chapter [7] considers properties of semi-hyperbolic mappings which can be considered analogous to structural stability results. More precisely, it is shown that topological entropy can only increase following to continuous perturbation of a Lipschitz semi-hyperbolic mapping. Then problems of conjugation and factorization of semi-hyperbolic mappings are be studied. and chaotic behavior of semi-hyperbolic mappings is discussed.

In Chaps. [8] and [9] we briefly consider several applications of semi-hyperbolicity and its consequences, in particular to delay differential equations, systems with hysteresis and the numerical approximation of chaotic attractors.

The first Appendix [A] is contributed by Marcin Mazur, Jacek Tabor and Piotr Kościeniak of the Jagiellonian University, Institute of Mathematics, Krakow, Poland. It contains recent work on some relationships between semi-hyperbolicity and hyperbolicity, and parameter estimations suitable for numerical verification of hyperbolicity using the semi-hyperbolic parameters.

The second Appendix [B] by Janosch Rieger of the Goethe Universität, Frankfurt am Main, considers extensions of the concepts of semi-hyperbolicity and bi-shadowing to set-valued dynamics generated by a set-valued mapping.
2 Smooth Dynamical Systems

This chapter briefly reviews background material on dynamical systems, particularly that involving differentiable hyperbolic mappings, and introduces terminology and recalls results that are required in later chapters.

2.1 Hyperbolic Mappings

Major theoretical advances in the theory of dynamical systems by Anosov in the early 1960s extended the concept of a hyperbolic set from a finite cyclic set to a general compact invariant subset $K$, opening the way for major advances in the theory. In particular, Smale used this generalization of hyperbolicity to describe highly non-trivial recurrent behavior in dynamical systems as well as the robustness of such behavior. Their work and that of many others was in the context of diffeomorphisms on smooth compact manifolds, motivated in part by the fact that time-one mappings and return mappings of smooth differential equations are diffeomorphisms, and developed differentiable dynamics. A principal objective of this monograph is to show that complicated dynamical behavior is also possible under far less stringent assumptions.

In what follows $\| \cdot \|$ will denote a fixed, but otherwise arbitrary norm on $\mathbb{R}^d$.

2.1.1 Hyperbolic Cycles

Let $X$ be an open subset of $\mathbb{R}^d$ and $f : X \to X$ a differentiable mapping. This mapping generates a discrete time dynamical system by successive iteration,

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \ldots,$$

determining the trajectory of the system starting at the point $x_0 \in X$. 
A trajectory starting at $x_0$ is called a cycle of $f$ with period $p \geq 1$, or $p$-cycle, if the successive points $x_0, x_1, \ldots, x_{p-1}$ are distinct and $x_p = x_0$. A point $x_k$ of a $p$-cycle is called a $p$-periodic point of $f$, and a hyperbolic periodic point if all eigenvalues $\lambda_j^{(k)}$ of the derivative $Df_{x_k}$ satisfy the non-unit modulus condition:

$$|\lambda_j^{(k)}| \neq 1.$$ 

The eigenvalues of $Df_{x_k}$ are actually the same at each of the points $\{x_0, x_1, \ldots, x_{p-1}\}$ of the cycle, so the superindex $k$ be omitted in what follows.

Suppose that the eigenvalues, after possible rearrangement, satisfy

$$|\lambda_j| < 1, \quad 0 \leq j \leq n_s, \quad |\lambda_j| > 1, \quad n_s + 1 \leq j \leq d$$

(2.1) for some $0 \leq n_s \leq d$. Then the linear subspaces $E^s_{x_k}$ and $E^u_{x_k}$ spanned by the eigenvectors, and generalized eigenvectors if necessary, of $Df_{x_k}$ corresponding to the eigenvalues with modulus less than 1 and modulus greater than 1, respectively, form a splitting or decomposition of $\mathbb{R}^d$, that is with

$$\mathbb{R}^d = E^s_{x_k} \oplus E^u_{x_k}.$$ 

with dimensions

$$\dim E^s_{x_k} = n_s, \quad \dim E^u_{x_k} = n_u := d - n_s$$

(2.2) and projection operators

$$P^s_{x_k} : \mathbb{R}^d \to E^s_{x_k}, \quad P^u_{x_k} : \mathbb{R}^d \to E^u_{x_k}.$$ 

Note that an equivalent norm $\| \cdot \|_{x_k}$, which may depend on $x_k$, can be found such that $Df_{x_k} | E^s_{x_k}$ is contractive and $Df_{x_k} | E^u_{x_k}$ is expansive in the sense that there is a constant $\lambda > 1$ such that

$$\|Df_{x_k} u\|_{f(x_k)} \leq \lambda^{-1} \|u\|_{x_k}, \quad \|Df_{x_k} v\|_{f(x_k)} \geq \lambda \|v\|_{x_k}$$

for any $u \in E^s_{x_k}$ and $v \in E^u_{x_k}$. The linear subspaces $E^s_{x_k}$ and $E^u_{x_k}$ are called the stable and unstable subspaces at $x_k$, respectively. Condition (2.1) is satisfied at each point of the cycle with the same $n_s$, so the corresponding stable and unstable spaces for each point of the cycle have the same dimension (2.2). They are related by the equivariance property:

$$Df_{x_k}(E^s_{x_k}) = E^s_{f(x_k)}, \quad Df_{x_k}(E^u_{x_k}) = E^u_{f(x_k)}$$

for each $k$.

It is a trivial, yet nevertheless important observation that the set $K = \{x_0, \ldots, x_{p-1}\}$ of points of a $p$-cycle of $f$ is invariant under $f$, or $f$-invariant, that is $f(K) = K$. In addition, the behavior of the dynamical system generated by the mapping $f$ is robust with respect to small perturbations in the vicinity of any cyclic set $K$ if its points are hyperbolic. This robustness manifests itself globally when the mapping $f$ has only a finite number of cycles each of which is hyperbolic.

A hyperbolic periodic point of period 1 is called a hyperbolic fixed point.
2.1.2 Anosov Systems

Anosov began the investigation of non-cyclic hyperbolic sets with what is now called an Anosov system, for a diffeomorphism $f$ on an $m$-dimensional closed differentiable manifold $M$ in $\mathbb{R}^d$. Recall that a diffeomorphism $f : M \to M$ is an invertible mapping for which both $f$ and its inverse $f^{-1}$ are continuously differentiable on $M$.

In an Anosov system the tangent space $T_x M$, isomorphic to $\mathbb{R}^m$ at each point $x \in M$, splits into a direct sum of stable and unstable subspaces of the tangent mapping $Tf_x$ at $x \in M$ with the separation

$$\delta(E^s_x, E^u_x) = \inf \{\|u - v\| : u \in E^s_x, v \in E^u_x, \|u\| = \|v\| = 1\}$$

between these linear subspaces assumed uniformly bounded away from zero.

**Definition 2.1.1 (Anosov System).** A diffeomorphism $f : M \to M$ where $M$ is an $m$-dimensional closed differentiable manifold in $\mathbb{R}^d$ generates an Anosov system on $M$ if

A1: for each $x \in M$ there exists a splitting $T_x M = E^s_x \oplus E^u_x$ such that:

$$\dim E^s_x = k \neq 0, \quad \dim E^u_x = m - k \neq 0,$$

for some integer $k$ independent of $x \in M$, and

$$Tf_x E^s_x = E^s_{f(x)}, \quad Tf_x E^u_x = E^u_{f(x)};$$

A2: there exist constants $\lambda > 1$ and $\rho_0 > 0$ independent of $x \in M$ such that:

$$\|Tf^n_x u\| \leq \rho_0 \lambda^{-n} \|u\|, \quad \|Tf^n_x v\| \geq \rho_0^{-1} \lambda^n \|v\|, \quad n \geq 0,$$

for $u \in E^s_x$ and $v \in E^u_x$;

A3: there exists a constant $\rho_1 > 0$ independent of $x \in M$ such that:

$$\delta(E^s_x, E^u_x) \geq \rho_1$$

for all $x \in M$.

An example of a linear mapping $f$ which generates an Anosov system and is called an Anosov automorphism will be considered in Chap. 3.

From A3 the splitting and consequent direct sum depend continuously on $x$, that is, in a neighborhood of any point $x_0 \in M$ there exists a set of vectors $e_1(x), e_2(x), \ldots, e_m(x) \in T_x M$ which depend continuously on $x \in M$, which can be chosen so that the vectors $e_1(x), e_2(x), \ldots, e_k(x)$ form a basis of the subspace $E^s_x \subseteq T_x M$ and the vectors $e_{k+1}(x), e_{k+2}(x), \ldots, e_m(x)$ form a basis of the subspace $E^u_x \subseteq T_x M$ (cf. [8,24]).

**Theorem 2.1.2.** For an Anosov system generated by a diffeomorphism on a closed differentiable manifold $M$ of $\mathbb{R}^d$ the splitting $T_x M = E^s_x \oplus E^u_x$ depends continuously on $x \in M$.

Note that only compact Riemannian manifolds $M$ were considered in [8,24], but the proof of Theorem 2.1.2 there does not make use of this extra structure and holds for any closed differentiable manifold.
2.1.3 Hyperbolic Diffeomorphisms

In an Anosov system the whole manifold $M$ is a non-cyclic hyperbolic set. Smale extended the idea to an $f$-invariant subset $K$ of the manifold. The term *hyperbolic mapping* will often be used for any mapping $f$ that has a hyperbolic invariant set $K$.

The usual definition of hyperbolicity now used refers to a diffeomorphism $f : M \to M$ where $M$ is a compact manifold and a compact $f$-invariant subset $K$ of $M$. As for an Anosov system, the splitting is of the tangent space $T_xM$ at each point $x \in K$, but the expansion and contraction on the stable and unstable subspaces of the tangent mapping $Tf_x$ are expressed in terms of a Riemannian norm $\| \cdot \|_x$ on $T_xM$, which may vary with the point $x \in K$, allowing the constant $\rho_0$ in Condition A2 of an Anosov system to be set equal to 1.

**Definition 2.1.3 (Hyperbolicity on a Manifold).** A closed subset $K$ of a compact manifold $M$ which is invariant for a diffeomorphism $f : M \to M$ is said to be hyperbolic if there exists a splitting $T_xM = E^s_x \oplus E^u_x$ for each $x \in K$, a constant $\lambda > 1$ and a Riemannian norm $\| \cdot \|_x$ on $T_xM$ such that

**H1*: For all $x \in K$:

$$Tf_x (E^s_x) = E^s_{f(x)}, \quad Tf_x (E^u_x) = E^u_{f(x)};$$

**H2*: For all $x \in K$, $u \in E^s_x$ and $v \in E^u_x$:

$$\|Tf_x u\|_{f(x)} \leq \lambda^{-1} \|u\|_x, \quad \|Tf_x v\|_{f(x)} \geq \lambda \|v\|_x.$$ 

Conditions H1* and H2* in Definition 2.1.3 together imply that the splitting is continuous as in Theorem 2.1.2, as well as the constancy of the dimension of the splitting subspaces along a trajectory of the mapping $f$.

**Remark 2.1.4.** The use of an equivalent adapted norm in Condition H2* in Definition 2.1.3 allows the constant $\rho_0$ in Condition A2 of an Anosov system to be set equal to 1. This is convenient for many theoretical purposes, but can be cumbersome for specific practical examples. The Whitney Embedding Theorem provides one way of avoiding the problem and using the same norm at each point of $x \in K$ in the inequalities in Condition H2*. Essentially the manifold $M$ is smoothly embedded in a higher dimensional space $\mathbb{R}^{d'}$ and the norm of this space can be used everywhere. Alternatively, Smale has shown that there is an iterate $f^l$ of a hyperbolic diffeomorphism $f$, such that $f^l$ is hyperbolic on the invariant set $K$, with the inequalities in Condition H2* satisfied with respect to a norm that does not depend on the point of $K$.

The previous remark motivates the following modification of the definition of hyperbolicity for a mapping $f : X \to \mathbb{R}^d$, where $X$ is an open subset of $\mathbb{R}^d$ containing the $f$-invariant compact set $K$ under consideration, which is a diffeomorphism on a neighborhood of the set $K$. 

Definition 2.1.5 (Hyperbolicity). A compact subset $K \subset X$ which is invariant for a diffeomorphism $f : X \to \mathbb{R}^d$ is said to be hyperbolic if there exists a splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ for each $x \in K$, varying continuously in $x \in K$, constants $\lambda > 1$ and $\rho_0 > 0$, and a norm $\| \cdot \|$ on $\mathbb{R}^d$ such that

H1: For all $x \in K$:

$$Df_x (E^s_x) = E^s_{f(x)}, \quad Df_x (E^u_x) = E^u_{f(x)};$$

H2: For all $u \in E^s_x$ and $v \in E^u_x$:

$$\|Df^n_x u\| \leq \rho_0 \lambda^{-n} \|u\|, \quad \|Df^n_x v\| \geq \rho_0^{-1} \lambda^n \|v\|, \quad n \geq 0.$$

Given a $\gamma_s, \gamma_u \geq 0$, associate with the split $\mathbb{R}^d = E^s_x \oplus E^u_x$ the family of cones called (the cone field)

$$K^s_x = K^s_x (\gamma_s) = \{ x \in \mathbb{R}^d : \|P^u_x x\| \leq \gamma_s \|P^s_x x\| \},$$

$$K^u_x = K^u_x (\gamma_u) = \{ x \in \mathbb{R}^d : \|P^s_x x\| \leq \gamma_u \|P^u_x x\| \}.$$

Definition 2.1.6 (Hyperbolic Cone Field). A differentiable mapping $f : X \to \mathbb{R}^d$ is said to have hyperbolic cone field on a compact $K \subset X$ if there exists a splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ with projectors $P^s_x, P^u_x$ for each $x \in K$ and a norm $\| \cdot \|$ on $\mathbb{R}^d$ such that

HCF1: For some family of cones $K^s_x, K^u_x$ associated with the split $\mathbb{R}^d = E^s_x \oplus E^u_x$ hold the conditions

$$Df_x K^u_x \subseteq \text{Int } K^u_{f(x)} \cup \{0\}, \quad Df^{-1}_x K^s_{f(x)} \subseteq \text{Int } K^s_x \cup \{0\};$$

HCF2: There exist $\lambda < 1 < \mu$ such that

$$\|Df_x x\| \geq \mu \|x\|, \quad x \in K^u_x,$$

$$\|Df^{-1}_x x\| \geq \lambda^{-1} \|x\|, \quad x \in K^s_{f(x)}.$$

The splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ in Definition 2.1.6 is not supposed to be equivariant with respect to $Df_x$. Conditions HCF1 and HCF2 mean that the cones $K^u_x$ are invariant under $Df_x$ which is expanding on $K^u_x$ while the cones $K^s_{f(x)}$ are invariant under $Df^{-1}_x$ which is expanding on $K^s_{f(x)}$.

As is known [65], the existence of the Hyperbolic Cone Field for a differentiable mapping enables the hyperbolicity of this mapping.

2.1.4 Generalizations

Various generalizations of hyperbolicity to mappings other than diffeomorphisms have been proposed, in particular for homeomorphisms and to mappings which may be neither invertible nor continuous. Instead of a splitting
into stable and unstable linear spaces, dynamically defined nonlinear stable and unstable sets are used.

Let \((X, \rho)\) be a compact metric space and let \(f : X \to X\) be a homeomorphism of \(X\) onto itself. For an arbitrary point \(x \in X\) and any \(\varepsilon > 0\), the local stable and unstable sets of \(f\) of size \(\varepsilon\) at \(x\) are defined as

\[
W_s^\varepsilon(x) = \{ y \in X : \rho(f^n x, f^n y) \leq \varepsilon, \ n \geq 0 \}
\]

and

\[
W_u^\varepsilon(x) = \{ y \in X : \rho(f^n x, f^n y) \leq \varepsilon, \ n \leq 0 \},
\]

respectively, while the stable and unstable sets of \(f\) at \(x\) are defined as

\[
W_s(x) = \{ y \in X : \rho(f^n x, f^n y) \to 0, \ n \to \infty \}
\]

and

\[
W_u(x) = \{ y \in X : \rho(f^n x, f^n y) \to 0, \ n \to -\infty \},
\]

respectively.

Note that cardinality \(#(W_s^\varepsilon(x) \cap W_u^\varepsilon(x)) \geq 1\) for any \(\varepsilon > 0\).

**Definition 2.1.7 (Hyperbolic Homeomorphism).** A homeomorphism \(f\) of \(X\) onto itself is hyperbolic if there exist constants \(\varepsilon_0 > 0\), \(K > 0\) and \(\lambda > 1\) such that

\[
\rho(f^n x, f^n y) \leq K \lambda^{-n} \quad \text{for all} \quad x, y \in W_s^\varepsilon_0(x) \quad \text{and} \quad n \geq 0,
\]

\[
\rho(f^{-n} x, f^{-n} y) \leq K \lambda^{-n} \quad \text{for all} \quad x, y \in W_u^\varepsilon_0(x) \quad \text{and} \quad n \geq 0,
\]

and there exists \(\delta_0 > 0\) such that

\[
#(W_s^\varepsilon_0(x) \cap W_u^\varepsilon_0(y)) = 1,
\]

for all \(x, y \in X\) with \(\rho(x, y) \leq \delta_0\).

The case of noninvertible \(f\) can be handled by a suggestion of Ruelle \[129\]. Let \(f : X \to X\) be a continuous mapping and for each \(x \in X\) define the sequence set

\[
X^\dagger = \left\{ \{x_n\}_{n=-\infty}^0 \in \prod_{n=-\infty}^0 X : f(x_{-n}) = x_{-n+1}, \ n = 1, 2, \ldots \right\}
\]

and a mapping \(f^\dagger : X^\dagger \to X^\dagger\) by \(f^\dagger(\{x_n\}) = \{f(x_n)\}\). Then \(f^\dagger\) is a homeomorphism on \(X^\dagger\) with respect to the metric

\[
D(\{x_n\}, \{y_n\}) = \sup_{n \leq 0} \rho(x_n, y_n)
\]

and \(f\) is said to be hyperbolic on \(X\) if \(f^\dagger\) is hyperbolic on \(X^\dagger\).
2.2 Fundamental Properties

Robustness properties of a dynamical system near a hyperbolic cycle is also a feature of dynamical systems with nontrivial hyperbolic sets, but the dynamical behavior can be much more complicated. Some fundamental properties of hyperbolic diffeomorphisms are summarized here without proof, focussing on those properties that extend, perhaps modified, to semi-hyperbolic mappings. Proofs of these results will be presented in later chapters. To simplify the exposition, results will be stated in terms of a diffeomorphism $f : X \to X \subset \mathbb{R}^d$ with a compact invariant hyperbolic subset $K$. Homeomorphisms will also be considered below.

2.2.1 Expansivity and Shadowing

An immediate consequence of hyperbolicity is that a diffeomorphism is expansive on a hyperbolic set. Given that a hyperbolic set is compact and invariant, this suggests complex dynamics within the set itself.

**Definition 2.2.1 (Expansivity).** A homeomorphism $f : K \to K$ is expansive on an invariant set $K$ if there exists an $\varepsilon > 0$ such that for all $x, y \in K$

\[ \| f^n(x) - f^n(y) \| \leq \varepsilon \quad \text{for all} \quad n \in \mathbb{Z} \]

implies that $x = y$.

**Theorem 2.2.2.** A diffeomorphism $f$ with a hyperbolic set $K$ is expansive on $K$.

Thus no two distinct trajectories $\{ f^n(x) : n \in \mathbb{Z} \}$, $\{ f^n(y) : n \in \mathbb{Z} \}$ in a hyperbolic set can be $\varepsilon$-close to each other, no matter how close the points $x, y$. Nevertheless, within the hyperbolic set there is always a true trajectory close to an approximate or pseudo-trajectory. A sequence $\{ y_n \} \subset X$ is called a $\delta$-pseudo-trajectory of a dynamical system generated by a mapping $f$ if

\[ \| y_{n+1} - f(y_n) \| < \delta \quad \text{for all} \quad n \in \mathbb{Z} \]

and a true trajectory $\{ x_n \}$ is said to $\varepsilon$-shadow a pseudo-trajectory $\{ y_n \}$ if

\[ \| y_n - x_n \| < \varepsilon \quad \text{for all} \quad n \in \mathbb{Z}. \]

**Theorem 2.2.3 (Shadowing Theorem).** If $f : X \to X$ is a diffeomorphism with an $f$-invariant hyperbolic set $K \subset X$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ and an open neighborhood $U$ of $K$ in $X$ such that every $\delta$-pseudo-trajectory of $f$ in $U$ is $\varepsilon$-shadowed by a true trajectory of $f$ in $K$. 
A dynamical system satisfying the assertion of Theorem 2.2.3 will be said to have the shadowing property. The following characterization of hyperbolic homeomorphisms holds.

**Theorem 2.2.4.** A homeomorphism \( f \) on a compact metric space \((X, \rho)\) is hyperbolic if and only if \( f \) is expansive and has the shadowing property.

The pseudo-trajectories in the Shadowing Theorem are often interpreted as being the true trajectories of a nearby approximating system, for example generated by computation of the given system on a digital computer. Furthermore, hyperbolic diffeomorphisms enjoy a stronger relationship with systems generated by nearby mappings.

### 2.2.2 Conjugate Systems

Let \( \mathcal{H}(K) \) denote the space of homeomorphisms \( f : K \to K \), where \( K \) is a compact subset of \( \mathbb{R}^d \). Note that \( (\mathcal{H}(K), \rho_0) \) is a complete metric space with the metric

\[
\rho_0(f, g) = \max_{x \in K} \{ \| f(x) - g(x) \|, \| f^{-1}(x) - g^{-1}(x) \| \}.
\]

A mapping \( g \in \mathcal{H}(K) \) is said to be topologically semi-conjugate to a mapping \( f \in \mathcal{H}(K) \) if there exists a continuous mapping \( h \) of \( K \) onto \( K \) such that \( f \circ h = h \circ g \) and topologically conjugate when the mapping \( h : K \to K \) is a homeomorphism. Many important dynamical properties are preserved under conjugacy. One illustration is the Hartman–Grobman theorem in which the mapping and its linearization about a hyperbolic point have locally similar behavior. Moreover, for a hyperbolic diffeomorphism \( f \) there is a neighborhood of homeomorphisms in \( (\mathcal{H}(K), \rho_0) \) homeomorphisms, each of which is semi-conjugate to \( f \) under a common semi-conjugacy \( h \) close to the identity mapping on \( K \).

**Theorem 2.2.5 (Topological Stability).** Let \( f \in (\mathcal{H}(K), \rho_0) \) be a hyperbolic diffeomorphism on \( K \). Then, given \( \varepsilon > 0 \) there exists a unique continuous mapping \( h : K \to K \) with \( \| x - h(x) \| < \varepsilon \) for all \( x \in K \) and a \( \delta = \delta(\varepsilon) > 0 \) such that \( h \circ g = f \circ h \) for all \( g \in (\mathcal{H}(K), \rho_0) \) with \( \rho_0(f, g) < \delta \). Moreover, if \( \varepsilon \) is small enough and \( K \) is a compact manifold, then \( h \) maps \( K \) onto \( K \).

This result shows the topological stability of the mapping \( f \) when \( \varepsilon \) is small enough to ensure that \( h \) is a surjective mapping. Neighboring dynamical systems \( g \) in \( (\mathcal{H}(K), \rho_0) \) are then also expansive mappings with the shadowing property. Whether or not diffeomorphisms \( g \) in this \( \delta \)-neighborhood of \( f \) in the space \( (\mathcal{H}(K), \rho_0) \) are also hyperbolic on \( K \) or a subset of \( K \) is a much deeper question and has lead to an extensive structural theory of differential dynamics which involves the stronger concept of structural stability in the space \( \mathcal{D}(K) \) of diffeomorphisms on \( K \) with a \( C^1 \)-metric \( \rho_1 \). In this theory generic properties
of diffeomorphisms in a residual subset, that is in a countable intersection of open dense subsets of \((\mathcal{D}(K), p_1)\), are of primary interest.

Topological conjugacies between mappings defined on different compact sets \(K_i\), which need not be subsets of a common space, are also useful in comparing the dynamics of systems when one of them is relatively simple enough to be well understood. For example, the topological conjugacy between the horseshoe diffeomorphism on the sphere, restricted to the hyperbolic horseshoe subset, and the shift operator \(\sigma\) on the space \(\Sigma\) of bi-infinite sequences \(x = \{x_n\}_{n \in \mathbb{Z}}\) where \(x_n \in \{0, 1\}\) with the metric

\[
\rho(x, \tilde{x}) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - \tilde{x}_n| ,
\]

where \((\sigma x)_n = x_{n+1}\) for each \(n \in \mathbb{Z}\). The dynamics of the system generated on \(\Sigma\) by the homeomorphism \(\sigma\) is expansive on \(\Sigma\), the set \(\text{Per}(\sigma)\) of all periodic points of \(\sigma\) is dense in \(\Sigma\) and there is a trajectory of \(\sigma\) which is dense in \(\Sigma\). These properties are carried over to the horseshoe diffeomorphism on its hyperbolic horseshoe set by the conjugacy homeomorphism. This is but one illustration of how symbolic dynamics has become an indispensable tool in the theory of dynamical systems.

### 2.2.3 Hyperbolic Sets

Often a hyperbolic subset \(K\) of a mapping \(f\) defined on a set \(X\) is not unknown in advance and must be determined. There are several natural candidates for such a subset, the most obvious of which is the set \(\text{Per}(f, X)\) of all periodic points \(x \in X\) of \(f\) whose cycles belong to \(X\). More interesting are \(\Omega(f, X)\), the set of non-wandering points of \(f\), and \(\text{CR}(f, X)\) of chain recurrent points of \(f\). A point \(x \in X\) is a non-wandering point of \(f\) if for every \(\varepsilon > 0\) there exists an \(\varepsilon\)-pseudo-trajectory \(\{x_n\} \subseteq X\) of \(f\) with \(x_0 = x\) and \(\|x_N - x\| \leq \varepsilon\) for some \(N > 0\). Sometimes a slightly different, though equivalent, definition of a chain recurrent point is used: a point \(x \in X\) is chain recurrent for \(f\) if for every \(\varepsilon, N > 0\) there exists an \(\varepsilon\)-pseudo-trajectory \(\{x_n\} \subseteq X\) of \(f\) with \(x_0 = x_K = x\) for some \(K \geq N\) (cf. [129]). The sets \(\text{Per}(f, X), \Omega(f, X)\) and \(\text{CR}(f, X)\) are closed in \(X\), and

\[
\text{Per}(f, X) \subseteq \Omega(f, X) \subseteq \text{CR}(f, X),
\]

where each is \(f\)-invariant.

Much of the structural theory of differential dynamics has focussed on the Axiom A diffeomorphisms, where \(\Omega(f)\) is a hyperbolic set and \(\text{Per}(f)\) is
dense in $\Omega(f)$, since a diffeomorphism is structurally stable if and only if it is an Axiom A diffeomorphism and its stable and unstable manifolds satisfy a strong transversality condition \([133]\). Note that $\text{Per}(f) = \Omega(f) = \text{CR}(f)$ for a generic system $f \in \mathcal{H}(X)$ and the mapping $f \mapsto \Omega(f)$ is upper-semicontinuous on $(\mathcal{H}(X), \rho_0)$ at each such $f$, in the sense that for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\Omega(g) \subset \mathcal{O}_\varepsilon(\Omega(f))$$

for all $\rho_0(g, f) < \delta$, where $\mathcal{O}_\varepsilon(\Omega(f))$ is the open $\varepsilon$-neighborhood of the subset $\Omega(f)$,

$$\mathcal{O}_\varepsilon(\Omega(f)) = \bigcup_{x \in K} \text{Int} B(\varepsilon, x),$$

where $\text{Int} B(\varepsilon, x)$ is the open ball of radius $\varepsilon$ centered at $x$.

A modified form of structural stability is $\Omega$-stability in which a topological conjugacy is established between the restricted mappings $f|_{\Omega(f)}$ and $g|_{\Omega(g)}$. For nonsmooth mappings to be considered later in this book conjugacies between $f|_{\text{CR}(f)}$ and $g|_{\text{CR}(g)}$, and between the sets of their trajectories, is important.

### 2.2.4 Entropy

The term entropy refers to a variety of related concepts in mathematics. Here, the principal concern is that of topological entropy, an invariant of topological conjugacy \([9]\). Like other concepts of entropy, topological entropy provides an index of how complicated the dynamics of a system are. There are several equivalent definitions \([21, 90, 139]\) of topological entropy and that used here is based on the Bowen scheme which allows entropy to be defined for a uniformly continuous mapping on a subset of a metric space that need not be compact (cf. \([21, 139]\)).

Let $f : X \to X$ be a continuous mapping where $X$ is an open bounded subset of $\mathbb{R}^d$ and let $K$ be a compact subset of $X$. For a fixed positive integer $N$ denote by $\text{Tr}_N(f, K)$ the totality of finite trajectories $x = \{x_{-N}, \ldots, x_0, \ldots, x_N\}$ of $f$ that are contained entirely in $K$ and introduce on $\text{Tr}_N(f, K)$ the metric

$$\rho_N(x, \tilde{x}) = \sup_{-N \leq n \leq N} \|x_n - \tilde{x}_n\|.$$

Let $\varepsilon > 0$ and denote by $C_\varepsilon(\text{Tr}_{\pm N}(f, K))$ the $\varepsilon$-capacity of the set of elements $x^{(1)}, \ldots, x^{(p)}$ in $\text{Tr}_N(f, K)$ such that $\rho_N(x^{(i)}, x^{(j)}) \geq \varepsilon$ for all $i \neq j$.

\[2\] If $(Y, d)$ is a metric space and $S$ is a compact subset of $Y$, then the $\varepsilon$-capacity of the set $S$, $C_\varepsilon(S) = \log s_\varepsilon(S)$, is the binary logarithm of the maximal number $s = s_\varepsilon(S)$ of elements $y_1, y_2, \ldots, y_s \in S$ satisfying $y_i \neq y_j$ for $i \neq j$. Hence $C_\varepsilon(\text{Tr}_N(f, K))$ is the $\varepsilon$-capacity of the compact metric space $\text{Tr}_N(f, K)$. 

Definition 2.2.6. The topological $\varepsilon$-entropy $h_\varepsilon(f, K)$ of $f : X \to X$ on a compact subset $K$ of $X$ is defined by

$$h_\varepsilon(f, K) = \limsup_{N \to \infty} \frac{1}{2N + 1} C_\varepsilon(\text{Tr}_N(f, K)).$$

The topological entropy $h(f, K)$ of $f$ on $K$ is defined to be the limit

$$h(f, K) = \lim_{\varepsilon \to 0} h_\varepsilon(f, K) = \sup_{\varepsilon > 0} h_\varepsilon(f, K).$$

Clearly, $h_\varepsilon(f, K)$ is nonincreasing in $\varepsilon$, so $h(f, K)$ is well defined.

One of the most difficult problems in investigating topological entropy is explicit evaluation. For an $\xi$-expansive mapping $f$ with $f(K) = K$ the topological entropy in some cases is given by the following formula

$$h(f, K) = h_\theta(f, K), \quad \theta < \xi$$

which will be proved in Lemma [7.1.1] of Chap. 7.

The following theorem establishes the relation between entropy and the number of periodic points of a mapping and is worth noting here.

Theorem 2.2.7. If $f$ is an expansive homeomorphism of a compact metric space $K$, then

$$h(f, K) \geq \limsup_{n \to \infty} \frac{1}{n} \log N_n(f)$$

where $N_n(f)$ is the number of $n$-periodic points of $f$.

If $f$ is a diffeomorphism of a compact manifold $M$ onto itself which satisfies Axiom A, then

$$h(f, K) = \limsup_{n \to \infty} \frac{1}{n} \log N_n(f).$$

2.2.5 Chaos

Chaos is a word that is now usually used to describe rather complicated behavior of a dynamical system that includes sensitive dependence on initial conditions, an abundance of unstable periodic trajectories and irregular mixing effects. Although the term is frequently used by many authors, there is no commonly accepted definition of chaos. Often the behavior of the shift mapping $\sigma$ on the space $\Sigma$ of binary sequences, described in Sect. 2.2.2, is used to provide a paradigm for chaotic behavior, which could be called symbolic dynamical chaos or shift mapping chaos. The behavior of the Smale horse-shoe system and the closely related behavior of a dynamical system near a transversal homoclinic point are famous examples of this kind of chaos.

Definition 2.2.8. Let $f : X \to X$ be a diffeomorphism, where $X$ is an open subset of $\mathbb{R}^d$. A point $y \in X$ is called homoclinic if there exists a fixed point $x$ of $f$ with $x \neq y$ such that $y \in W^s(x) \cap W^u(x)$. A homoclinic point $y \in X$ is called transversal homoclinic if $T_y W^s(x) \oplus T_y W^u(x) = \mathbb{R}^d$. 

The following theorem shows that chaotic behavior in the above sense occurs in a neighborhood of any homoclinic point.

**Theorem 2.2.9 (Smale Homoclinic Theorem).** Let $f$ be a $C^1$ diffeomorphism and $y$ a transversal homoclinic point for a fixed point $x$ of $f$. Then there is an integer $n$ such that $g = f^n$ has a hyperbolic compact invariant set $K$ which is homeomorphic to a Cantor set and contains $x$ and $y$. Moreover, the mapping $g|_K$ is topologically conjugate to the shift $\sigma$ acting on the space $\Sigma$ of binary sequences.

Chaotic behavior of the shift mapping kind is, however, quite restrictive. A more descriptive and broadly applicable definition which retains the essential features of shift mapping chaos was proposed by Li and Yorke [84].

**Definition 2.2.10 (Li–Yorke Chaos).** Let $X$ be a subset of $\mathbb{R}^d$. A continuous mapping $f : X \to X$ is called chaotic if

**CH1:** There exists a positive integer $N$ such $f$ has a $p$-periodic point for any integer $p \geq N$;

**CH2:** There exists an uncountable $f$-invariant set $S \subseteq X$ containing no periodic points, called a scrambled set, such that:

$$\limsup_{n \to \infty} \| f^n(x) - f^n(y) \| > 0$$

for every $x, y \in S$ with $x \neq y$, and for every $x \in S$ and any periodic point $y$;

**CH3:** There exists an uncountable subset $S_0$ of $S$ such that:

$$\liminf_{n \to \infty} \| f^n(x) - f^n(y) \| = 0$$

for every $x, y \in S_0$.

A sufficient condition for the chaos in the sense of Definition 2.2.10 for one-dimensional mappings gives the following Li and Yorke [84] criterion called ‘period three implies chaos’.

**Theorem 2.2.11 (Period Three Implies Chaos).** Let $I$ be an interval in $\mathbb{R}^1$ and $f : I \to I$ be a continuous mapping. Let there is a point, $a \in I$, for which the points $b = f(a)$, $c = f^2(a)$ and $d = f^3(a)$ satisfy

$$d \leq a < b < c \quad \text{or} \quad d \geq a > b > c.$$

Then the mapping $f$ is chaotic.

The following theorem due to Shiraiwa and Kurata [131] expands the ‘Period Three Implies Chaos Theorem’ in $\mathbb{R}^1$ to higher dimensional mappings. See also Kloeden [71].
Theorem 2.2.12. Let $X$ be a subset of $\mathbb{R}^d$ and $f : X \to X$ be a $C^1$-map. Let $x_0 \in X$ be a hyperbolic fixed point of $f$ and assume that the following three conditions are satisfied:

(i) $\dim E^u_{x_0} > 0$;
(ii) there exist an $\varepsilon > 0$, a point $x_1 \in W^u_\varepsilon(x_0)$, $x_1 \neq x_0$, and a positive integer $m$ such that $f^m(x_1) \in W^s_\varepsilon(x_0)$;
(iii) there exists a disk $D^u \subset W^u_\varepsilon(x_0)$ such that $D^u$ is a neighborhood of $x_1$ in $W^u_\varepsilon(x_0)$, $f^m|_{D^u} : D^u \to X$ is an embedding, and $f^m(D^u)$ intersects $W^u_\varepsilon(x_0)$ transversally at $f^m(x_1)$.

Then the mapping $f$ is chaotic.

Motivated by the work of Li and Yorke [84], Marotto [89] further generalized Theorem 2.2.11 to the multidimensional setting. Consider the following $d$-dimensional system:

$$x_{n+1} = f(x_n), \quad x_n \in \mathbb{R}^d, \quad n = 0, 1, 2, \ldots,$$

where the map $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable. Denote by $B_r(x)$ the closed ball in $\mathbb{R}^d$ of radius $r$ centered at a point $x \in \mathbb{R}^d$ and let $f$ be differentiable in the interior of $B_r(x)$. The point $x \in \mathbb{R}^d$ is an expanding fixed point of $f$ in $B_r(x)$, if $f(x) = x$ and all eigenvalues of $Df(y)$ exceed 1 in absolute value for all $y \in B_r(x)$.

Definition 2.2.13 (Snap-Back Repeller). Let $x$ be an expanding fixed point of $f$ in $B_r(x)$ for some $r > 0$. Then, $x$ is said to be a snap-back repeller of $f$, if there exists a point $x_0 \in B_r(x)$ with $x_0 \neq x$, such that $f^m(x_0) = x$ and the determinant $|Df^m(x_0)| \neq 0$ for an integer $m > 0$.

The following theorem is due to Marotto [89].

Theorem 2.2.14. If $f$ possesses a snap-back repeller, then the system (2.3) is chaotic.

A simpler definition of chaos was given by Devaney [40].

Definition 2.2.15 (Devaney Chaos). Let $S$ be a set in a metric space $(X, \rho)$. A mapping $f : S \to S$ is said to be chaotic, if

DCH1: The map $f$ has sensitive dependence on initial conditions, in the sense that there exists $\delta > 0$, such that for any $x \in S$ and any neighborhood $U$ of $x$ in $S$, $\rho(f^m(x), f^m(y)) > \delta$ for some $y \in U$ and some $m \geq 0$;

DCH2: The map $f$ is topologically transitive, in the sense that for any pair of nonempty open subsets $U, V \in S$, there exists an integer $m > 0$, such that $f^m(U) \cap V \neq \emptyset$;

DCH3: The periodic points of the map $f$ are dense in $S$.

---

A disk is a homeomorphic image of a ball in a normed linear space.
This definition has some redundancy and can be further simplified. If Condition DCH3 above is dropped, then it is called chaos in the sense of Wiggins [140]. For further references and discussion of chaos concepts see in [27, 69, 71].
Semi-Hyperbolic Mappings

The concept of semi-hyperbolicity is introduced in this chapter and some examples are considered. The first definition generalizes Definition 2.1.3 to a differentiable mapping, which need not be invertible, on a compact manifold and involves variable norms at each point of the compact invariant subset of the manifold. The second and third definitions apply to a general compact subset of $\mathbb{R}^d$ as in Definition 2.1.5, first for a differentiable mapping and then for a Lipschitz mapping. In both cases a fixed norm is used and invariance of the subset is not assumed.

Further generalizations of the concept of semi-hyperbolicity to mappings in Banach spaces will be given in Chaps. 8 and 9, and Appendix A.

3.1 Definitions

Consider a mapping $f : X \to \mathbb{R}^d$ where $X$ is an open subset of $\mathbb{R}^d$ and let $K$ be a nonempty compact subset of $X$ such that $K \cap f(K) \neq \emptyset$, or consider a mapping $f : M \to M$ where $M$ is a compact manifold and let $K$ be an $f$-invariant subset of $M$. The following definition is for a splitting of $\mathbb{R}^d$. An analogous definition holds for a splitting $T_x M = E^s_x \oplus E^u_x$ of the tangent space $T_x M$ of a compact manifold $M$, where the projectors are $P^s_x : T_x M \to E^s_x$ and $P^u_x : T_x M \to E^u_x$.

**Definition 3.1.1.** A splitting or decomposition $\mathbb{R}^d = E^s_x \oplus E^u_x$ with corresponding projectors $P^s_x : \mathbb{R}^d \to E^s_x$ and $P^u_x : \mathbb{R}^d \to E^u_x$ defined by

\[
P^s_x \mathbb{R}^d = E^s_x, \quad P^s_x E^u_x = 0, \quad (3.1)
\]
\[
P^u_x \mathbb{R}^d = E^u_x, \quad P^u_x E^s_x = 0, \quad (3.2)
\]

is said to be uniform on the compact subset $K$ with respect to a mapping $f$ if

SH0: $\dim E^u_x = \dim E^u_{f(x)}$ for $x \in K$ with $f(x) \in K$;
there exists a positive real number $h$ such that

\[ \text{SH1: } \sup_{x \in K} \{ \|P^s_x\|, \|P^u_x\| \} \leq h. \]

Note that neither the invariance of the set $K$ with respect to the mapping $f$ nor the continuity in $x$ of the splitting subspaces $E^s_x, E^u_x$ or of the projectors $P^s_x, P^u_x$ are assumed in Definition 3.1.1.

To explore further ramifications of this definition, rates of expansion and contraction for the projections are needed, in the form of a split. See Definition 3.1.2 below.

**Definition 3.1.2.** A 4-tuple $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers is called a split if

\[ \lambda_s < 1 < \lambda_u, \quad (1 - \lambda_s)(\lambda_u - 1) > \mu_s\mu_u. \]

The split $s$ is called positive if all the values $\lambda_s, \lambda_u, \mu_s$ and $\mu_u$ are positive.

Clearly, for any given $\lambda_s$ and $\lambda_u$ satisfying $\lambda_s < 1 < \lambda_u$ a 4-tuple of nonnegative numbers $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ will be a split if the product $\mu_s\mu_u$ is small enough.

An alternative definition of a split with a geometrical interpretation is possible: a 4-tuple $(\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split if and only if the eigenvalues $\Delta_1$ and $\Delta_2$ of every matrix $\Delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix}$ with components satisfying $|\delta_{11}| \leq \lambda_s$, $|\delta_{12}| \leq \mu_s$, $|\delta_{21}| \leq \mu_u$ and $|\delta_{22}| \geq \lambda_u$ are real and satisfy

\[ |\Delta_1| < 1 < |\Delta_2|. \]

It is of practical importance to know how the elements of a split can be perturbed and remain a split. Set

\[ \nu(s) = \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s\mu_u}{\lambda_u - \lambda_s + \mu_s + \mu_u}. \]

Note that

\[ \nu(s) \leq \frac{(1 - \lambda_s)(\lambda_u - 1)}{\lambda_u - \lambda_s} \leq \min \{1 - \lambda_s, \lambda_u - 1\}, \]

so $\nu(s) < 1$.

**Lemma 3.1.3.** If $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split, then for

\[ \tilde{\lambda}_s = \lambda_s + \delta_1, \quad \tilde{\lambda}_u = \lambda_u - \delta_2, \quad \tilde{\mu}_s = \mu_s + \delta_3, \quad \tilde{\mu}_u = \mu_u + \delta_4 \]

and real $\delta_1, \delta_2, \delta_3, \delta_4$ satisfying $0 \leq \delta_1, \delta_2, \delta_3, \delta_4 < \nu(s)$ the 4-tuple $\tilde{s} = (\tilde{\lambda}_s, \tilde{\lambda}_u, \tilde{\mu}_s, \tilde{\mu}_u)$ is also a split.
Proof. Clearly the elements of the 4-tuple \( \tilde{s} \) are non-negative and by (3.3), (3.4) we have \( \delta_1 < 1 - \lambda_s \) and \( \delta_2 < \lambda_u - 1 \). Hence

\[
\tilde{\lambda}_s = \lambda_s + \delta_1 < \lambda_s + 1 - \lambda_s = 1, \quad \tilde{\lambda}_u = \lambda_u - \delta_2 > \lambda_u - (\lambda_u - 1) = 1,
\]

and by (3.3)

\[
(1 - \tilde{\lambda}_s)(\tilde{\lambda}_u - 1) - \tilde{\mu}_s \tilde{\mu}_u
> (1 - \lambda_s - \nu(s))(\lambda_u - 1 - \nu(s)) - (\mu_s + \nu(s))(\mu_u + \nu(s))
= (1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u - \nu(s)(\lambda_u - \lambda_s + \mu_s + \mu_u) = 0.
\]

The lemma is proved. \( \Box \)

### 3.1.1 Differentiable Mappings on Compact Manifolds

Here, the variable norm version of hyperbolicity on a manifold of Definition 2.1.3 is generalized to semi-hyperbolic differentiable mappings on a manifold.

**Definition 3.1.4.** Let \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) be a split and let \( M \) be a compact manifold in \( \mathbb{R}^d \). A differentiable mapping \( f : M \to M \) is said to be \( s \)-semi-hyperbolic on a compact invariant subset \( K \) of \( M \) if there exists a uniform splitting \( T_x M = E^s_x \oplus E^u_x \) with projectors \( P^s_x, P^u_x \) and a norm \( \|\cdot\|_x \) on \( T_x M \) for each \( x \in K \) such that

\[
\begin{align*}
\|P^s_{f(x)} T_{f(x)} u\|_{f(x)} &\leq \lambda_s \|u\|_x, & u \in E^s_x, \\
\|P^s_{f(x)} T_{f(x)} v\|_{f(x)} &\leq \mu_s \|v\|_x, & v \in E^u_x, \\
\|P^u_{f(x)} T_{f(x)} u\|_{f(x)} &\leq \mu_u \|u\|_x, & u \in E^s_x, \\
\|P^u_{f(x)} T_{f(x)} v\|_{f(x)} &\geq \lambda_u \|v\|_x, & v \in E^u_x,
\end{align*}
\]

where \( T_{f(x)} \) denotes the tangent mapping of \( f \) at the point \( x \).

Nonzero values of \( \mu_s \) and \( \mu_u \) allow for a possible leakage away from the strict equivariance of the splitting subspaces \( E^s_x \) and \( E^u_x \) of Condition H1, that holds when the \( f \) is a hyperbolic diffeomorphism on \( K \) as in Definition 2.1.3.

Essentially, the notion of semi-hyperbolicity is very close to that of hyperbolic cone field, see Definition 2.1.6.

Comparing Definition 3.1.4 with Definition 2.1.3, it is clear that a differentiable hyperbolic mapping on \( K \) with expansivity parameter \( \lambda > 1 \) is \( s \)-semi-hyperbolic on \( K \) with the same splitting and split \( s = (\lambda^{-1}, \lambda, 0, 0) \), that is with \( \mu_s = \mu_u = 0 \).
3.1.2 Differentiable Mappings on Compact Sets

Consider now a differentiable mapping \( f : X \to \mathbb{R}^d \) where \( X \) is an open subset of \( \mathbb{R}^d \). Definition 2.1.5 of a hyperbolic diffeomorphism on a compact subset \( K \) of \( \mathbb{R}^d \) will be generalized to a differentiable mapping, which need not be invertible, and to a nonempty subset \( K \) of \( X \) such that only satisfies \( K \cap f(K) \neq \emptyset \) rather than being \( f \)-invariant.

**Definition 3.1.5.** Let \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) be a split \( K \) a compact subset of an open subset \( X \) of \( \mathbb{R}^d \). A differentiable mapping \( f : X \to \mathbb{R}^d \) is said to be \( s \)-semi-hyperbolic on \( K \) if there exists a uniform splitting \( \mathbb{R}^d = E^s_x \oplus E^u_x \) with projectors \( P^s_x, P^u_x \) for each \( x \in K \) and a norm \( \| \cdot \| \) on \( \mathbb{R}^d \) such that

\[
\begin{align*}
\| P^s_{f(x)} Df_x u \| &\leq \lambda_s \| u \|, & u \in E^s_x, \\
\| P^u_{f(x)} Df_x v \| &\leq \mu_u \| v \|, & v \in E^u_x, \\
\| P^u_{f(x)} Df_x u \| &\leq \mu_u \| u \|, & u \in E^s_x, \\
\| P^u_{f(x)} Df_x v \| &\geq \lambda_u \| v \|, & v \in E^u_x,
\end{align*}
\]

where \( Df_x \) denotes the derivative of the mapping \( f \) at the point \( x \).

Comparing Definitions 3.1.5, Definition 2.1.5 clearly a hyperbolic diffeomorphism on \( K \) with expansivity parameter \( \lambda > 1 \) is \( s \)-semi-hyperbolic on \( K \) with the same splitting and split \( s = (\lambda^{-1}, \lambda, 0, 0) \), that is \( \mu_s = \mu_u = 0 \).

Since the splitting \( \mathbb{R}^d = E^s_x \oplus E^u_x \) in Definition 3.1.5 is uniform, then projectors \( P^s_x, P^u_x \) satisfy SH1 of Definition 3.1.1 for some constant \( h \). To stress this dependency on \( h \), the \( s \)-semi-hyperbolic map \( f : X \to \mathbb{R}^d \) may occasionally be called \((s, h)\)-semi-hyperbolic.

3.1.3 Lipschitz Mappings on Compact Sets

Turning to nonsmooth applications, the following generalization of Definition 2.1.5 to Lipschitz mappings \( f : X \to \mathbb{R}^d \) is given. Generalization of Definition 2.1.3 to Lipschitz mappings on a manifold, while possible is rather cumbersome and is omitted as it is not be used elsewhere below.

As in Definition 3.1.5, the subset \( K \) is a nonempty compact subset of \( X \) such that \( K \cap f(K) \neq \emptyset \) and a fixed norm is used.

**Definition 3.1.6.** Let \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) be a split and \( K \) a compact subset of an open set \( X \subseteq \mathbb{R}^d \). A Lipschitz mapping \( f : X \to \mathbb{R}^d \) is said to be \( s \)-semi-hyperbolic on \( K \) if there exists a splitting \( \mathbb{R}^d = E^s_x \oplus E^u_x \) on \( K \) with projectors \( P^s_x \) and \( P^u_x \) for each \( x \in K \), a norm \( \| \cdot \| \) on \( \mathbb{R}^d \) and a positive real number \( \delta \) such that

\[
\text{SH0(Lip): } \dim E^u_x = \dim E^u_y \text{ for all } x, y \in K \text{ with } \| f(x) - y \| \leq \delta;
\]
3.1 Definitions

SH1(Lip): \( \sup_{x \in K} \{ \| P_x^s \|, \| P_x^u \| \} \leq h; \)

SH2(Lip): The inclusion
\[
x + u + v \in X
\]
and the inequalities
\[
\begin{align*}
\| P_y^s (f(x + u + v) - f(x + \tilde{u} + v)) \| &\leq \lambda_s \| u - \tilde{u} \|, \quad (3.13) \\
\| P_y^u (f(x + u + v) - f(x + u + \tilde{v})) \| &\leq \mu_u \| v - \tilde{v} \|, \quad (3.14) \\
\| P_y^u (f(x + u + v) - f(x + \tilde{u} + v)) \| &\leq \mu_u \| u - \tilde{u} \|, \quad (3.15) \\
\| P_y^u (f(x + u + v) - f(x + u + \tilde{v})) \| &\geq \lambda_u \| v - \tilde{v} \|, \quad (3.16)
\end{align*}
\]
hold for all \( x, y \in K \) with \( \| f(x) - y \| \leq \delta \) and all \( u, \tilde{u} \in E_x^s \) and \( v, \tilde{v} \in E_x^u \) such that \( \| u \|, \| \tilde{u} \|, \| v \|, \| \tilde{v} \| \leq \delta \).

The first three inequalities in SH2(Lip) of are just local Lipschitz conditions on the projections of \( f \) while the last is an expansivity condition which implies local invertibility in the unstable direction of \( f \) at \( x \). As in the two differentiable cases above (Definitions 3.1.4 and 3.1.5), nonzero values of \( \mu_s \) and \( \mu_u \) allow possible leakage away from the equivariance of the splitting subspaces \( E_x^s \) and \( E_x^u \).

Remark 3.1.7. Condition SH0(Lip) of Definition 3.1.6 is more restrictive than in the uniform splitting Definition 3.1.1 while SH1(Lip) is the same as SH1.

Comparing the inequalities (3.13)–(3.16) with the corresponding inequalities (3.5)–(3.8) or (3.9)–(3.12), note that in the first case, Lipschitz mappings on compact manifolds, projectors \( P_y^s \) and \( P_y^u \) are considered for \( y \) from some neighborhood of the point \( f(x) \), while in (3.5)–(3.8) or (3.9)–(3.12) they are considered only at the point \( y = f(x) \). This additional restriction can be regarded as a weaker form of continuous dependence of splitting \( \mathbb{R}^d = E_x^s \oplus E_x^u \) in \( x \). See also Sect. 3.1.4 and Remark 3.1.11.

3.1.4 Continuous Splittings

Continuous dependence on \( x \) of the subspaces \( E_x^s \) and \( E_x^u \) of a splitting \( T_x M = E_x^s \oplus E_x^u \) of a compact smooth manifold \( M \), or of a splitting \( \mathbb{R}^m = E_x^s \oplus E_x^u \) of the space \( \mathbb{R}^m \) is a fundamental property of hyperbolic mappings, but does not follow automatically for semi-hyperbolic mappings.

Definition 3.1.8. A splitting \( T_x M = E_x^s \oplus E_x^u \) of a compact smooth manifold \( M \) is said to depend continuously on \( x \) if in a neighborhood of any point \( x_0 \in M \) a set of vectors

\[
e_1(x), e_2(x), \ldots, e_m(x) \in T_x M
\]
in local coordinates, depending continuously on \( x \), can be chosen such that the vectors \( e_1(x), e_2(x), \ldots, e_k(x) \) form a basis of the subspace \( E_x^s \subseteq T_x M \) and the vectors \( e_{k+1}(x), e_{k+2}(x), \ldots, e_m(x) \) form a basis of the subspace \( E_x^u \subseteq T_x M \).
For a splitting of the space $\mathbb{R}^m$ an alternative definition, which is sometimes more convenient to use, is:

**Definition 3.1.9.** The splitting $\mathbb{R}^m = E^s_x \oplus E^u_x$ is said to depend continuously on $x$ if $P^s_x$ and $P^u_x$ of (3.1) and (3.2) respectively, are continuous in $x$ as linear operator-valued functions.

Note that if the linear operator-valued functions $x \mapsto P^s_x$, $x \mapsto P^u_x$ of (3.1) and (3.2) are continuous in $x$ as linear operator-valued functions, neither the invariance of the set $K$ with respect to $f$, nor continuity in $x$ of the splitting subspaces $E^s_x, E^u_x$ and of the projectors $P^s_x, P^u_x$, are assumed in Definition 3.1.1 of uniform splitting.

For hyperbolic mappings continuity of the splitting is a direct consequence of the definition, but this does not always follow from the weaker assumptions of semi-hyperbolicity. In some cases continuity may hold and, as will be seen in later chapters, stronger results are obtained. The following definition applies for semi-hyperbolic mappings with respect to either a fixed norm or variable adapted norms.

**Definition 3.1.10.** A mapping $f$ is said to be continuously semi-hyperbolic on $K$ if it is $s$-semi-hyperbolic on $K$ as in any of Definitions 3.1.4–3.1.6 for some split $s$ for which the uniform splitting is continuous in $x$ on $K$ in the sense of Definitions 3.1.8 or 3.1.9.

**Remark 3.1.11.** A continuously $s$-semi-hyperbolic differentiable mapping is $s_\varepsilon$-semi-hyperbolic as in Definition 3.1.6 for $s_\varepsilon = (\lambda_s + \varepsilon, \lambda_u - \varepsilon, \mu_s + \varepsilon, \mu_u + \varepsilon)$ for any sufficiently small $\varepsilon > 0$ and an appropriate choice of $\delta = \delta(\varepsilon) > 0$.

Continuously semi-hyperbolic mappings form an open set in the class of all semi-hyperbolic mappings. A stronger result holds for Lipschitz semi-hyperbolic mappings and is stated below to illustrate the property overall.

Denote by $C(X, \mathbb{R}^d)$, $X \subseteq \mathbb{R}^d$, the Banach space of all bounded continuous mappings $f : X \to \mathbb{R}^d$ with norm

$$
\|f\|_C = \sup_{x \in X} \|f(x)\|.
$$

Let $\text{Lip}(X, \mathbb{R}^d)$, $X \subseteq \mathbb{R}^d$ be the Banach space of all bounded Lipschitz mappings $f : X \to \mathbb{R}^d$ with norm

$$
\|f\|_{\text{Lip}} = \|f\|_C + \sup_{x, y \in X, x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}.
$$

Denote by $\mathcal{O}_\varepsilon(K)$ the open $\varepsilon$-neighborhood of the subset $K$, that is

$$
\mathcal{O}_\varepsilon(K) = \bigcup_{x \in K} \text{Int} B(\varepsilon, x),
$$

where $\text{Int} B(\varepsilon, x)$ is the open ball of radius $\varepsilon$ centered at $x$. 
Lemma 3.1.12. Let $X \subseteq \mathbb{R}^d$ be an open set and let the mapping $f \in \text{Lip}(X, \mathbb{R}^d)$ be continuously semi-hyperbolic on a compact set $K \subset X$. Then there exists an $\eta = \eta(f, X) > 0$, a split $s$, constants $h, \delta$ and a uniform splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ such that $\overline{O}_\eta(K) \subset X$ and every mapping from the set

$$F_\eta = \{ g \in \text{Lip}(X, \mathbb{R}^d) : \| g - f \|_{\text{Lip}} \leq \eta \}$$

is semi-hyperbolic on $\overline{O}_\eta(K)$ with the split $s$, constants $h, \delta$ and the splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$.

Proof. Let $E^s_x \oplus E^u_x$ be a continuous splitting of $\mathbb{R}^d$, for the mapping $f$, satisfying Conditions SH0(Lip)–SH2(Lip) of Definition 3.1.6 with the split $s^*$ and positive constants $h^*$ and $\delta^*$.

For each $x \in X$ let $\pi(x)$ be any one of closest points in $K$ to $x$. Since $K$ is compact, such a function $\pi(x)$ is well defined, possibly nonuniquely, and generally will not be continuous. For what follows it is important that $\pi(x) = x$, $x \in K$, and $\| \pi(x) - x \| < \eta$, $x \in \overline{O}_\eta(K)$.

Extend the splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ to the whole of $X$ by defining

$$E^s_x = E^s_{\pi(x)}, \quad E^u_x = E^u_{\pi(x)}, \quad x \in X.$$  

Then the extended splitting

$$\mathbb{R}^d = E^s_x \oplus E^u_x \equiv E^s_{\pi(x)} \oplus E^u_{\pi(x)}, \quad x \in X,$$  

will satisfy Conditions SH0(Lip)–SH1(Lip) of Definition 3.1.6 with the constant $h = h^*$.

Let $\eta_0 > 0$ be such that $\overline{O}_{\eta_0}(K) \subset X$, take

$$\eta < \min \left\{ \eta_0, \frac{1}{2} \delta^*, h^{-1} \nu(s) \right\}$$

where $\nu(s)$ is as in (3.3), and set

$$\delta = \delta^* - 2\eta, \quad s = \{ \lambda^*_s + h\eta, \lambda^*_u - h\eta, \mu^*_s + h\eta, \mu^*_u + h\eta \}$$

Then each mapping $g$ in the set $F_\eta$ satisfies Condition SH2(Lip) of Definition 3.1.6 on the compact set $\overline{O}_\eta(K)$ with the splitting (3.17), the split $s$ and constants $h, \delta$.

Remark 3.1.13. In spite of continuity of the splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ for $f$ in Lemma 3.1.12, the splitting (3.17) is generally not continuous. Continuity in (3.17) may be possible, for example, when the compact $K$ is a retract of a neighborhood of itself.
3.2 Examples

All diffeomorphisms with a hyperbolic invariant set are semi-hyperbolic on that set provided the appropriate corresponding definitions are used. Consequently, mappings which are semi-hyperbolic, but not hyperbolic, illustrate the significance of the theory, especially when the invariant set is uncountable.

3.2.1 Elementary Examples

Differential dynamics concentrates on the behavior of a diffeomorphism within an invariant set. When dynamical behavior of a mapping is approximated by a computational method, what occurs close to the invariant set is also of some interest, particularly when the set quite simple. Consider, for example, the problem of replicating a phase portrait about a saddle point using finite machine arithmetic. The shadowing results for semi-hyperbolic mappings in Chap. 6 are useful here, while those for hyperbolic diffeomorphisms are not applicable.

Example 3.2.1. Let $A$ be a hyperbolic matrix, that is $|\lambda_i| \neq 1$ for the eigenvalues $\lambda_1, \ldots, \lambda_d$, and consider the splitting

$$\mathbb{R}^d = E^s \oplus E^u$$

(3.18)

where $E^s$ is the eigenspace of $A$ of the eigenvalues satisfying $|\lambda_i| < 1$ and $E^u$ the eigenspace of the eigenvalues satisfying $|\lambda_i| > 1$. For simplicity of exposition, let $d = 2$. The linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x) = Ax$ is hyperbolic on the singleton set $\{0\}$ with splitting (3.18), but is only semi-hyperbolic on, say, the unit disk $D_2$ since $D_2$ is not invariant under $f$. The splitting on $D_2$ is obtained by translating that at $\{0\}$ to each point $x \in D_2$, and is thus both uniform and continuous.

Nonsmooth perturbations of hyperbolic mappings are also another source of examples of semi-hyperbolic mappings. These are easily described as in the following example where the hyperbolic set is a saddle point.

Example 3.2.2. Let $A$ be a $2 \times 2$ hyperbolic matrix as in Example 3.2.1 and consider a nonlinear mapping $f_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ obtained as a perturbation of the linear mapping $f_0(x) = Ax$, such as

$$f_\varepsilon \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = A \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) + \left( \varepsilon |x_1| \varepsilon |x_2| \right)$$

for all $x = (x_1, x_2)^T \in \mathbb{R}^2$ and some sufficiently small $\varepsilon > 0$. This mapping is Lipschitz everywhere, but is not differentiable at the origin. It is semi-hyperbolic in any set containing the origin in its interior with $E^s_x = \{0\} \oplus \mathbb{R}^1$ and $E^u_x = \mathbb{R}^1 \oplus \{0\}$ for all $x$. 
Mappings defined by switching between members of a set of maps, often linear, on adjoining subsets are common in electronic and control systems and can behave very erratically. The switching points or sets are not necessarily semi-hyperbolic.

Example 3.2.3. The piecewise linear mapping $f : \mathbb{R}^1 \to \mathbb{R}^1$ with

$$f(x) = \begin{cases} \frac{1}{2}x, & x < 0, \\ 4x, & x \geq 0 \end{cases}$$

is not semi-hyperbolic on any set $K$ containing the switching point $x = 0$ because there is no appropriate splitting of $\mathbb{R}^1$ at $x = 0$ satisfying the inequalities of Condition SH2(Lip). Each component mapping is semi-hyperbolic on its subset of definition, with the splitting $E_x^s \oplus E_x^u = \{0\} \oplus \mathbb{R}^1$ for $x < 0$ and with $E_x^s \oplus E_x^u = \mathbb{R}^1 \oplus \{0\}$ for $x \geq 0$.

The piecewise linear mapping $f : \mathbb{R}^1 \to \mathbb{R}^1$ with

$$f(x) = \begin{cases} \frac{1}{2}x, & x < 0, \\ 4x, & x \geq 0 \end{cases}$$

is also not semi-hyperbolic on any set $K$ containing the switching point $x = 0$, but any iterate $f^j$, $j \geq 2$, is semi-hyperbolic. An admissible splitting is $E_x^s \oplus E_x^u = \{0\} \oplus \mathbb{R}^1$ for $x \in K$ and split $s$ is

$$(\lambda_s, \lambda_u, \mu_s, \mu_u) = (0, 2^{2j-3}, 0, 0).$$

Here $P_x^s$ is the zero mapping and $P_x^u$ the identity mapping.

Observe that the tent mapping

$$f(x) = 1 - |1 - 2x|,$$

which is Lipschitz, is not semi-hyperbolic on any subset of $\mathbb{R}^1$ containing the point $x = \frac{1}{2}$.

3.2.2 Anosov Endomorphisms

Examples of semi-hyperbolic mappings with a uniform splitting that is either not continuous or lacks equivariance of a splitting for a hyperbolic mapping are more difficult to describe explicitly. In the following examples, algebraic Anosov automorphisms and noninvertible Anosov endomorphisms are introduced and an example with the desired properties is constructed as a differentiable perturbation of an Anosov endomorphism on a torus. Structural stability is not contradicted here since neither the endomorphism which is being perturbed nor the perturbations themselves are diffeomorphisms, although differentiable. Write the elements of $\mathbb{R}^d$ as vectors $x$ with coordinates
Let $x_1, x_2, \ldots, x_d$ and let $\mathbb{T}^d$ be the standard $d$-dimensional torus, that is the quotient $\mathbb{R}^d/\mathbb{Z}^d$. $\mathbb{T}^d$ is a closed differentiable manifold in $\mathbb{R}^d$ with respect to the locally Euclidean metric

$$\rho(x, y) = \sqrt{|x_1 - y_1|^2_{\text{mod } 1} + |x_2 - y_2|^2_{\text{mod } 1} + \cdots + |x_d - y_d|^2_{\text{mod } 1}}$$
on T^d$ where

$$|t - s|_{\text{mod } 1} = \min \{|t - s + 2k|: k = 0, \pm 1\}, \quad 0 \leq t, s < 1.$$ The tangent space $T_x \mathbb{T}^d$ can then be identified with $\mathbb{R}^d$ by an appropriate choice of natural coordinates generated by those of $\mathbb{R}^d$, so $T_x \mathbb{T}^d = \mathbb{R}^d$ for each $x \in \mathbb{T}^d$.

Denote the natural projection from $\mathbb{R}^d$ onto $\mathbb{T}^d$ by

$$\Pi(x) = (x_1 \mod 1, x_2 \mod 1, \ldots, x_d \mod 1), \quad x = (x_1, x_2, \ldots, x_d)$$

and consider the mapping $f: \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$f(x) = \Pi(Ax) \quad (3.19)$$

for some fixed $d \times d$ matrix $A$ with integer components $a_{ij}$.

Further, suppose that the matrix $A$ is hyperbolic and $\mathbb{R}^d = E^s \oplus E^u$ is the splitting of Example 3.2.1. Since the eigenvalues of the restriction $A|_{E^s}$ lie inside the unit circle while those of the restriction $A|_{E^u}$ lie outside the unit circle, there exist constants $C > 0$ and $\lambda > 1$ such that

$$\|A^n u\| \leq C \lambda^{-n} \|u\|, \quad \|A^n v\| \geq C^{-1} \lambda^n \|v\| \quad (3.20)$$

for all $u \in E^s$ and $v \in E^u$, where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$.

**Example 3.2.4 (Anosov Automorphisms and Endomorphisms).** Let $A$ be an invertible hyperbolic matrix with integer components. The mapping $f$ defined by (3.19) is called an *algebraic hyperbolic automorphism* of the torus $\mathbb{T}^d$ if $|\det A| = 1$ and an *algebraic hyperbolic endomorphism* of $\mathbb{T}^d$ otherwise.

An algebraic hyperbolic automorphism of a torus $\mathbb{T}^d$ is clearly a globally invertible mapping and is a diffeomorphism on $\mathbb{T}^d$, but an algebraic hyperbolic endomorphism is generally not a diffeomorphism since it is only locally invertible.

Since $T_x \mathbb{T}^d$ has been identified with $\mathbb{R}^d$, the tangent mapping $Tf_x$ can be identified with $A$ and the linear subspaces $E^s_x = E^s, E^u_x = E^u$ for $x \in \mathbb{T}^d$ are invariant under $Tf_x \equiv A$. Thus

$$T_x \mathbb{T}^d = \mathbb{R}^d = E^s \oplus E^u = E^s_x \oplus E^u_x \quad (3.21)$$

is an equivariant splitting for the tangent mapping $Tf_x$.

Let $P^s_x$ and $P^u_x$ be projectors corresponding to the splitting (3.21) (see Definition 3.1.1) and for each $x \in \mathbb{T}^d$ define a norm
\[ \|v\|_x = \max_{n \geq 0} \lambda^n \|A^n P_s x v\| + \min_{n \geq 0} \lambda^{-n} \|A^n P_u x v\|, \]

on \( T_x \mathbb{T}^d = \mathbb{R}^d \), where \( \lambda \) is as in (3.20) (in fact, this norm \( \|\cdot\|_x \) does not depend on \( x \)). Then

\[ \|Tf_x u\|_x \leq \lambda^{-1} \|u\|_x, \quad \|Tf_x v\|_x \geq \lambda \|v\|_x \quad (3.22) \]

for all \( u \in E^s_x, v \in E^u_x \) and \( x \in K \), that is Conditions H1* and H2* of Definition 2.1.3 are valid for the endomorphism \( f \) on the torus \( \mathbb{T}^d \); if the mapping \( f \) is an automorphism, then it is a hyperbolic diffeomorphism on \( \mathbb{T}^d \).

The relations (3.21) and (3.22) for the Anosov hyperbolic endomorphism \( f \) defined in Example 3.2.4 can be rewritten in the form

\[
\begin{align*}
\|P_s f(x) T f_x u\|_{f(x)} &\leq \lambda^{-1} \|u\|_x, \quad u \in E^s_x, \\
\|P_s f(x) T f_x v\|_{f(x)} &= 0, \quad v \in E^u_x, \\
\|P_u f(x) T f_x v\|_{f(x)} &= 0, \quad u \in E^s_x, \\
\|P_u f(x) T f_x v\|_{f(x)} &\geq \lambda \|v\|_x, \quad v \in E^u_x.
\end{align*}
\]

By Definition 3.1.4 this means that every Anosov hyperbolic endomorphism \( f \) is semi-hyperbolic on the torus \( \mathbb{T}^d \). This property of semi-hyperbolicity is robust under small \( C^1 \)-perturbations, as is shown in the more general context of Theorem 5.2.1 in Chap. 5.

**Lemma 3.2.5.** Given an Anosov hyperbolic endomorphism \( f : \mathbb{T}^d \rightarrow \mathbb{T}^d \), there exists an \( \varepsilon > 0 \) such that every differentiable mapping \( f_\varepsilon : \mathbb{T}^d \rightarrow \mathbb{T}^d \) which is \( \varepsilon \)-close to the \( f \) in \( C^1 \)-metric is semi-hyperbolic.

This extends a classical result of Anosov on the robustness of the hyperbolicity property when \( f \) is an automorphism, that is, an invertible endomorphism, where every nearby diffeomorphism \( f_\varepsilon : \mathbb{T}^d \rightarrow \mathbb{T}^d \) is hyperbolic and in particular has a hyperbolic splitting. Theorem 5.1.1 will show that invertibility is essential in this case.

It is sometimes more convenient to consider semi-hyperbolic mappings on a compact subset of \( \mathbb{R}^d \) rather than on a manifold. This is also possible for Anosov mappings and their perturbations, but the transition from one approach to the other is often not trivial in practice. The following example illustrates the procedure for one particular mapping, and the general construction is later used in the proof of Theorem 5.1.1.

**Example 3.2.6.** Consider the \( 3 \times 3 \) matrix

\[
A = \begin{bmatrix}
2 & 3 & 0 \\
1 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

which has eigenvalues...
\[ \lambda_1 = 2 - \sqrt{3}, \quad \lambda_2 = 2 + \sqrt{3}, \quad \lambda_3 = 2 \]

with corresponding eigenvectors
\[ v_1 = (1, -1/\sqrt{3}, 0), \quad v_2 = (1, 1/\sqrt{3}, 0), \quad v_3 = (0, 0, 1), \]
respectively. Denote by \( f : T^3 \to T^3 \) the continuously differentiable mapping defined by \( f(x) = \Pi(Ax) \).

Interpret a point \( z \in \mathbb{R}^6 \) as an ordered triplet \((z_1, z_2, z_3)\) of complex numbers with the norm \( \|z\| = \sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2} \) and let \( \mathcal{J} : T^3 \to \mathbb{R}^6 \) be the immersion which maps the point \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \in T^3 \) to the point
\[ z = (e^{2\pi i \varphi_1}, e^{2\pi i \varphi_2}, e^{2\pi i \varphi_2}) \in K, \]
where
\[ K = \{ z \in \mathbb{R}^6 : |z_1| = |z_2| = |z_3| = 1 \}. \]

Consider the mapping
\[ F(z) = f(P(z)), \quad z_1, z_2, z_3 \neq 0, \]
where \( P \) is the natural projection on \( K \) defined by
\[ P(z) = \left( \frac{z_1}{|z_1|}, \frac{z_2}{|z_2|}, \frac{z_3}{|z_3|} \right), \quad z_1, z_2, z_3 \neq 0. \]

The restriction of the mapping \( F \) to \( K \) is then topologically conjugate by the immersion \( \mathcal{J} \) to the mappings \( f \) and \( f_\varepsilon \), respectively.

It is not easy to find an explicit expression for \( F \), but its restriction to \( K \) has the form
\[ F(z) = (z_1^{a_{11}} z_2^{a_{12}} z_3^{a_{13}}, z_1^{a_{21}} z_2^{a_{22}} z_3^{a_{23}}, z_1^{a_{31}} z_2^{a_{32}} z_3^{a_{33}}), \quad z \in K, \]
where the \( a_{ij} \) are the components of the matrix \( A \) generating the Anosov endomorphism \( f \).

For each point \( z \in K \) the sets of vectors
\[ e_1^t(z) = \left( iz_1, -\frac{i}{\sqrt{3}} z_2, 0 \right), \quad e_2^t(z) = \left( iz_1, \frac{i}{\sqrt{3}} z_2, 0 \right), \quad e_3^t(z) = (0, 0, iz_3), \]
\[ e_1^n(z) = (z_1, 0, 0), \quad e_2^n(z) = (0, z_2, 0), \quad e_3^n(z) = (0, 0, z_3) \]
are a basis for \( \mathbb{R}^6 \), for which the differential \( DF_z \) of \( F \) satisfies
\[
\begin{align*}
DF_z e_1^t(z) & = (2 - \sqrt{3}) e_1^t(F(z)), \\
DF_z e_2^t(z) & = (2 + \sqrt{3}) e_2^t(F(z)), \\
DF_z e_3^t(z) & = 2 e_3^t(F(z)), \\
DF_z e_1^n(z) & = DF_z e_2^n(z) = DF_z e_3^n(z) = 0.
\end{align*}
\]
Consider the splitting $T_z\mathbb{R}^6 = E^s_z \oplus E^u_z$ for $z \in K$ with subspaces

\[ E^s_z = \text{span}\{e^t_1(z), e^n_1(z), e^n_2(z), e^n_3(z)\}, \]

\[ E^u_z = \text{span}\{e^t_2(z), e^t_3(z)\} \]

with corresponding projectors $P^s_z$ and $P^u_z$ satisfying

\[ P^s_z \mathbb{R}^6 = E^s_z, \quad P^s_z E^u_z = 0, \quad P^u_z \mathbb{R}^6 = E^u_z, \quad P^u_z E^s_z = 0. \]

The relations (3.23) imply that the splitting $T_z\mathbb{R}^6 = E^s_z \oplus E^u_z$ is invariant for the differential $DF_z$. On the other hand, the inequalities

\[ \|P^s_{F(z)} DF_z u\| \leq (2 - \sqrt{3})\|u\|, \quad u \in E^s_z, \]

\[ \|P^s_{F(z)} DF_z v\| = 0, \quad v \in E^u_z, \]

\[ \|P^u_{F(z)} DF_z v\| = 0, \quad u \in E^s_z, \]

\[ \|P^u_{F(z)} DF_z v\| \geq 2\|v\|, \quad v \in E^u_z, \]

hold, so the differentiable mapping $F$ is both hyperbolic and semi-hyperbolic on the compact subset set $K$ of $\mathbb{R}^6$ in the sense of Definitions 2.1.5, 3.1.5.
Semi-Hyperbolic Sequences of Matrices

The utility of first approximation methods in investigating the properties of semi-hyperbolic mappings and their trajectories naturally leads to consideration of linear operators in spaces of sequences generated by derivatives of semi-hyperbolic mappings. Some elementary properties of such linear operators that will be needed, in this and later chapters, are considered here.

Throughout this chapter $\| \cdot \|$ will denote a fixed but otherwise arbitrary norm on $\mathbb{R}^d$ and $\ell^\infty(I, \mathbb{R}^d)$ will denote the space of all bounded sequences of vectors $x_n \in \mathbb{R}^d$ with indices $n$ taking values in a subset, or interval $I$ in $\mathbb{Z}$ which can be finite, uni- or bi-directionally infinite depending on context. The norm on $\ell^\infty(I, \mathbb{R}^d)$ is defined as

$$\| \{x_n\} \|_\infty = \sup_{n \in I} \|x_n\|.$$  

4.1 The Split Matrix

Recall from Definition 3.1.2 that a split is a four-tuple $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ of nonnegative real numbers for which

$$\lambda_s < 1 < \lambda_u, \quad (1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u. \quad (4.1)$$

Recall also that the split $s$ is positive if all of the numbers $\lambda_s, \lambda_u, \mu_s$ and $\mu_u$ are positive. Clearly, the splits in Definitions 3.1.4, 3.1.5 and 3.1.6 can also have positive counterparts.

For a given positive split $s$ define the $2 \times 2$ split matrix $M(s)$ by

$$M(s) := \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{bmatrix}, \quad (4.2)$$

with spectral radius $\sigma(s)$
\[ \sigma(s) = \frac{1}{2} \left( \frac{1}{\lambda_u} + \lambda_s + \sqrt{\left( \frac{1}{\lambda_u} - \lambda_s \right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}} \right). \tag{4.3} \]

A simple but cumbersome calculation shows that
\[ \sigma(s) < 1 - \nu(s) < 1 \tag{4.4} \]
where
\[ \nu(s) = \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s\mu_u}{\lambda_u - \lambda_s + \mu_s + \mu_u} \]
(see (3.3)). Moreover, since the entries of the matrix \( M(s) \) are positive for a positive split \( s \), it follows by the Perron–Frobenius Theorem that \( \sigma(s) \) is the maximal eigenvalue of \( M(s) \) and that the corresponding eigenvector has positive components. This eigenvector will be written here as \( (1, \gamma(s))^T \), where
\[ \gamma(s) := \frac{1}{2\mu_s} \left( \frac{1}{\lambda_u} - \lambda_s + \sqrt{\left( \frac{1}{\lambda_u} - \lambda_s \right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}} \right). \tag{4.5} \]

When the split \( s \) is fixed, we shall write \( M = M(s) \), \( \gamma = \gamma(s) \) and \( \sigma = \sigma(s) \), so
\[ M \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u/\lambda_u & 1/\lambda_u \end{bmatrix} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \sigma \begin{pmatrix} 1 \\ \gamma \end{pmatrix}, \tag{4.6} \]
and introduce the norm \( \| \cdot \|_* \) on \( \mathbb{R}^2 \) defined by
\[ \|(y_1, y_2)\|_* := \max \{ \gamma |y_1|, |y_2| \}. \]
The corresponding norm \( \|M\|_* \) of the linear operator with the matrix (4.2) clearly coincides with the spectral radius \( \sigma \) of \( M \), so \( \|M\|_* = \sigma < 1 \) by (4.4) and \( \|Mx\|_* \leq \sigma \|x\|_* \) for all \( x \in \mathbb{R}^2 \).

**Lemma 4.1.1.** The following hold
\[ \min \{ \gamma, 1 \} \|(y_1, y_2)\|_\infty \leq \|(y_1, y_2)\|_* \leq \max \{ \gamma, 1 \} \|(y_1, y_2)\|_\infty. \]

**Proof.** By direct calculation. \( \square \)

The next lemma will play an important role in what follows.

**Lemma 4.1.2.** For \( m = 1, 2, \ldots \) let the following inequalities hold:
\[ a_m^s \leq \lambda_s a_{m-1}^s + \mu_s a_{m-1}^u + h^s, \]
\[ a_m^u \leq \frac{\mu_u}{\lambda_u} a_{m-1}^s + \frac{1}{\lambda_u} a_{m-1}^u + h^u, \tag{4.7} \]
with \( a_m^s, a_m^u, h^s, h^u \geq 0 \) and \( a_0^s = a_0^u = 0 \). Then
\[ \limsup_{m \to \infty} (a_m^s + a_m^u) \leq \frac{\max \{ h^s, \lambda_u h^u \}}{\nu(s)}. \tag{4.8} \]
Proof. Define vectors \( a_m = (a^s_m, a^u_m)^T \) for \( m = 0, 1, \ldots \) and \( h = (h^s, h^u)^T \). Then the inequalities (4.7) can be rewritten in the vector form

\[
\begin{align*}
a_m & \leq M a_{m-1} + h, & m = 1, 2, \ldots,
\end{align*}
\]

(4.9)

where \( a_0 = 0 \) and the inequality between vectors is interpreted component-wise. Since the entries of the matrix \( M = M(s) \) are non-negative, we preserve inequality direction when applying the matrix \( M \) to the both sides of inequality (4.9). Hence

\[
\begin{align*}
a_m & \leq M^m a_0 + (I + M + \cdot + M^{m-1}) h, & m = 1, 2, \ldots,
\end{align*}
\]

from which, in view of equality \( a_0 = 0 \), we obtain

\[
\limsup_{m \to \infty} a_m \leq (I - M)^{-1} h
\]

(4.10)

where the \( \limsup \) and inequality apply to the respective components. By direct calculation

\[
(I - M)^{-1} = \frac{1}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} \begin{bmatrix} \lambda_u - 1 & \mu_s \lambda_u \\ \mu_u & (1 - \lambda_s) \lambda_u \end{bmatrix},
\]

so (4.10) can be rewritten componentwise as

\[
\begin{align*}
\limsup_{m \to \infty} a^s_m & \leq \frac{(\lambda_u - 1) h^s + \mu_s \lambda_u h^u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}, \\
\limsup_{m \to \infty} a^u_m & \leq \frac{\mu_u h^s + (1 - \lambda_s) \lambda_u h^u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u},
\end{align*}
\]

from which inequality (4.8) is an immediate consequence. \( \square \)

4.1.1 A Perturbation Theorem

Various proofs for semi-hyperbolic mappings will involve a matrix or a sequence of matrices that have similar semi-hyperbolic properties. This motivates the next definition where \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) is an arbitrary but otherwise fixed split.

**Definition 4.1.3 (Semi-Hyperbolic Matrix).** A \( d \times d \) matrix \( A \) is called an \( (s, h) \)-semi-hyperbolic matrix if the corresponding linear mapping is \( s \)-semi-hyperbolic on \( \mathbb{R}^d \) with respect to a splitting \( \mathbb{R}^d = E^s \oplus E^u \) and projectors \( P^s \) and \( P^u \) satisfying \( \|P^s\| \leq h \) and \( \|P^u\| \leq h \).

The following Perturbation Theorem, which asserts that the class of semi-hyperbolic matrices is an open set in the space of all matrices, indicates the robustness of semi-hyperbolic matrices and provides explicit estimates of possible perturbations which preserve the semi-hyperbolicity of a matrix.
Theorem 4.1.4 (Perturbation Theorem). Let a \( d \times d \) matrix \( A \) be \((s, h)\)-hyperbolic. Then every \( d \times d \) matrix \( \tilde{A} \) satisfying
\[
\|\tilde{A} - A\| \leq \delta < \frac{\nu(s)}{h}
\]
is \((\tilde{s}, h)\)-hyperbolic with the split \( \tilde{s} = (\lambda_s + \delta h, \lambda_u - \delta h, \mu_s + \delta h, \mu_u + \delta h) \).

Proof. In view of \((s, h)\)-hyperbolicity of \( A \) there is a decomposition \( \mathbb{R}^d = E^s \oplus E^u \) with corresponding projections \( P^s : \mathbb{R}^d \to E^s \) and \( P^u = I - P^s : \mathbb{R}^d \to E^u \) such that
\[
\|P^s\| \leq h, \quad \|P^u\| \leq h
\]
and
\[
\|P^s AP^s x\| \leq \lambda_s \|P^s x\|,
\|P^s AP^u x\| \leq \mu_s \|P^u x\|,
\|P^u AP^s x\| \leq \mu_u \|P^s x\|,
\|P^u AP^u x\| \geq \lambda_u \|P^u x\|,
\]
for all \( x \in \mathbb{R}^d \). Hence
\[
\|P^s \tilde{A} P^s x\| \leq \|P^s (\tilde{A} - A) P^s x\| + \|P^s AP^s x\| \leq (\delta h + \lambda_s) \|P^s x\|
\]
and
\[
\|P^u \tilde{A} P^u x\| \geq \|P^u AP^u x\| - \|P^u (\tilde{A} - A) P^u x\| \geq (\lambda_u - \delta h) \|P^u x\|.
\]
Similarly,
\[
\|P^s \tilde{A} P^u x\| \leq (\mu_s + \delta h) \|P^u x\|,
\|P^u \tilde{A} P^s x\| \leq (\mu_u + \delta h) \|P^s x\|.
\]
Define
\[
\tilde{\lambda}_s = \lambda_s + \delta h, \quad \tilde{\lambda}_u = \lambda_u - \delta h, \quad \tilde{\mu}_s = \mu_s + \delta h, \quad \tilde{\mu}_u = \mu_u + \delta h.
\]
By Lemma 3.1.3 these numbers form a split \( \tilde{s} \), which completes the proof. \( \square \)

4.2 Semi-Hyperbolicity Implies Hyperbolicity

In this section some properties of sequences of semi-hyperbolic mappings, required later, are investigated. Let \( \{A_n\} \) be a sequence of \( d \times d \) matrices defined for \( n \in \mathbb{I} \), a given interval of \( \mathbb{Z} \) which could be finite or infinite. The main objective of this section is to show that semi-hyperbolicity of a matrix sequence \( \{A_n\} \) implies the existence of an equivariant splitting \( \mathbb{R}^d = \tilde{E}_n^s \oplus \tilde{E}_n^u \).
Definition 4.2.1 (Semi-Hyperbolic Sequence of Matrices). A bounded sequence of $d \times d$ matrices $\{A_n\}$ will be called $(s, h)$-semi-hyperbolic if for each matrix $A_n$, $n \in \mathbb{I}$, there is a splitting $\mathbb{R}^d = E^s_n \oplus E^u_n$ with
\[
\dim E^s_n = \dim E^s_{n+1}, \quad \dim E^u_n = \dim E^u_{n+1} \tag{4.11}
\]
and projections $P^s_n : \mathbb{R}^d \to E^s_n$, $P^u_n = I - P^s_n : \mathbb{R}^d \to E^u_n$, with the uniform matrix norm bounds
\[
\|P^s_n\| \leq h, \quad \|P^u_n\| \leq h, \quad n \in \mathbb{I}, \tag{4.12}
\]
such that
\[
\|P^s_{n+1} A_n P^s_n x\| \leq \lambda_s \|P^s_n x\|, \\
\|P^s_{n+1} A_n P^u_n x\| \leq \mu_s \|P^u_n x\|, \\
\|P^u_{n+1} A_n P^s_n x\| \leq \mu_u \|P^s_n x\|, \\
\|P^u_{n+1} A_n P^u_n x\| \geq \lambda_u \|P^u_n x\|, \tag{4.13}
\]
for all $n \in \mathbb{I}$ and $x \in \mathbb{R}^d$.

Conditions (4.11)–(4.13) are the obvious analogs of Conditions SH0–SH1 in Definition 3.1.1 of a uniform splitting and of Condition SH2 in the Definitions 3.1.4 and 3.1.5 for a semi-hyperbolic mapping. With the index $n$ dropped, the projection bounds (4.12) and the inequalities (4.13) apply for a semi-hyperbolic matrix as in Definition 4.1.3.

Henceforth let $\{A_n\}$ be an $(s, h)$-semi-hyperbolic bi-sequence of invertible and uniformly bounded $d \times d$ matrices (cf. Definition 4.2.1, with $\mathbb{I} = \mathbb{Z}$ here).

The inequalities (4.13) can be interpreted as conditions that, roughly speaking, ensure the existence of almost equivariant splittings $\mathbb{R}^d = E^s_n \oplus E^u_n$ for the sequence $\{A_n\}$ with contracting subspaces $E^s_n$ and expanding subspaces $E^u_n$. Here ‘almost’ indicates that the subspaces $A_n E^s_{n+1}$ and $A_n E^u_{n+1}$ are only close to $E^s_n$ and $E^u_n$, respectively, but generally do not coincide with them.

4.2.1 Sequence Operators $\mathcal{R}$ and $\mathcal{L}$

For each integer $k \in \mathbb{Z}$, define on the sequence space $\ell^\infty([k, \infty), \mathbb{R}^d)$ the norm
\[
\|x\|_\infty^* = \sup_{n \geq k} \{\|P^s_n x_n\|, \|P^u_n x_n\|\},
\]
which is clearly equivalent to the norm $\|\cdot\|_\infty$. Similarly, on the sequence space $\ell^\infty((-\infty, k], \mathbb{R}^d)$, define the norm
\[
\|x\|_\infty^* = \sup_{n \leq k} \{\|P^s_n x_n\|, \|P^u_n x_n\|\}.
\]

Define the operator
with parameters $\alpha > 0$, $k \in \mathbb{Z}$ and $v \in P^s_k \mathbb{R}^d$, mapping a sequence $x = \{x_n\} \in \ell^\infty([k, \infty), \mathbb{R}^d)$ to the sequence $v = \{v_n\} \in \ell^\infty([k, \infty), \mathbb{R}^d)$ defined by

$$
P^s_k v_k = v, \\
P^u_n v_n = \alpha P^s_n A_{n-1} x_{n-1}, \\
P^u_n v_n = \alpha^{-1}(P^u_{n+1} A_n P^u_n)^{-1} P^u_{n+1} x_{n+1} - (P^u_{n+1} A_n P^u_n)^{-1} P^u_{n+1} A_n P^n_s x_n, \\
\text{for } n \geq k + 1,
$$

(4.14)

From the split inequalities (4.1) there exists $\omega > 0$ such that for all $\alpha \in [1 - \omega, 1 + \omega]$ the strict inequalities

$$
\alpha \lambda_s < 1 < \alpha \lambda_u, \quad (1 - \alpha \lambda_s)(\alpha \lambda_u - 1) > \alpha^2 \mu_s \mu_u
$$

(4.15)

hold. Hence

$$
\rho = \sup_{\alpha \in [1 - \omega, 1 + \omega]} \left\{ \frac{\alpha \mu_u + \alpha \lambda_u - 1}{\alpha \lambda_u - 1}, \frac{1 - \alpha \lambda_s + \alpha \mu_s}{1 - \alpha \lambda_s} \right\}
$$

is well defined and for all $\alpha \in [1 - \omega, 1 + \omega]$ the spectral radius $\sigma(M_\alpha)$ of the matrix

$$
M_\alpha = \begin{bmatrix}
\alpha \lambda_s & \alpha \mu_s \\
\mu_u / \lambda_u & 1 / (\alpha \lambda_u)
\end{bmatrix}
$$

is, from (4.3),

$$
\sigma(M_\alpha) = \frac{1}{2} \left( \frac{1}{\alpha \lambda_u} + \alpha \lambda_s + \sqrt{\left( \frac{1}{\alpha \lambda_u} - \alpha \lambda_s \right)^2 + \frac{4 \alpha \mu_s \mu_u}{\lambda_u}} \right) < 1.
$$

Since the entries of the matrix $M_\alpha$ are positive (recall that we have restricted ourselves to positive splits $s$), it follows by the Perron–Frobenius Theorem that the spectral radius $\sigma(M_\alpha)$ is an eigenvalue and that the corresponding eigenvector has positive coordinates. Without loss of generality this vector has the form $(1, \gamma_\alpha)^T$. Now, note that for each $\alpha \in [1 - \omega, 1 + \omega]$ the norm

$$
\|x\|_{\infty}^\alpha = \sup_{n \geq k} \left\{ \gamma_\alpha \|P^n_s x_n\|, \|P^n_u x_n\| \right\}
$$

on the space $\ell^\infty([k, \infty), \mathbb{R}^d)$ is equivalent to the norm $\| \cdot \|_\infty$.

**Lemma 4.2.2.** For any $\alpha \in [1 - \omega, 1 + \omega]$, any integer $k$ and any $v \in P^s_k \mathbb{R}^d$, the operator $\mathbb{R}_{[\alpha, k, v]}$ is contracting with constant $\sigma(M_\alpha)$ in the norm $\| \cdot \|_\infty$. Moreover, the set

$$
\mathcal{Y}_{[\alpha, k, v]} = \left\{ x : \|P^n_s x_n\| \leq \|v\|, \|P^n_u x_n\| \leq \frac{\alpha \mu_u}{\alpha \lambda_u - 1} \|v\|, \quad n \geq k \right\}
$$

(4.16)

is invariant under $\mathbb{R}_{[\alpha, k, v]}$. 
Corollary 4.2.3. For any \( v \in P^s R^d \) the operator \( R[\alpha, k, v] \) has a unique fixed point \( x^* = x^*_{[\alpha, k, v]} = \{ x^*_n \} \) for which \( P^s_k x^*_n = v \) and

\[
\| P^s_n x^*_n \| \leq \| v \|, \quad \| P^u_n x^*_n \| \leq \frac{\alpha \mu_u}{\alpha \lambda_u - 1} \| v \|, \quad n \geq k.
\]

Corollary 4.2.3 follows immediately from Lemma 4.2.2 by the Banach Contraction Mapping Theorem, so it suffices to prove Lemma 4.2.2.

Proof. Fix a value of \( \alpha \in [1 - \omega, 1 + \omega] \). The definition of the vector \((1, \gamma_\alpha)^T\) gives the equality

\[
\left[ \frac{\alpha \lambda_s}{\mu_u / \lambda_u} \frac{\alpha \mu_s}{1/(\alpha \lambda_u)} \right] \left( \frac{1}{\gamma_\alpha} \right) = \sigma(M_\alpha) \left( \frac{1}{\gamma_\alpha} \right),
\]

from which it follows that

\[
\alpha \lambda_s + \alpha \mu_s \gamma_\alpha = \sigma(M_\alpha), \quad \frac{\mu_u}{\lambda_u} + \frac{1}{\alpha \lambda_u} \gamma_\alpha = \sigma(M_\alpha) \gamma_\alpha. \quad (4.17)
\]

Given arbitrary \( x, \tilde{x} \in \ell^\infty([k, \infty), R^d) \) write

\[
v = R[\alpha, k, v](x), \quad \hat{v} = R[\alpha, k, v](\tilde{x}).
\]

Then from (4.14) it follows that

\[
P^s_k(v_k - \tilde{v}_k) = 0,
\]

\[
P^s_n(v_n - \tilde{v}_n) = \alpha P^s_n A_{n-1}(x_{n-1} - \tilde{x}_{n-1}), \quad n \geq k + 1,
\]

\[
P^u_n(v_n - \tilde{v}_n) = \alpha^{-1} \left(P^u_{n+1} A_n P^u_n\right)^{-1} P^u_{n+1}(x_{n+1} - \tilde{x}_{n+1})
- \left(P^u_{n+1} A_n P^u_n\right)^{-1} P^u_{n+1} A_n P^s_n(x_n - \tilde{x}_n), \quad n \geq k.
\]

From (4.13) the following relations hold.

\[
\| P^s_k(v_k - \tilde{v}_k) \| = 0,
\]

\[
\| P^s_n(v_n - \tilde{v}_n) \| \leq \alpha \lambda_s \| P^s_n(x_{n-1} - \tilde{x}_{n-1}) \|
+ \alpha \mu_s \| P^u_n(x_{n-1} - \tilde{x}_{n-1}) \|, \quad n \geq k + 1,
\]

\[
\| P^u_n(v_n - \tilde{v}_n) \| \leq \frac{\mu_u}{\lambda_u} \| P^s_n(x_n - \tilde{x}_n) \|
+ \frac{1}{\alpha \lambda_u} \| P^u_{n+1}(x_{n+1} - \tilde{x}_{n+1}) \|, \quad n \geq k.
\]

From these and (4.17) it is clear that

\[
\| R[\alpha, k, v](x) - R[\alpha, k, v](\tilde{x}) \|_\infty \leq \sigma(M_\alpha) \| x - \tilde{x} \|_\infty,
\]

and the operator \( R[\alpha, k, v] \) is contracting in the norm \( \| \cdot \|_\infty \) with contraction constant \( \sigma(M_\alpha) \).
It remains only to show that the set (4.16) is invariant for \( R_{[\alpha,k,v]} \). Given an arbitrary \( x = \{x_n\} \in \mathcal{V}_{[\alpha,k,v]} \), denote \( v = R_{[\alpha,k,v]}(x) \). From (4.13) and (4.14) it is apparent that
\[
\|P^s_k v_k\| = \|v\|,
\|P^s_n v_n\| \leq \alpha \lambda_s \|P^s_n x_{n-1}\| + \alpha \mu_s \|P^u_n x_{n-1}\|, \quad n \geq k + 1,
\|P^u_n v_n\| \leq \frac{\mu_u}{\lambda_u} \|P^u_n x_n\| + \frac{1}{\alpha \lambda_u} \|P^u_{n+1} x_{n+1}\|, \quad n \geq k.
\]
Hence, in view of the definition (4.16) of the set \( \mathcal{V}_{[\alpha,k,v]} \), we have
\[
\|P^s_k v_k\| = \|v\|,
\|P^s_n v_n\| \leq \alpha \lambda_s \|v\| + \frac{\alpha^2 \mu_s \mu_u}{\alpha \lambda_u - 1} \|v\|, \quad n \geq k + 1,
\|P^u_n v_n\| \leq \frac{\mu_u}{\lambda_u} \|v\| + \frac{\alpha \mu_u}{\alpha \lambda_u(\alpha \lambda_u - 1)} \|v\|, \quad n \geq k,
\]
and thus, from (4.15), can conclude that
\[
\|P^s_k v_k\| = \|v\|,
\|P^s_n v_n\| \leq \|v\| + \frac{\alpha^2 \mu_s \mu_u - (1 - \alpha \lambda_s)(\alpha \lambda_u - 1)}{\alpha \lambda_u - 1} \|v\| \leq \|v\|, \quad n \geq k + 1,
\|P^u_n v_n\| \leq \frac{\alpha \mu_u}{\alpha \lambda_u - 1} \|v\|, \quad n \geq k.
\]
This completes the proof of invariance of the set (4.16) under \( R_{[\alpha,k,v]} \). Lemma 4.2.2 is proved.

The next lemma follows immediately from Lemma 4.2.2 and from the definition of the operator \( R_{[\alpha,k,v]} \).

**Lemma 4.2.4.** The only bounded solution of the difference equation
\[
x_{n+1} = \alpha A_n x_n, \quad n \geq k, \quad (4.18)
\]
which satisfies the condition \( P^s_k x_k = v \) is the fixed point \( x_{[\alpha,k,v]} \) of the operator \( R_{[\alpha,k,v]} \).

Lemmas 4.2.2 and 4.2.4 can be combined to show that the fixed point sequence \( x_{[1,k,v]} \) of \( R_{[1,k,v]} \) is not only bounded one but is also exponentially convergent to \( 0 \in \mathbb{R}^d \).

**Corollary 4.2.5.** The fixed point sequence \( x^* = x_{[\alpha,k,v]} = \{x_n^*\} \) of the operator \( R_{[1,k,v]} \) satisfies
\[
\|x_n^*\| \leq \frac{\rho}{(1 + \omega)|n-k|} \|v\|, \quad n \geq k. \quad (4.19)
\]
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Proof. By the definition of the constant $\omega > 0$ the operator $R_{[1+\omega,k,v]}$ has a fixed point $x_{1+\omega}^* = \{x_{1+\omega,n}^*\}$. By Lemma 4.2.4 $\{x_{1+\omega,n}^*\}$ is the bounded solution of equation (4.18) for $\alpha = 1 + \omega$. Hence the sequence $x^* = \{x_n^*\}$, where

$$x_n^* = (1 + \omega)^{k-n}x_{1+\omega,n}^*, \quad n \geq k,$$

is the solution of equation (4.18) for $\alpha = 1$. In view of Corollary 4.2.3 we have for any $n \geq k$

$$\|x_{1+\omega,n}^*\| \leq \|P^s x_{1+\omega,n}^*\| + \|P^n x_{1+\omega,n}^*\| \leq \frac{\alpha \mu_s + \alpha \lambda_s - 1}{\alpha \lambda_s - 1}\|v\| \leq \rho\|v\|$$

and so by the definition of $x^*$ the inequalities (4.19) are valid.

Thus, $x^*$ is the bounded solution of equation (4.18) for $\alpha = 1$ and by Lemma 4.2.4 it is the fixed point of the operator $R_{[1,k,v]}$ for which estimate (4.19) is valid. \hfill \Box

Define the operator

$$L := L_{[\alpha,k,w]} : \ell^\infty((-\infty, k], \mathbb{R}^d) \rightarrow \ell^\infty((-\infty, k], \mathbb{R}^d),$$

again with parameters $\alpha > 0$, $k \in \mathbb{Z}$ and $w \in P^u_k \mathbb{R}^d$, mapping a sequence $y = \{y_n\} \in \ell^\infty((-\infty, k], \mathbb{R}^d)$ to the sequence $w = \{w_n\} \in \ell^\infty((-\infty, k], \mathbb{R}^d)$ defined by

$$P^s_n w_n = \alpha P^s_n A_{n-1} y_{n-1}, \quad n \leq k,$$

$$P^u_k w_k = w,$$

$$P^u_n w_n = \alpha^{-1} (P^u_{n+1} A_n P^u_n)^{-1} P^u_{n+1} y_{n+1} - (P^u_{n+1} A_n P^u_n)^{-1} P^u_{n+1} A_n P^s_n y_n, \quad n \leq k - 1.$$

The proofs of the following two lemmas and corollaries are essentially repetitions of those for Lemmas 4.2.2 and 4.2.4 and for Corollaries 4.2.3 and 4.2.5 and are omitted.

Lemma 4.2.6. For any $\alpha \in [1 - \omega, 1 + \omega]$, any integer $k$ and any $w \in P^u_k \mathbb{R}^d$ the operator $L_{[\alpha,k,w]}$ is contracting in the norm $\|\cdot\|_\alpha$ with constant $\sigma(M_\alpha)$. Moreover, the set

$$\mathcal{W}_{[\alpha,k,w]} = \left\{ \mathcal{x} : \|P^s_n x_n\| \leq \frac{\alpha \mu_s}{1 - \alpha \lambda_s}\|w\|, \|P^u_n x_n\| \leq \|w\|, \quad n \leq k \right\}$$

is invariant under $L_{[\alpha,k,w]}$.

Corollary 4.2.7. For any $w \in P^u \mathbb{R}^d$ the operator $L_{[\alpha,k,w]}$ has a unique fixed point $y^* = y_{[\alpha,k,w]} = \{y_n^*\}$ for which $P^u_k y_k^* = w$ and

$$\|P^s_n y_n^*\| \leq \frac{\alpha \mu_s}{1 - \alpha \lambda_s}\|w\|, \quad \|P^u_n y_n^*\| \leq \|w\|, \quad n \leq k.$$
Lemma 4.2.8. The only bounded solution of the difference equation
\[ y_{n+1} = \alpha A_n y_n, \quad n \leq k - 1, \]
which satisfies the condition \( P_k^u y_k = w \) is the fixed point \( y_{[\alpha,k,w]} \) of the operator \( \mathcal{L}_{[\alpha,k,w]} \).

Corollary 4.2.9. The fixed point \( y^* = y_{[\alpha,k,w]} = \{y_n^*\} \) of the operator \( \mathcal{L}_{[1,k,w]} \) satisfies
\[ \|y_n^*\| \leq \frac{\rho}{(1 + \omega)^{|n-k|}} \|w\|, \quad n \leq k. \]

4.2.2 An Equivariant Splitting

Let \( x_{[\alpha,k,v]} = \{x_{[\alpha,k,v],n}\} \) denote the fixed point of the operator \( \mathcal{R}_{[\alpha,k,v]} \) and define the operator, \( \mathcal{X}_{[\alpha,k]} : P_k^s \mathbb{R}^d \to \mathbb{R}^d \) by
\[ \mathcal{X}_{[\alpha,k]} v = x_{[\alpha,k,v],k}. \]

Lemma 4.2.10. The operator \( \mathcal{X}_{[\alpha,k]} \) is linear, \( P_k^s \mathcal{X}_{[\alpha,k]} P_k^s = P_k^s \) and
\[ \|P_k^u \mathcal{X}_{[\alpha,k]} v\| \leq \frac{\alpha \mu_u}{\alpha \lambda_u - 1} \|v\|, \quad v \in P_k^s \mathbb{R}^d. \]  \( (4.20) \)

Proof. Let \( x^* = \{x_n\} = x_{[\alpha,k,v]} \) be the fixed point of the operator \( \mathcal{R}_{[\alpha,k,v]} \) and \( \tilde{x}^* = \{\tilde{x}_n\} = x_{[\alpha,k,\tilde{v}]} \) be the fixed point of the operator \( \mathcal{R}_{[\alpha,k,\tilde{v}]} \) for \( v, \tilde{v} \in P_k^s \mathbb{R}^d \). Then by Lemma 4.2.4 \( x^* = \{x_n\} \) is the solution of equation (4.18) satisfying the condition \( P_k^s x_k = v \), while \( \tilde{x}^* = \{\tilde{x}_n\} \) is the solution of equation (4.18) satisfying the condition \( P_k^s \tilde{x}_k = \tilde{v} \). Hence for any real numbers \( \xi \) and \( \zeta \), the sequence
\[ z^* = \xi x^* + \zeta \tilde{x}^* = \{\xi x_n + \zeta \tilde{x}_n\} \in \ell^\infty([k, \infty), \mathbb{R}^d) \]
is the solution of equation (4.18) satisfying the condition \( P_k^s z_k = \xi v + \zeta \tilde{v} \). Using Lemma 4.2.4 again, we find that \( z^* \) is the fixed point of the operator \( \mathcal{R}_{[\alpha,k,\xi v + \zeta \tilde{v}]} \), that is \( z^* = x_{[\alpha,k,\xi v + \zeta \tilde{v}]} \). Thus
\[ \xi x_{[\alpha,k,v]} + \zeta x_{[\alpha,k,\tilde{v}]} = x_{[\alpha,k,\xi v + \zeta \tilde{v}]}, \]
so the mapping \( x_{[\alpha,k,\cdot]} \) is linear and hence the operator \( \mathcal{X}_{[\alpha,k]}(\cdot) = x_{[\alpha,k,\cdot]} \) is linear.

Now note that \( P_k^s \mathcal{X}_{[\alpha,k]} v = P_k^s x_{[\alpha,k,v],k} = v \) for \( v \in P_k^s \mathbb{R}^d \), which means that \( P_k^s \mathcal{X}_{[\alpha,k]} P_k^s = P_k^s \), as required.

Estimate (4.20) then follows immediately from the definition of the operator \( \mathcal{X}_{[\alpha,k]} \) and from Corollary 4.2.3. \[ \square \]
Similarly, denote the fixed point of the operator $\mathcal{L}_{[\alpha,k,w]}$ by
\[ y_{[\alpha,k,w]} = \{ y_{[\alpha,k,w],n} \} \]
and define an operator $\mathcal{Y}_{[\alpha,k]} : \mathbb{R}^d_{\alpha,k} \rightarrow \mathbb{R}^d$ by
\[ \mathcal{Y}_{[\alpha,k]} w = y_{[\alpha,k,w],k}. \]
The proof of the next lemma is essentially the same as that of Lemma 4.2.10.

**Lemma 4.2.11.** The operator $\mathcal{Y}_{[\alpha,k]}$ is linear, $P^u_k \mathcal{Y}_{[\alpha,k]} P^u_k = P^u_k$ and
\[ \| P^s_k \mathcal{Y}_{[\alpha,k]} w \| \leq \frac{\alpha \mu_s}{1 - \alpha \lambda_s} \| w \|, \quad w \in P^u_k \mathbb{R}^d. \]

Now define $\hat{E}^s_k$ as the set of those $v \in \mathbb{R}^d$ for which the equation
\[ v_{n+1} = A_n v_n, \quad n \geq k, \quad (4.21) \]
has a bounded solution satisfying $v_k = v$, and define $\hat{E}^u_k$ as the set of those $w \in \mathbb{R}^d$ for which the equation
\[ w_{n+1} = A_n w_n, \quad n < k, \quad (4.22) \]
has a bounded solution satisfying $w_k = w$. Further, define subspaces $\hat{E}^s_k$ and $\hat{E}^u_k$ of $\mathbb{R}^d$ by
\[ \hat{E}^s_k = X_{[1,k]} P^s_k \mathbb{R}^d, \quad \hat{E}^u_k = Y_{[1,k]} P^u_k \mathbb{R}^d, \quad k \in \mathbb{Z}. \quad (4.23) \]
Finally, recall that the *separation* $\delta(E_1, E_2)$ between two subspaces $E_1, E_2 \subseteq \mathbb{R}^d$ is defined as
\[ \delta(E_1, E_2) = \inf \{ \| x + y \| : x \in E_1, y \in E_2, \| x \| = \| y \| = 1 \}. \]

**Lemma 4.2.12.** Suppose that matrices $A_n, n \in \mathbb{Z}$, are invertible. Then for any integer $k$ the following relations are valid:
\[ \hat{E}^s_k = \hat{E}^s_k, \quad (4.24) \]
\[ \hat{E}^u_k = \hat{E}^u_k, \quad (4.25) \]
\[ \hat{E}^s_k \cap \hat{E}^u_k = \{ 0 \}, \quad (4.26) \]
\[ \hat{E}^s_k \oplus \hat{E}^u_k = \mathbb{R}^d, \quad \hat{A}_k \hat{E}^s_k = \hat{E}^s_{k+1}, \quad \hat{A}_k \hat{E}^u_k = \hat{E}^u_{k+1}. \quad (4.27) \]
\[ \delta(\hat{E}^s_k, \hat{E}^u_k) \geq \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{h(\lambda_u - 1 + \mu_u)(1 - \lambda_s + \mu_s)} > 0. \quad (4.28) \]

**Proof.** We first prove that $\hat{E}^s_k = \hat{E}^s_k$ from (4.24). Indeed, if $v \in \hat{E}^s_k$ then, by definition of $\hat{E}^s_k$, there exists $v^* \in P^s_k \mathbb{R}^d$ such that $v = X_{[1,k]} v^*$ and, by definition of the operator $X_{[1,k]}$, $v^* = P^s_k v_k$ where $\{ v_n \}$, $n \geq k$, is the fixed point of the operator $\mathcal{R}_{[1,k,v^*]}$ satisfying $v_k = v$. Hence by Lemma 4.2.4, $\{ v_n \}$
is a bounded solution of equation (4.21) and so \( v \in \hat{E}_k^s \), from which it follows that \( \hat{E}_k^s \subseteq \hat{E}_k^s \). Similarly, if \( v \in \hat{E}_k^s \) then there exist a bounded solution \( \{v_n\} \) of equation (4.21) satisfying \( v_k = v \). So by Lemma 4.2.4 \( \{v_n\} \) is the fixed point of the operator \( \mathcal{R}[1,k,p^s_k v] \). Therefore, by the definition of the operator \( X[1,k] \), the vector \( v \) can be represented in the form \( v = X[1,k]P^s_k v \) and so \( v \in \hat{E}_k^s \), from which it follows that \( \hat{E}_k^s \subseteq \hat{E}_k^s \). Combining the two results gives the desired identity \( \hat{E}_k^s = \hat{E}_k^s \). That \( \hat{E}_u^s \cap \hat{E}_u^u \neq \{0\} \) follows similarly.

To prove the first part of (4.25) suppose that \( \hat{E}_k^s \cap \hat{E}_k^u \neq \{0\} \) for some \( k \). Then there exist nonzero \( v \in \hat{P}_k^s \mathbb{R}^d \), \( w \in \hat{P}_k^u \mathbb{R}^d \) and \( \{v_n\} \in \ell^\infty([k, \infty), \mathbb{R}^d) \), \( \{w_n\} \in \ell^\infty((-\infty, k], \mathbb{R}^d) \) such that \( \{v_n\} \) is the fixed point of the operator \( \mathcal{R}[1,k,v] \), \( \{w_n\} \) is the fixed point of the operator \( \mathcal{L}[1,k,w] \) and

\[
v_k = w_k \in \hat{E}_k^s \cap \hat{E}_k^u, \quad v_k = w_k \neq 0. \tag{4.28}
\]

By Lemmas 4.2.4 and 4.2.10 the sequence \( \{v_n\} \) is a bounded solution of equation (4.21), and the sequence \( \{w_n\} \) is a bounded solution of equation (4.22). From (4.28), the sequence \( \{x_n\} \) defined for all integer \( k \) by

\[
x_n = \begin{cases} v_n, & n \geq k, \\ w_n, & n < k \end{cases}
\]

is thus a nonzero bounded solution of

\[
x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}.
\]

Set \( \chi = \text{sup}_{n \in \mathbb{Z}} \|x_n\| \). Then by Corollary 4.2.5

\[
\|x_k\| \leq \frac{\rho}{(1 + \omega)^{k-n}} \|x_n\| \leq \frac{\rho \chi}{(1 + \omega)^{|k-n|}}
\]

for any \( n < k \). Letting \( n \) converge to \( -\infty \), from the above relations we obtain \( x_k = v_k = w_k = 0 \), which contradicts (4.28). The first equation of (4.25) is proved.

To prove the second of (4.25) it suffices to note that by Lemmas 4.2.10 and 4.2.11

\[
dim \hat{E}_k^s \geq \dim P^s_k \mathbb{R}^d, \quad \dim \hat{E}_k^u \geq \dim P^u_k \mathbb{R}^d,
\]

and so

\[
dim \hat{E}_k^s + \dim \hat{E}_k^u \geq \dim P^s_k \mathbb{R}^d + \dim P^u_k \mathbb{R}^d = \dim \mathbb{R}^d = d.
\]

From this and from the first of (4.25), the second of (4.25) follows.

Now from the definition of the subspaces \( \hat{E}_k^s \), \( \hat{E}_k^u \) and from invertibility of the matrices \( A_n, n \in \mathbb{Z} \), it follows immediately that
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\[ A_k \hat{E}_k^s = \hat{E}_{k+1}^s, \quad A_k \hat{E}_k^u = \hat{E}_{k+1}^u. \]

Then by (4.24)

\[ A_k \hat{E}_k^s = A_k \tilde{E}_k^s = \tilde{E}_{k+1}^s = \hat{E}_{k+1}^s, \]
\[ A_k \hat{E}_k^u = A_k \tilde{E}_k^u = \tilde{E}_{k+1}^u = \hat{E}_{k+1}^u. \]

from which (4.26) follows.

Finally, turn to the inequality (4.27). Set

\[ \theta_s = \frac{\mu_s}{1 - \lambda_s}, \quad \theta_u = \frac{\mu_u}{\lambda_u - 1}. \]

Fix an arbitrary vector \( x \in \hat{E}_k^s \), \( \|x\| = 1 \), and denote \( v = P_k^s x \). Then, by the definition of the subspace \( \hat{E}_k^s \), \( x = X_{[1,k]} v \) and from Lemma 4.2.10

\[ \|P_k^u x\| \leq \theta_u \|v\| = \theta_u \|P_k^s x\|. \]  

Similarly, for an arbitrary vector \( y \in \hat{E}_k^u \), \( \|y\| = 1 \), from the definition of the subspace \( \hat{E}_k^u \) and from Lemma 4.2.11 the inequality

\[ \|P_k^u y\| \leq \theta_s \|P_k^u y\| \]  

can be easily obtained.

By (4.12) and (4.30)

\[ \|x + y\| \geq \frac{1}{h} \|P_k^s(x + y)\| \geq \frac{1}{h} \left( \|P_k^s x\| - \|P_k^s y\| \right) \geq \frac{1}{h} \left( \|P_k^s x\| - \theta_s \|P_k^u y\| \right), \]

and also from (4.12) and (4.29)

\[ \|x + y\| \geq \frac{1}{h} \|P_k^u(x + y)\| \geq \frac{1}{h} \left( \|P_k^u y\| - \|P_k^u x\| \right) \geq \frac{1}{h} \left( \|P_k^u y\| - \theta_u \|P_k^s x\| \right). \]

From (4.29) and (4.30) it follows that in the above inequalities the terms \( \|P_k^s x\| \) and \( \|P_k^u y\| \) satisfy

\[ \|P_k^s x\| \geq \frac{1}{1 + \theta_u} \|x\| = \frac{1}{1 + \theta_u}, \quad \|P_k^u y\| \geq \frac{1}{1 + \theta_s} \|y\| = \frac{1}{1 + \theta_s}. \]

Hence

\[ \|x + y\| \geq \frac{1}{h} \inf_{s \geq (1 + \theta_u)^{-1}, \ t \geq (1 + \theta_s)^{-1}} \max \{s - \theta_s t, \ t - \theta_u s\}. \]

Note that the function \( \max \{s - \theta_s t, \ t - \theta_u s\} \) can attain its infimum on the set \( s \geq (1 + \theta_u)^{-1}, \ t \geq (1 + \theta_s)^{-1} \) only in the case when \( s - \theta_s t = t - \theta_u s \), that is when

\( (1 + \theta_u)s = (1 + \theta_s)t. \)

Let \( \tau = (1 + \theta_u)s = (1 + \theta_s)t \), so that
\[ s = \frac{\tau}{1+\theta_u}, \quad t = \frac{\tau}{1+\theta_s}. \]

Then
\[ \|x + y\| \geq \frac{1}{h} \inf_{\tau \geq 1} \left\{ \frac{\tau}{1+\theta_u} - \theta_s \frac{\tau}{1+\theta_s} \right\} \geq \frac{1 - \theta_u \theta_s}{h(1+\theta_u)(1+\theta_s)}. \]

From this and the definitions of \( \delta(\hat{E}_k^s, \hat{E}_k^u), \theta_s \) and \( \theta_u \), (4.27) follows. \( \Box \)

**Remark 4.2.13.** If the invertibility of the matrices \( A_n, n \in \mathbb{Z} \), in Lemma 4.2.12 is not assumed, then (4.26) is, in general, replaced by
\[ A_k \hat{E}_k^s \subseteq \hat{E}_{k+1}^s, \quad A_k \hat{E}_k^u = \hat{E}_{k+1}^u. \]

That is, both the spaces \( \hat{E}_k^s \) and \( \hat{E}_k^u \) do not necessarily have symmetric properties if invertibility is not present.

The results of this section are summarized by the following theorem.

**Theorem 4.2.14 (Equivariant Splitting Theorem).** Let \( \{A_n\}, n \in \mathbb{Z}, \) be an \((s, h)\)-semi-hyperbolic sequence of invertible \( d \times d \) matrices. Then the subspaces \( \hat{E}_n^s, \hat{E}_n^u \) defined by (4.23) form a splitting \( \mathbb{R}^d = \hat{E}_n^s \oplus \hat{E}_n^u \) with corresponding projections \( \hat{P}_n^s : \mathbb{R}^d \to \hat{E}_n^s \) and \( \hat{P}_n^u = I - \hat{P}_n^s : \mathbb{R}^d \to \hat{E}_n^u \) satisfying the uniform matrix bounds
\[ \|\hat{P}_n^s\| \leq \hat{h}, \quad \|\hat{P}_n^u\| \leq \hat{h}. \] (4.31)

This splitting is equivariant with respect to the matrix sequence \( \{A_n\} \),
\[ A_n \hat{E}_n^s = \hat{E}_{n+1}^s, \quad A_n \hat{E}_n^u = \hat{E}_{n+1}^u. \]

Moreover, there exist constants \( \rho > 0 \) and \( 0 < \lambda < 1 \), depending only on the split \( s \) and the parameter \( h \), such that for any integer \( k \) and \( v_k \in \hat{E}_k^s \) the sequence \( \{v_n\} \) defined by
\[ v_{n+1} = A_nv_n, \quad n \geq k, \]
satisfies
\[ v_n \in \hat{E}_n^s, \quad \|v_n\| \leq \rho \lambda^{-|n-k|} \|v_k\|, \quad n \geq k. \] (4.32)

Similarly, for any integer \( k \) and \( w_k \in \hat{E}_k^u \) the sequence \( \{w_n\} \) defined by
\[ w_{n+1} = A_nw_n, \quad n < k, \]
satisfies
\[ w_n \in \hat{E}_n^w, \quad \|w_n\| \leq \rho \lambda^{-|n-k|} \|w_k\|, \quad n \leq k. \] (4.33)
Proof. Existence of the equivariant splitting $\mathbb{R}^d = \hat{E}_s \oplus \hat{E}_u$ and inclusions in (4.32), (4.33) follow immediately from Lemma 4.2.12. The inequality for $\|v_n\|$ in (4.32) follows from Corollary 4.2.5, while that for $\|w_n\|$ in (4.33) follows from Corollary 4.2.9. It remains only to prove the inequalities (4.31) for the projections $\hat{P}_n^s : \mathbb{R}^d \to \hat{E}_s^\prime$ and $\hat{P}_n^u = I - \hat{P}_n^s : \mathbb{R}^d \to \hat{E}_u^\prime$.

Let
$$\hat{h} = \frac{2(\lambda_u - 1 + \mu_u)(1 - \lambda_s + \mu_s)}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h$$
and consider (4.31) for this constant. Clearly
$$\|x\| = \|\hat{P}_n^s x + \hat{P}_n^u x\| \geq \|\hat{P}_n^s x\| - \|\hat{P}_n^u x\|. \quad (4.34)$$

If
$$\|\hat{P}_n^s x\| - \|\hat{P}_n^u x\| \geq \frac{1}{h} \max \left\{ \|\hat{P}_n^s x\|, \|\hat{P}_n^u x\| \right\}$$
then from this and from (4.34), obtain (4.31). So, suppose that
$$\|\hat{P}_n^s x\| - \|\hat{P}_n^u x\| < \frac{1}{h} \max \left\{ \|\hat{P}_n^s x\|, \|\hat{P}_n^u x\| \right\}. \quad (4.35)$$

Without loss of generality it may be assumed that
$$\hat{P}_n^s x \neq 0, \quad \hat{P}_n^u x \neq 0 \quad \text{and} \quad \|\hat{P}_n^s x\| \geq \|\hat{P}_n^u x\|,$$
which implies
$$\|\hat{P}_n^s x\| = \max \left\{ \|\hat{P}_n^s x\|, \|\hat{P}_n^u x\| \right\}. \quad (4.36)$$

Now consider $\|x\|$ in the following way
$$\|x\| = \|\hat{P}_n^s x + \hat{P}_n^u x\|$$
$$= \left| \left( \frac{\hat{P}_n^s x}{\|\hat{P}_n^s x\|} + \frac{\hat{P}_n^u x}{\|\hat{P}_n^u x\|} \right) \|\hat{P}_n^s x\| - \frac{\hat{P}_n^u x}{\|\hat{P}_n^u x\|} \left( \|\hat{P}_n^s x\| - \|\hat{P}_n^u x\| \right) \right|$$
$$\geq \left| \frac{\hat{P}_n^s x}{\|\hat{P}_n^s x\|} + \frac{\hat{P}_n^u x}{\|\hat{P}_n^u x\|} \right| \|\hat{P}_n^s x\| - \|\hat{P}_n^s x\| - \|\hat{P}_n^u x\|.$$ 

Here, by (4.27) and (4.36), the first term in the right-hand part can be bounded from below by the value $2\hat{h}^{-1} \max \left\{ \|\hat{P}_n^s x\|, \|\hat{P}_n^u x\| \right\}$, while for the second term the inequality (4.35) holds. Thus
$$\|x\| \geq \frac{1}{h} \max \left\{ \|\hat{P}_n^s x\|, \|\hat{P}_n^u x\| \right\},$$
which completes the proof of (4.31). \qed
Remark 4.2.15. By statements (4.32) and (4.33) of Theorem 4.2.14, the subspaces $\hat{E}^s_n$ and $\hat{E}^u_n$ in the splitting $\mathbb{R}^d = \hat{E}^s_n \oplus \hat{E}^u_n$ can be interpreted as asymptotically stable and asymptotically unstable, respectively. In such a situation it seems not unreasonable to conjecture from Theorem 4.2.14 that if $\{A_n\}$ is an $(s, h)$-semi-hyperbolic sequence of invertible matrices, then it is $(\hat{s}, \hat{h})$-semi-hyperbolic with a split $\hat{s} = (\hat{\lambda}_s, \hat{\lambda}_u, \hat{\mu}_s, \hat{\mu}_u)$ satisfying

$$\hat{\lambda}_s < 1, \quad \hat{\lambda}_u > 1, \quad \hat{\mu}_s = \hat{\mu}_u = 0,$$

so that the subspaces $\hat{E}^s_n$ and $\hat{E}^u_n$ in the equivariant splitting $\mathbb{R}^d = \hat{E}^s_n \oplus \hat{E}^u_n$ are stable and unstable, respectively, with respect to a unique fixed norm $\| \cdot \|$ in $\mathbb{R}^d$. Unfortunately, a proof of such a generalization of Theorem 4.2.14 is not available. Indeed, Example 4.2.17 suggests invalidity if the class of semi-hyperbolic mappings is too restrictive. A possible way to generalize Theorem 4.2.14 in a similar way, by widening the class of semi-hyperbolic mappings by introducing a slightly weaker condition, is discussed in the next section.

### 4.2.3 Hyperbolicity of Sequences of Matrices

The results of Sects. 4.2.1 and 4.2.2 can be generalized by defining a version of the semi-hyperbolic matrix sequence that uses a variable norm replacing that of Definition 3.1.4 and (3.5)–(3.8).

**Definition 4.2.16 (Semi-Hyperbolic Sequence of Matrices).** A bounded sequence of $d \times d$ matrices $\{A_n\}$ will be called $(s, h)$-semi-hyperbolic if there exists a set of norms $\{\| \cdot \|_n\}$ on $\mathbb{R}^d$ satisfying the uniform boundedness condition

$$q^{-1} \| x \| \leq \| x \|_n \leq q \| x \|, \quad x \in \mathbb{R}^d$$

for some constant $q \geq 1$ such that for each matrix $A_n, n \in \mathbb{I}$, there exists a splitting $\mathbb{R}^d = E^s_n \oplus E^u_n$ with

$$\dim E^s_n = \dim E^s_{n+1}, \quad \dim E^u_n = \dim E^u_{n+1}$$

and projections $P^s_n : \mathbb{R}^d \to E^s_n, P^u_n = I - P^s_n : \mathbb{R}^d \to E^u_n$, with the uniform matrix norm bounds, in the usual $\mathbb{R}^d$ operator norm,

$$\| P^s_n \| \leq h, \quad \| P^u_n \| \leq h,$$

such that, in the norms $\| \cdot \|_n$,

$$\| P^s_{n+1} A_n P^s_n x \|_n \leq \lambda_s \| P^s_n x \|_n, \quad \| P^u_{n+1} A_n P^u_n x \|_n \leq \mu_s \| P^u_n x \|_n,$$

$$\| P^u_{n+1} A_n P^s_n x \|_n \leq \mu_u \| P^s_n x \|_n, \quad \| P^u_{n+1} A_n P^u_n x \|_n \geq \lambda_u \| P^u_n x \|_n.$$  

(4.40)
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The following simple example clearly illustrates that this transition from the fixed norm definition to one involving variable norms broadens the defined class of semi-hyperbolic mappings.

**Example 4.2.17.** Consider the following sequence of linear operators

\[ A_{2n}x = 2x, \quad A_{2n+1}x = \frac{1}{4}x, \quad x \in \mathbb{R}^1, \quad n \in \mathbb{Z}, \]

acting in the one-dimensional space \( \mathbb{R}^1 \). Clearly, the matrix sequence \( \{A_n\} \) is not semi-hyperbolic in the sense of Definition 4.2.1 for any possible choice of fixed norm in \( \mathbb{R}^1 \), but if we define the norms \( \| \cdot \|_n \) on \( \mathbb{R}^1 \) by

\[ \|x\|_{2n} = |x|, \quad \|x\|_{2n+1} = \frac{1}{3}|x|, \quad n \in \mathbb{Z}, \]

where \( |x| \) denotes the absolute value of the number \( x \), then

\[ \|A_{2n}x\|_{2n+1} = \frac{1}{3}|2x| = \frac{2}{3}|x| = \frac{2}{3}\|x\|_{2n} \]

and

\[ \|A_{2n-1}x\|_{2n} = \left|\frac{1}{4}x\right| = \frac{1}{4}|x| = \frac{3}{4}\|x\|_{2n-1}. \]

Hence the matrix sequence \( \{A_n\} \) is semi-hyperbolic in the sense of Definition 4.2.16.

The special case of a semi-hyperbolic sequence of matrices corresponding to the situation in which the parameters \( \mu_s \) and \( \mu_u \) vanish is called a hyperbolic sequence of matrices. In this case the spaces \( E^s_n \) and \( E^u_n \) are equivariant for the sequence \( \{A_n\} \), so the formal definition is simpler than Definition 4.2.16.

**Definition 4.2.18 (Hyperbolic Sequence of Matrices).** A bounded sequence \( \{A_n\} \) of \( d \times d \) matrices will be called hyperbolic if for each \( n \in \mathbb{I} \) there exists a norm \( \| \cdot \|_n \) in \( \mathbb{R}^d \) satisfying the uniform boundedness condition (4.37) and a splitting \( \mathbb{R}^d = E^s_n \oplus E^u_n \) with corresponding projections \( P^s_n : \mathbb{R}^d \to E^s_n \), \( P^u_n = I - P^s_n : \mathbb{R}^d \to E^u_n \) satisfying (4.38) and (4.39) such that

\[ A_n E^s_n = E^s_{n+1}, \quad A_n E^u_n = E^u_{n+1} \tag{4.41} \]

and the inequalities

\[ \|A_n P^s_n x\|_{n+1} \leq \lambda_s \|P^s_n x\|_n, \quad \|A_n P^u_n x\|_{n+1} \geq \lambda_u \|P^u_n x\|_n \tag{4.42} \]

hold for some constants \( 0 \leq \lambda_s < 1 < \lambda_m \).

Using Definitions 4.2.16 and 4.2.18, the statement of Theorem 4.2.14 now becomes more compact and elegant.
Theorem 4.2.19. Every semi-hyperbolic sequence of invertible matrices \( \{A_n\}, \ n \in \mathbb{Z} \), is hyperbolic and conversely.

Proof. Clearly, every hyperbolic sequence of invertible matrices \( \{A_n\}, \ n \in \mathbb{Z} \), is semi-hyperbolic, so we need only prove that the semi-hyperbolicity of a matrix sequence implies its hyperbolicity.

An adjustment of the proof of Theorem 4.2.14 to the ‘variable norm’ case can be done in the same manner as was explained in the proof of Theorem 4.3.11. In this way we obtain that for a given \((s,h)\)-semi-hyperbolic sequence of invertible matrices \( \{A_n\}, \ n \in \mathbb{Z} \), in the sense of Definition 4.2.16 there exists a splitting \( \mathbb{R}^d = \hat{E}^s_n \oplus \hat{E}^u_n \) with corresponding projections \( \hat{P}^s_n : \mathbb{R}^d \to \hat{E}^s_n \) and \( \hat{P}^u_n = I - \hat{P}^s_n : \mathbb{R}^d \to \hat{E}^u_n \) with uniform matrix bounds (4.31). This splitting is equivariant with respect to the matrix sequence \( \{A_n\} \), that is it satisfies (4.41). Moreover, there exist constants \( \rho > 0 \) and \( \lambda \in (0,1) \) such that for any integer \( k \in \mathbb{Z} \) and \( v_k \in \hat{E}^s_k \) the sequence \( \{v_n\} \) defined by

\[
  v_{n+1} = A_n v_n, \quad n \geq k, \tag{4.43}
\]

also satisfies

\[
  v_n \in \hat{E}^s_n, \quad \|v_n\|_n \leq \rho \lambda^{-|n-k|} \|v_k\|_k, \quad n \geq k. \tag{4.44}
\]

Similarly, for any integer \( k \in \mathbb{Z} \) and \( w_k \in \hat{E}^u_k \) the sequence \( \{w_n\} \) defined by

\[
  w_{n+1} = A_n w_n, \quad n < k, \tag{4.45}
\]

also satisfies

\[
  w_n \in \hat{E}^u_n, \quad \|w_n\|_n \leq \rho \lambda^{-|n-k|} \|w_k\|_k, \quad n \leq k. \tag{4.46}
\]

Thus, to complete the proof of theorem it suffices to establish the existence of norms \( \|\cdot\|^*_n \) in \( \mathbb{R}^d \) satisfying uniform boundedness condition

\[
  \frac{1}{q^*} \|x\| \leq \|x\|^*_n \leq q^* \|x\|, \quad x \in \mathbb{R}^d, \tag{4.47}
\]

with some constant \( q^* \geq 1 \) such that

\[
  \|A_n \hat{P}^s_n x\|^*_n \leq \lambda \|\hat{P}^s_n x\|^*_n, \quad \|A_n \hat{P}^u_n x\|^*_n \geq \lambda^{-1} \|\hat{P}^u_n x\|^*_n \tag{4.48}
\]

To construct the required norms, fix an integer \( k \in \mathbb{Z} \) and \( x \in \mathbb{R}^d \) and choose \( v_k = \hat{P}^s_k x, \ w_k = \hat{P}^u_k x \). Then define sequences \( \{v_n\}, \ n \leq k \), and \( \{w_n\}, \ n \geq k \), satisfying (4.43) and (4.45), respectively. Finally, define

\[
  \|x\|^*_k = \max \left\{ \sup_{n \geq k} \lambda^{|k-n|} \|v_n\|_n, \sup_{n \leq k} \lambda^{|n-k|} \|w_n\|_n \right\}.
\]

From (4.44) and (4.46), inequalities (4.48) and (4.47) immediately follow with an appropriately chosen \( q^* \). The theorem is proved. \( \square \)
4.3 A More Abstract Approach

In this section an abstract approach in terms of linear operators in Banach spaces is used to investigate properties of sequences of semi-hyperbolic mappings.

4.3.1 Semi-Hyperbolic Linear Operators

An abstract approach is able to clarify to a deeper extent the basic relationships between conditions of semi-hyperbolicity and hyperbolicity for sequences of matrices. To implement such an approach, begin by investigating how the property of semi-hyperbolicity of a linear operator on a Banach space is related to that of hyperbolicity.

Let $A : E \to E$ be a bounded linear operator on a Banach space $E$ and let $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ be a split.

**Definition 4.3.1 (Semi-Hyperbolic Linear Operator).** A linear bounded operator $A$ on a Banach space $E$ with a norm $\| \cdot \|$ is called semi-hyperbolic, with respect to a split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, a constant $h$ and the norm $\| \cdot \|$, if there is a splitting $E = E^s \oplus E^u$ with corresponding bounded projectors $P^s : E \to E^s$ and $P^u = I - P^s : E \to E^u$ satisfying

$$\|P^s\| \leq h, \quad \|P^u\| \leq h,$$

(4.49)

and

$$\|P^s AP^s x\| \leq \lambda_s \|P^s x\|,$$

$$\|P^s AP^u x\| \leq \mu_s \|P^u x\|,$$

$$\|P^u AP^s x\| \leq \mu_u \|P^s x\|,$$

$$\|P^u AP^u x\| \geq \lambda_u \|P^u x\|.$$  

(4.50)

**Remark 4.3.2.** Above, it is assumed that the linear operator $A$ is real, that is, acting on a real Banach space $E$. Often, however in studying spectral properties of linear operators it is convenient to treat them as if they were acting on complex Banach spaces. The natural way to pass from real to complex operators is the complexification of a linear operator. For the sake of completeness of presentation the necessary details are briefly reviewed.

For a real Banach space $E$ with a norm $\| \cdot \|$ define the complexification of $E$, the complex Banach space $E_c$, as the set of complex vectors $z = x + iy$ with $x, y \in E$ endowed with the norm

$$\|z\|_c = \|x + iy\|_c := \max_{|\alpha + i\beta| = 1} \|\alpha x + \beta y\|, \quad \alpha, \beta \in \mathbb{R}^1.$$  

For a linear operator $A : E \to E$ we then define its complexification $A_c : E_c \to E_c$ by
\[ A_c(x + iy) := Ax + iAy, \quad x, y \in E. \]

With such a definition of complexification, spectral properties of a real operator \( A \) are exactly the same as those of its complexification \( A_c \), both operators have the same spectrum, the same set of eigenvalues, and so on.

For what follows, it is important that the passage from a real linear operator \( A \) to its complexification \( A_c \) preserves all the essential features of semi-hyperbolicity (see Definition 4.3.1). More precisely, if \( A \) is a semi-hyperbolic linear operator satisfying (4.49) and (4.50) then its complexification \( A_c : E_c \to E_c \) must satisfy

\[
\| P_s^c A_c P_s^c z \|_c \leq \lambda_s \| P_s^c z \|_c, \\
\| P_c^u A_c P_c^u z \|_c \leq \mu_s \| P_c^u z \|_c, \\
\| P_c^u A_c P_s^c z \|_c \leq \mu_u \| P_c^u z \|_c, \\
\| P_c^u A_c P_c^u z \|_c \geq \lambda_u \| P_c^u z \|_c,
\]

for all \( z \in E_c \) with corresponding bounded projectors \( P_s^c : E_c \to E_c^s \) and \( P_c^u = I - P_s^c : E_c \to E_c^u \) satisfying

\[
\| P_c^s \|_c \leq h, \quad \| P_c^u \|_c \leq h.
\]

It seems quite reasonable now to define the concept of a hyperbolic linear operator by introducing the additional requirement of invariance of the splitting subspaces \( E^s \) and \( E^u \) with respect to the operator \( A \). Such an invariance condition can be expressed by one of the following three equivalent conditions:

\[ AE^s \subseteq E^s, \quad AE^s \subseteq E^s, \]

or

\[ P^s A P^u = 0, \quad P^u A P^s = 0, \]

or

\[ P^s A P^u x \equiv 0, \quad P^u A P^s x \equiv 0. \]

**Definition 4.3.3 (Hyperbolic Linear Operator: Metric Definition).**

A linear bounded operator \( A \) on a Banach space \( E \) with a norm \( \| \cdot \| \) is called hyperbolic in the norm \( \| \cdot \| \) if there is an invariant splitting \( E = E^s \oplus E^u \) with corresponding bounded projectors \( P^s : E \to E^s \) and \( P^u = I - P^s : E \to E^u \) satisfying

\[
\| P^s A P^s x \| \leq \lambda_s \| P^s x \|, \quad \| P^u A P^u x \| \geq \lambda_u \| P^u x \|,
\]

with appropriate constants \( \lambda_s < 1 \) and \( \lambda_u > 1 \).

The concept of hyperbolicity for a linear operator \( A \) on a Banach space can also be defined in a clearer and more constructive way via a description of the spectral properties of the operator \( A \).
**Definition 4.3.4 (Hyperbolic Linear Operator: Spectral Definition).**
A linear bounded operator $A$ on a Banach space is called spectrally hyperbolic if its spectrum does not intersect the unit disc $|\lambda| = 1$ in the complex plane.

The link between the concepts of semi-hyperbolicity and hyperbolicity, in both metric and spectral versions, is established by the following theorem. For linear operators the concepts of semi-hyperbolicity and hyperbolicity, metric or spectral, are equivalent.

The following theorem will be proved in Appendix A and is that of 93.

**Theorem 4.3.5.** If the operator $A$ is semi-hyperbolic with respect to a split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and a norm $\| \cdot \|$, then $A$ is spectrally hyperbolic.

**Theorem 4.3.6.** For a linear bounded operator $A$ on a Banach space $E$ with a norm $\| \cdot \|$ the following three statements are equivalent:

(i) The operator $A$ is semi-hyperbolic with respect to some split $s$, constant $h$ and a norm $\| \cdot \|_*$ equivalent to the norm $\| \cdot \|$.  
(ii) The operator $A$ is metrically hyperbolic in some norm $\| \cdot \|_*$ equivalent to the norm $\| \cdot \|$.  
(iii) The operator $A$ is spectrally hyperbolic.

**Proof.** As was remarked earlier, throughout the proof of theorem we can treat $E$ as a complex Banach space. To prove the theorem the following implications are shown.

(i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Note that (i) $\Rightarrow$ (iii) is given by the previously stated result, the proof of which is in Appendix A.

Now turning to (iii) $\Rightarrow$ (ii), that spectral hyperbolicity of the operator $A$ implies hyperbolicity in the sense of Definition 4.3.3. By supposition, the spectrum $\sigma$ of the operator $A$ does not intersect the unit disc $|\lambda| = 1$. Then, since the spectrum of any operator is a closed set, the set $\sigma$ can be represented as the union $\sigma = \sigma_s \cup \sigma_u$ of closed subsets $\sigma_s$ and $\sigma_u$ where $\sigma_s$ lies entirely inside the unit disc and $\sigma_u$ lies entirely outside the unit disc.

Denote by $E^s \subseteq E$ the invariant spectral subspace of the operator $A$ corresponding to the part $\sigma_s$ of its spectrum, and denote by $E^u \subseteq E$ the invariant spectral subspace of the operator $A$ corresponding to the part $\sigma_u$ of its spectrum. Then, from spectral theory,

$$E^s \cap E^u = \emptyset, \quad E^s \oplus E^u = E$$

hold. Denote by $P^s : E \to E^s$ and $P^u = I - P^s : E \to E^u$ the projections corresponding to the splitting $E = E^s \oplus E^u$, which are bounded in this case.

Now, the spectral radius $\sigma(A)$ of a linear operator $A$ on a Banach space satisfies

$$\sigma(A) = \limsup_{n \to \infty} \|A^n\|^{1/n},$$
from which it follows that for any $\varepsilon > 0$ there exists a number $c_\varepsilon$ such that
\[
\|A^n\| \leq c_\varepsilon (\sigma(A) + \varepsilon)^n, \quad n \geq 0.
\] (4.51)

By definition, the closed set $\sigma_s$ lies inside the disc $|\lambda| < 1$. Thus numbers $\lambda_s \in (0,1)$ and $\varepsilon > 0$ can be found such that $\sigma_s$ is contained in the disc $|\lambda| \leq \lambda_s - \varepsilon$. Then, applying the formula (4.51) to the restriction $A|_{E^s}$, of the linear operator $A$ to its invariant subspace $E^s$, obtain
\[
\|(A|_{E^s})^n\| \leq c_{s,\varepsilon} \lambda_s^n, \quad n \geq 0.
\] (4.52)

Similarly, numbers $\lambda_u \in (0,1)$ and $\varepsilon > 0$ can be found such that $\sigma_u$ is contained in the exterior of the disc $|\lambda| \leq \lambda_u + \varepsilon$. The restriction $A|_{E^u}$, of the linear operator $A$ to its invariant subspace $E^u$, is an invertible operator, and applying the formula (4.51) to $(A|_{E^u})^{-1}$, obtain
\[
\|(A|_{E^u})^{-n}\| \leq c_{u,\varepsilon} \lambda_u^{-n}, \quad n \geq 0.
\] (4.53)

Define the norm $\|\cdot\|_*$ in $E$ by
\[
\|x\|_* = \max_{n \geq 0} \lambda_s^{-n} \|(A|_{E^s})^n P^s x\| + \max_{n \geq 0} \lambda_u^{-n} \|(A|_{E^u})^{-n} P^u x\|.
\]

Clearly
\[
\|x\|_* \geq \|P^s x\| + \|P^u x\| \geq \|P^s x + P^u x\| = \|x\|,
\]
and, at the same time, by (4.52), (4.53)
\[
\|x\|_* \leq c_{s,\varepsilon} \|P^s x\| + c_{u,\varepsilon} \|P^u x\| \leq \max \{c_{s,\varepsilon} \|P^s\|, c_{u,\varepsilon} \|P^u\|\} \|x\|.
\]

Hence the norms $\|\cdot\|_*$ and $\|\cdot\|$ are equivalent. To complete the proof of the implication (iii) $\Rightarrow$ (ii), it remains only to note from (4.52), (4.53) that the inequalities
\[
\|P^s A P^s x\|_* \leq \lambda_s \|P^s x\|_*,
\]
\[
\|P^u A P^u x\|_* \geq \lambda_u \|P^u x\|_*,
\]
then follow, which, together with invariance of the subspaces $E^s$ and $E^u$ with respect to $A$, mean that $A$ is a linear hyperbolic operator in the sense of Definition 4.3.3. The implication (iii) $\Rightarrow$ (ii) is then proved.

Finally, the implication (ii) $\Rightarrow$ (i) is clear by definition of the split $s$ and constant $h$, namely $s = (\lambda_s, \lambda_u, 0, 0)$, $h = \max \{\|P^s\|, \|P^u\|\}$. The proof of the theorem is complete.

\[\square\]

### 4.3.2 Equivalent Operators in Sequence Spaces

Let $\{A_n\}$ be a sequence of $d \times d$ matrices defined for $n$ belonging to some interval $I \subseteq \mathbb{Z}$ of the form $[0,N)$ or $[0,\infty)$ or $(-\infty,0]$ or $\mathbb{Z}$. Now, define
some linear operators induced by the sequence \( \{A_n\} \), acting on an appropriate sequence space. Consider first the linear operator \( \mathcal{A} : X \rightarrow Y \) defined by

\[
(\mathcal{A}x)_{n+1} = A_n x_n, \quad n \in \mathbb{I}.
\]  

(4.54)

Here the definition of the spaces \( X \) and \( Y \) depend on \( \mathbb{I} \). If \( \mathbb{I} = [0, N] \), then it is convenient to set

\[
X = \ell^\infty([0, N+1], \mathbb{R}^d), \quad Y = \ell^\infty([1, N+1], \mathbb{R}^d);
\]  

(4.55)

if \( \mathbb{I} = (-\infty, 0] \), then

\[
X = Y = \ell^\infty((-\infty, 1], \mathbb{R}^d);
\]  

(4.56)

if \( \mathbb{I} = [0, \infty) \), then set

\[
X = \ell^\infty([0, \infty), \mathbb{R}^d), \quad Y = \ell^\infty([1, \infty), \mathbb{R}^d);
\]  

(4.57)

and, finally, if \( \mathbb{I} = \mathbb{Z} \) set

\[
X = Y = \ell^\infty(\mathbb{Z}, \mathbb{R}^d).
\]  

(4.58)

In each case \( Y \subseteq X \), but only in the cases \( \mathbb{I} = (-\infty, 0] \) and \( \mathbb{I} = \mathbb{Z} \) do we have \( Y = X \). Note that the domain of \( \mathcal{A} \) is strictly contained in \( X \) in (4.57) and (4.58), although its range is all of \( Y \).

Consider also the linear operator \( \mathcal{D} : X \rightarrow Y \) defined by

\[
(\mathcal{D}x)_{n+1} = x_{n+1} - A_n x_n, \quad n \in \mathbb{I},
\]  

(4.59)

that is, \( \mathcal{D} = I - \mathcal{A} \).

### 4.3.3 An Inversion Theorem

Consider the invertibility problem for the operator \( \mathcal{D} \). It turns out that this problem depends heavily on which set of indices, \( \mathbb{I} = [0, N] \) or \( \mathbb{I} = [0, \infty) \) or \( \mathbb{I} = (-\infty, 0] \) or \( \mathbb{I} = \mathbb{Z} \), the sequence of matrices \( A_n \) is defined.

For example, to find the inverse operator of \( \mathcal{D} \) in the case \( \mathbb{I} = [0, N] \) we need to find for each \( z = \{z_n\} \in \ell^\infty([1, N+1], \mathbb{R}^d) \) an \( x = \{x_n\} \in \ell^\infty([0, N+1], \mathbb{R}^d) \) which satisfies

\[
x_{n+1} = A_n x_n + z_{n+1}, \quad n \in \mathbb{I}.
\]  

(4.60)

Clearly, the set of equations (4.60) is not sufficient to define \( x \) uniquely when \( \mathbb{I} = [0, N] \), as \( x_0 \) can be chosen arbitrarily here. A similar kind of problem, not quite so obviously stated, also occurs when we try to find a bounded sequence \( x \) satisfying (4.60) in the case \( \mathbb{I} = [0, \infty) \) or \( \mathbb{I} = (-\infty, 0] \).

Let \( \mathbb{R}^d = E^s_n \oplus E^u_n \) be the splitting associated with the matrix \( A_n \), for which the projectors are \( P^s_n : \mathbb{R}^d \rightarrow E^s_n \) and \( P^u_n = I - P^s_n : \mathbb{R}^d \rightarrow E^u_n \). Rewrite (4.60) in the decomposed form
\[ P_{n+1}^{s}x_{n+1} = P_{n+1}^{s}A_{n}P_{n}^{s}x_{n} + P_{n+1}^{s}A_{n}P_{n}^{u}x_{n} + P_{n+1}^{s}z_{n+1}, \]
\[ P_{n+1}^{u}x_{n+1} = P_{n+1}^{u}A_{n}P_{n}^{s}x_{n} + P_{n+1}^{u}A_{n}P_{n}^{u}x_{n} + P_{n+1}^{u}z_{n+1}. \]

Note that the linear operator \( U_{n} = P_{n+1}^{u}A_{n}P_{n}^{u} : E_{n}^{u} \to E_{n+1}^{u} \) is invertible by (4.11) and by the last condition (4.13) so we can rewrite these equations as
\[ P_{n+1}^{s}x_{n+1} = P_{n+1}^{s} (A_{n}P_{n}^{s}x_{n} + A_{n}P_{n}^{u}x_{n} + z_{n+1}), \]  
\[ P_{n}^{u}x_{n} = U_{n}^{-1} P_{n+1}^{u} (x_{n+1} - A_{n}P_{n}^{s}x_{n} - z_{n+1}). \]  

Now introduce the linear operator \( \mathcal{H}_{z} : X \to X \), which, for a fixed sequence \( z = \{z_{n}\} \in Y \), transforms every sequence \( x = \{x_{n}\} \in X \) into a sequence \( w = \{w_{n}\} \in X \) satisfying
\[ P_{n+1}^{s}w_{n+1} = P_{n+1}^{s} (A_{n}P_{n}^{s}x_{n} + A_{n}P_{n}^{u}x_{n} + z_{n+1}), \]
\[ P_{n}^{u}w_{n} = U_{n}^{-1} P_{n+1}^{u} (x_{n+1} - A_{n}P_{n}^{s}x_{n} - z_{n+1}), \]
where both equations hold for \( n \in \mathbb{I} \).

Observe that (4.63) does not define \( P_{0}^{s}w_{0} \) in the case when 0 is the left end of the interval \( \mathbb{I} \), while (4.64) fails to define \( P_{N+1}^{u}w_{N+1} \) when \( N \) is the right end of the interval \( \mathbb{I} \). Hence, the operator \( \mathcal{H}_{z} \) is well-defined only if equations (4.63) and (4.64) are supplemented by boundary conditions, which may be chosen to be one of the following:
\[ P_{0}^{s}w_{0} = 0, \quad P_{N+1}^{u}w_{N+1} = 0 \quad \text{if} \quad \mathbb{I} = [0, N], \]  
\[ P_{1}^{u}w_{1} = 0 \quad \text{if} \quad \mathbb{I} = (-\infty, 0], \]
\[ P_{0}^{s}w_{0} = 0, \quad \text{if} \quad \mathbb{I} = [0, \infty). \]

The system of equations (4.63)–(4.64) is sufficient to well-define \( \mathcal{H}_{z} \) without boundary conditions only in the case when \( \mathbb{I} = \mathbb{Z} \).

Now introduce the norm \( || \cdot ||_{\infty} \) on the space \( X \) by
\[ ||x||_{\infty} = \sup_{n} \{ \gamma ||P_{n}^{s}x_{n}||, ||P_{n}^{u}x_{n}|| \}, \]
where \( \gamma = \gamma(s) \) as in (4.5) and the indices \( n \) are taken from \([0, N + 1]\) if \( \mathbb{I} = [0, N] \), from \(({-\infty, 1}] \) if \( \mathbb{I} = (-\infty, 0] \), and from \( \mathbb{I} \) if \( \mathbb{I} = [0, \infty) \) or \( \mathbb{I} = \mathbb{Z} \). As mentioned in Sect. 4.1, the split \( s \) under consideration may be treated without loss of generality as positive, in which case the norm \( || \cdot ||_{\infty} \) in \( X \) is equivalent to the norm \( || \cdot ||_{\infty} \) of (4.12) for any admissible \( n \). To see this, note that for any \( x_{n} \in \mathbb{R}^{d} \) we have
\[ \frac{1}{2} ||x_{n}|| \leq \max \{ ||P_{n}^{s}x_{n}||, ||P_{n}^{u}x_{n}|| \} \leq h ||x_{n}||, \]
so by Lemma 4.1.1.
\[
\frac{1}{2} \min \{ \gamma, 1 \} \| x_n \| \leq \max \{ \gamma \| P_n^s x_n \|, \| P_n^u x_n \| \} \leq h \max \{ \gamma, 1 \} \| x_n \|,
\]
from which it follows that
\[
\frac{1}{2} \min \{ \gamma, 1 \} \| x \|_\infty \leq \| x \|_\infty \leq h \max \{ \gamma, 1 \} \| x \|_\infty. \tag{4.69}
\]

**Lemma 4.3.7.** For each \( z \in Y \) the operator \( H_z : X \to X \) has a unique fixed point \( x := Bz \in X \) for which
\[
\| x \|_\infty \leq h \nu(s) \| z \|_\infty. \tag{4.70}
\]

**Proof.** We show first that the operator \( H_z : X \to X \) is contracting in the norm \( \| \cdot \|_\infty \) with contraction constant \( \sigma = \sigma(s) \) as in \( (4.3) \).

Given \( x, \tilde{x} \in X \), write \( y = x - \tilde{x} \) and \( v = H_z(x) - H_z(\tilde{x}) \). Then by \( (4.63), (4.64) \)
\[
P_{n+1}^s v_{n+1} = P_{n+1}^s (A_n P_n^s y_n + A_n P_n^u y_n),
\]
\[
P_n^u v_n = U_n^{-1} P_{n+1}^u (y_{n+1} - A_n P_n^s y_n),
\]
to which may be added, if needed, one or the both following boundary conditions
\[
P_0^s v_0 = 0, \quad P_{N+1}^u v_{N+1} = 0,
\]
by \( (4.68) \) if \( \mathbb{I} = (-\infty, 0] \), or \( (4.69) \) if \( \mathbb{I} = [0, \infty) \). Hence, by definition of the operator \( U_n \) and by \( (4.13) \) obtain
\[
\| P_{n+1}^s v_{n+1} \| \leq \lambda_s \| P_n^s y_n \| + \mu_s \| P_n^u y_n \|, \tag{4.71}
\]
\[
\| P_n^u v_n \| \leq \frac{\mu_u}{\lambda_u} \| P_n^s y_n \| + \frac{1}{\lambda_u} \| P_{n+1}^u y_{n+1} \|, \tag{4.72}
\]
again to be supplemented, if necessary, by
\[
\| P_0^s v_0 \| = 0, \quad \| P_{N+1}^u v_{N+1} \| = 0. \tag{4.73}
\]

Write
\[
V_1 = \sup_n \gamma \| P_n^s v_n \|, \quad V_2 = \sup_n \| P_n^u v_n \|,
\]
\[
Y_1 = \sup_n \gamma \| P_n^s y_n \|, \quad Y_2 = \sup_n \| P_n^u y_n \|.
\]
Then
\[
\| v \|_\infty = \max \{ V_1, V_2 \}, \quad \| y \|_\infty = \max \{ Y_1, Y_2 \}
\]
and so, from \( (4.71), (4.72) \), and from \( (4.73) \), if necessary, obtain
\[
V_1 \leq \lambda_s Y_1 + \gamma \mu_s Y_2 \leq (\lambda_s + \gamma \mu_s) \| y \|_\infty, \tag{4.74}
\]
\[
V_2 \leq \frac{\mu_u}{\gamma \lambda_u} Y_1 + \frac{1}{\lambda_u} V_2 \leq \left( \frac{\mu_u}{\gamma \lambda_u} + \frac{1}{\lambda_u} \right) \| y \|_\infty. \tag{4.75}
\]
But from (4.6),
\[ \lambda_s + \gamma \mu_s = \sigma, \quad \frac{\mu_u}{\gamma \lambda_u} + \frac{1}{\lambda_u} = \sigma, \]
so, by (4.74) and (4.75) obtain
\[ \|v\|_\infty = \max \{V_1, V_2\} \leq \sigma \|y\|_\infty, \]
that is
\[ \|H_z(x) - H_z(\tilde{x})\|_\infty \leq \sigma \|x - \tilde{x}\|_\infty. \]

By the Banach Contraction Mapping Theorem the operator \( H_z \) for any \( z \in Y \) has a unique fixed point \( x := Bz \in X \) for which
\[ \|Bz\|_\infty \leq \frac{1}{1 - \sigma} \|H_z(0)\|_\infty. \]
Unfortunately, the obvious bound on \( \|x\|_\infty \) provided by Lemma [4.1.1] is only a rough bound, so another approach is required to give the tighter inequality (4.72).

Since by (4.69) the norms \( \| \cdot \|_\infty \) and \( \| \cdot \|_\infty \) are equivalent then, from the Contraction Mapping Theorem, the sequence of successive iterates
\[ x^{(m)} = H_z(x^{(m-1)}), \quad m = 1, 2, \ldots, \]
with \( x^{(0)} = 0 \), converges in the norm \( \| \cdot \|_\infty \) to the fixed point \( x = Bz \) of \( H_z \). In particular,
\[ \|x\|_\infty \leq \limsup_{m \to \infty} \|x^{(m)}\|_\infty. \] (4.76)

Set
\[ a_m = \sup_n \|P^s_n x^{(m)}_n\|, \quad b_m = \sup_n \|P^u_n x^{(m)}_n\| \] (4.77)
and
\[ h^s = \sup_n \|P^s_{n+1} z_{n+1}\|, \quad h^u = \sup_n \|U^{-1}_n P^u_{n+1} z_{n+1}\|. \] (4.78)
Then, by taking the supremum of the norms of the left and right hand sides of equations (4.63), (4.64) and, if necessary, of one or both boundary conditions of (4.65), obtain the system of inequalities
\[ a_m \leq \lambda_s a_{m-1} + \mu_s b_{m-1} + h^s, \]
\[ b_m \leq \frac{\mu_u}{\lambda_u} a_{m-1} + \frac{1}{\lambda_u} b_{m-1} + h^u, \]
from which, by Lemma [4.1.2] it follows that
\[ \limsup_{m \to \infty} \|x^{(m)}\|_\infty \leq \limsup_{m \to \infty} (a_m + b_m) \leq \frac{\max \{h^s, \lambda_u h^u\}}{\nu(s)}. \] (4.79)
But clearly, by (4.12) and the definition of the operator \( U_n \),
\[ h^s \leq h\|z\|_\infty, \quad h^u \leq \frac{h}{\lambda_u}\|z\|_\infty. \]  

(4.80)

The required bound \( (4.70) \) follows from this and from \( (4.76) \) and \( (4.79) \). The lemma is proved. \( \square \)

Now consider the invertibility of the bounded linear operator \( D \) that was defined by \( (4.59) \) for a semi-hyperbolic sequence of matrices \( \{A_n\} \).

**Theorem 4.3.8.** If \( \{A_n\} \) is a \((s, h)\)-semi-hyperbolic sequence of \( d \times d \) matrices, then the linear operator \( D : X \to Y \) defined by \( (4.59) \), with spaces \( X, Y \) as in any of \( (4.55), (4.56), (4.57) \) or \( (4.58) \), is bounded and has a bounded right inverse \( D^{-1} : Y \to X \) satisfying

\[ \|D^{-1}\|_\infty \leq \frac{h}{\nu(s)}. \]  

(4.81)

**Proof.** By Lemma 4.3.7, for each \( z \in Y \) the operator \( H_z \) has a unique fixed point \( x = Bz \) for which

\[ \|Bz\|_\infty \leq \frac{h}{\nu(s)}\|z\|_\infty \]  

(4.82)

holds.

The operator \( B \) is obviously linear. By comparing equations \( (4.63) \) and \( (4.64) \) with \( (4.61) \) and \( (4.62) \), conclude that \( x \) is the solution of \( (4.60) \). Thus, the operator \( B \) is the right inverse to \( D \), that is \( B = D^{-1} \), and the inequality \( (4.81) \) follows from \( (4.82) \). \( \square \)

**Remark 4.3.9.** When the sequence of matrices \( A_n \) is defined for the set of indices \( I = \mathbb{Z} \) the bounded right inverse to \( D \) operator \( D^{-1} \) satisfying \( (4.81) \) is defined uniquely, while in the cases \( I = [0, N], I = [0, \infty) \) and \( I = (-\infty, 0] \) it is not.

**Remark 4.3.10.** In Definition 4.2.16 and hence in Theorem 4.3.8, the matrix sequence \( \{A_n\} \) is supposed to be bounded. This requirement has been imposed only to simplify proofs and can be easily removed, with obvious changes in the formulation of assertions and proofs.

With Definition 4.2.16 of semi-hyperbolic sequences of matrices an analogue of Theorem 4.3.8 is valid with some insignificant changes resulting from using the variable norm inequalities \( (4.40) \).

**Theorem 4.3.11.** If \( \{A_n\} \) is a \((s, h)\)-semi-hyperbolic sequence of \( d \times d \) matrices as in Definition 4.2.16, then the linear operator \( D : X \to Y \) defined by \( (4.59) \), with spaces \( X, Y \) defined by any of \( (4.55), (4.56), (4.57) \) or \( (4.58) \), is bounded and has a bounded right inverse \( D^{-1} : Y \to X \) satisfying

\[ \|D^{-1}\|_\infty \leq \frac{q^2 h}{\nu(s)}, \]

where \( q \) is as in the inequalities \( (4.37) \) of Definition 4.2.16.
Proof. In the proof of Theorem 4.3.8, we adjust definition (4.68) of the norm \( \| \cdot \|_\infty \) and definitions (4.77), (4.78) of \( a_n, b_n, h^s, h^u \) by replacing the norms \( \| P^s_n x_n \|, \| P^u_n x_n \| \) by the norms \( \| P^s_n x_n \|_n, \| P^u_n x_n \|_n \) in the following way

\[
\| x \|_\infty = \sup_n \{ \gamma \| P^s_n x_n \|_n, \| P^u_n x_n \|_n \}.
\]

With such an adjustment the proof of Theorem 4.3.11 is essentially the same as that of Theorem 4.3.8. It needs only to multiply by the factor \( q \) the corresponding sides of inequalities (4.79) and (4.80) involving estimation of the \( \| \cdot \|_\infty \)-norms of elements using \( a_n, b_n, h \), and conversely.

\[\square\]

4.3.4 Hyperbolicity of the Linear Operator \( \mathcal{A} \)

To complete Sect. 4.3 consider the relationship between the hyperbolicity of a matrix sequence and hyperbolicity of the linear operator \( \mathcal{A} \) defined by (4.54) in Sect. 4.3.2. Then, using Theorem 4.2.19, reveals the relationship between the semi-hyperbolicity of a matrix sequence and hyperbolicity of the linear operator \( \mathcal{A} \) and, by Theorem 4.3.6, the relationship between the semi-hyperbolicity of a matrix sequence and semi-hyperbolicity of the linear operator \( \mathcal{A} \).

Theorem 4.3.12. The sequence of invertible matrices \( \{ A_n \}, n \in \mathbb{Z} \), is hyperbolic if and only if the linear operator \( \mathcal{A} : \ell^\infty(\mathbb{Z}, \mathbb{R}^d) \to \ell^\infty(\mathbb{Z}, \mathbb{R}^d) \) defined by (4.54) is hyperbolic.

Proof. To simplify notation, denote \( \mathcal{L} = \ell^\infty(\mathbb{Z}, \mathbb{R}^d) \). Suppose that the sequence of invertible matrices \( \{ A_n \}, n \in \mathbb{Z} \), is hyperbolic. For each integer \( n \), let the projections \( P^s_n, P^u_n \) and norm \( \| \cdot \|_n \) be as in Definition 4.2.18 and define linear operators \( \mathcal{P}^s \) and \( \mathcal{P}^u \) on \( \mathcal{L} \) by

\[
(\mathcal{P}^s x)_n = P^s_n x_n, \quad (\mathcal{P}^u x)_n = P^u_n x_n, \quad n \in \mathbb{Z}.
\]

Clearly, the operators \( \mathcal{P}^s \) and \( \mathcal{P}^u \) are bounded projections and, moreover, \( \mathcal{P}^s + \mathcal{P}^u = I \). Hence the subspaces

\[
\mathcal{E}^s = \mathcal{P}^s \mathcal{L} \subseteq \mathcal{L}, \quad \mathcal{E}^u = \mathcal{P}^u \mathcal{L} \subseteq \mathcal{L}
\]

have trivial intersection and their direct sum coincides with \( \mathcal{L} \),

\[
\mathcal{E}^s \cap \mathcal{E}^u = \{ 0 \}, \quad \mathcal{E}^s \oplus \mathcal{E}^u = \mathcal{L}.
\]

In addition, it follows from the equivariance relations (4.41) and the definition of projections \( \mathcal{P}^s \) and \( \mathcal{P}^u \) that the subspaces \( \mathcal{E}^s \) and \( \mathcal{E}^u \) are invariant under the linear operator \( \mathcal{A} \),

\[
\mathcal{A} \mathcal{E}^s \subseteq \mathcal{E}^s, \quad \mathcal{A} \mathcal{E}^u \subseteq \mathcal{E}^u.
\]
Introduce the norm \( \| x \|_{\infty}^* = \sup_{n \in \mathbb{Z}} \{ \| P_n^s x_n \|_n, \| P_n^u x_n \|_n \} \) on the space \( \mathcal{L} \). Using (4.37) and (4.39) obtain the chain of inequalities

\[
\frac{1}{2q} \| x_n \| \leq \frac{1}{2} \| x_n \|_n \leq \frac{1}{2} (\| P_n^s x_n \|_n + \| P_n^u x_n \|_n) \leq \max \{ \| P_n^s x_n \|_n, \| P_n^u x_n \|_n \} \leq h \| x_n \|_n \leq hq \| x_n \|
\]

for any integer \( n \), from which it follows that

\[
\frac{1}{2q} \| x \|_{\infty} \leq \| x \|_{\infty}^* \leq hq \| x \|_{\infty},
\]

and the norms \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{\infty}^* \) in the space \( \mathcal{L} \) are equivalent.

From the inequalities (4.42), which hold for the matrix sequence \( \{ A_n \} \) because of hyperbolicity, it follows immediately that

\[
\| A P_n^s x \|_{\infty}^* \leq \lambda_s \| P_n^s x \|_{\infty}^*, \quad \| A P_n^u x \|_{\infty}^* \geq \lambda_u \| P_n^u x \|_{\infty}.
\]

Together with (4.83), these inequalities imply that the spectrum \( \sigma_s \), of the restriction \( \mathcal{A}|_{\mathcal{E}_s} \) of the linear operator \( \mathcal{A} \) to its invariant subspace \( \mathcal{E}_s \), lies entirely in the disc \( |\lambda| \leq \lambda_s < 1 \). Similarly, the spectrum \( \sigma_u \), of the restriction \( \mathcal{A}|_{\mathcal{E}_u} \) of the linear operator \( \mathcal{A} \) to its invariant subspace \( \mathcal{E}_u \), lies entirely outside the disc \( |\lambda| \leq \lambda_u \). Now, from (4.83) and (4.84) it follows that the spectrum of the operator \( \mathcal{A} \) coincides with the union of the sets \( \sigma_s \) and \( \sigma_u \), and thus does not intersect the disc \( |\lambda| = 1 \). Hence, the linear operator \( \mathcal{A} \) is hyperbolic.

Now consider the converse, namely that hyperbolicity of the operator \( \mathcal{A} \) implies hyperbolicity of the matrix sequence \( \{ A_n \} \). By supposition, the spectrum \( \sigma \) of the operator \( \mathcal{A} \) does not intersect the unit disc \( |\lambda| = 1 \). Since the spectrum of any operator is a closed set, the set \( \sigma \) can then be represented as union \( \sigma = \sigma_s \cup \sigma_u \) of closed subsets, \( \sigma_s \) which lies entirely inside the unit disk, and \( \sigma_u \) which lies entirely outside the unit disk.

Denote by \( \mathcal{E}_s \subseteq \mathcal{L} \) the invariant spectral subspace of the operator \( \mathcal{A} \) corresponding to the subset \( \sigma_s \) of its spectrum, and denote by \( \mathcal{E}_u \subseteq \mathcal{L} \) the invariant spectral subspace of the operator \( \mathcal{A} \) corresponding to \( \sigma_u \). Then the relations (4.84) hold. Denote the projections corresponding to the splitting (4.84) by \( \mathcal{P}_s : \mathcal{L} \rightarrow \mathcal{E}_s \) and \( \mathcal{P}_u = I - \mathcal{P}_s : \mathcal{L} \rightarrow \mathcal{E}_u \), which are bounded in this case.

The spectral radius \( \sigma(A) \) of a linear operator \( A \) on a Banach space satisfies

\[
\sigma(A) = \limsup_{n \to \infty} \| A^n \|^{1/n},
\]

from which follows the existence for each \( \varepsilon > 0 \) of a number \( c_\varepsilon \) such that
\[ \|A^n\| \leq c_\varepsilon (\sigma(A) + \varepsilon)^n, \quad n \geq 0. \] (4.85)

By definition, the closed set \( \sigma_s \) lies in the disc \( |\lambda| < 1 \), so numbers \( \lambda_s \in (0,1) \) and \( \varepsilon > 0 \) can be found such that \( \sigma_s \) will, in fact, belong to the disc \( |\lambda| \leq \lambda_s - \varepsilon \). Applying (4.85) to the restriction \( \mathcal{A}|_{\mathcal{E}^s} \), of the linear operator \( \mathcal{A} \) to its invariant subspace \( \mathcal{E}^s \), and taking \( \| \cdot \|_\infty \) as the operator norm \( \| \cdot \| \) in (4.85), it follows that
\[ \left\| (\mathcal{A}|_{\mathcal{E}^s})^n \right\|_\infty \leq c_\varepsilon \lambda_s^n, \quad n \geq 0, \]
for an appropriate constant \( c_\varepsilon \), from which
\[ \| \mathcal{A} P^s x \|_\infty \leq c_\varepsilon \lambda_s^n \| x \|_\infty, \quad x \in \mathcal{L}, \quad n \geq 0. \] (4.86)

Similarly, numbers \( \lambda_u > 1 \) and \( \varepsilon > 0 \) can be found such that the closed set \( \sigma_u \) belongs to the exterior of the disc \( |\lambda| \leq \lambda_u + \varepsilon \). Then the restriction \( \mathcal{A}|_{\mathcal{E}^u} \), of the linear operator \( \mathcal{A} \) to its invariant subspace \( \mathcal{E}^u \), is invertible and, applying the formula (4.85) to \( (\mathcal{A}|_{\mathcal{E}^u})^{-1} \), obtain
\[ \left\| (\mathcal{A}|_{\mathcal{E}^u})^{-n} \right\|_\infty \leq c_\varepsilon \lambda_u^{-n}, \quad n \geq 0, \]
for an appropriate constant \( c_\varepsilon \), from which it follows that
\[ \| \mathcal{A} P^u x \|_\infty \geq c_\varepsilon^{-1} \lambda_u^n \| x \|_\infty, \quad x \in \mathcal{L}, \quad n \geq 0. \] (4.87)

Define the norm on \( \mathcal{L} \) by
\[ \| x \|_* = \sup_{n \geq 0} \lambda_s^{-n} \| \mathcal{A} P^s x \|_\infty + \inf_{n \geq 0} \lambda_u^{-n} \| \mathcal{A} P^u x \|_\infty. \] (4.88)

From (4.86) and (4.87) the norms \( \| \cdot \|_\infty \) and \( \| \cdot \|_* \) are equivalent, so there exists a number \( Q \) such that
\[ Q^{-1} \| x \|_\infty \leq \| x \|_* \leq Q \| x \|_\infty \] (4.89)
for all \( x \in \mathcal{L} \). From (4.88) it thus follows that
\[ \| \mathcal{A} P^s x \|_* \leq \lambda_s \| P^s x \|_* , \quad \| \mathcal{A} P^u x \|_* \geq \lambda_u \| P^u x \|_* . \] (4.90)

Now fix an integer \( k \in \mathbb{Z} \) and define \( E^s_k \subseteq \mathbb{R}^d \) to be the set of those \( v \in \mathbb{R}^d \) for which the equation
\[ v_{n+1} = A_n v_n, \quad n \geq k, \] (4.91)
has a bounded solution satisfying \( v_k = v \). Similarly, define \( E^u_k \subseteq \mathbb{R}^d \) to be the set of those \( w \in \mathbb{R}^d \) for which the equation
\[ w_{n+1} = A_n w_n, \quad n < k, \] (4.92)
has a bounded solution satisfying \( w_k = w \). It is not hard to see that \( E^s_k \) and \( E^u_k \) are subspaces of \( \mathbb{R}^d \). It remains to show that these form a splitting of \( \mathbb{R}^d \), that is with
\[
E^s_k \cap E^u_k = \{ 0 \}, \quad E^s_k \oplus E^u_k = \mathbb{R}^d.
\]
(4.93)

Let \( x \in E^s_k \cap E^u_k \). By definition of the subspace \( E^s_k \) there exists a bounded sequence \( \{ v_n \} \), \( n \geq k \), satisfying (4.91) and \( v_k = x \). Similarly, by definition of the subspace \( E^u_k \) there exists a bounded sequence \( \{ w_n \} \), \( n \leq k \), satisfying (4.92) and \( w_k = x \). Then the sequence \( x = \{ x_n \} \) defined by
\[
x_n = \begin{cases} v_n, & n \geq k, \\ w_n, & n \leq k \end{cases}
\]
will satisfy the equation
\[
x_{n+1} = A_n x_n, \quad n \in \mathbb{Z},
\]
and hence
\[
x = A x.
\]
In view of hyperbolicity of the operator \( A \), this implies that \( x = 0 \). The first part of (4.93) is thus proved.

To prove the remainder of (4.93) choose an arbitrary but fixed \( x \in \mathbb{R}^d \) and for any fixed \( k > 0 \) define the sequence \( x = \{ x_n \} \in \mathcal{L} \) by
\[
x_k = x, \quad x_n = 0 \quad \text{for} \quad n \neq k, \quad (4.94)
\]
and let
\[
x^s = \mathcal{P}^s x, \quad x^u = \mathcal{P}^u x. \quad (4.95)
\]
Then \( x = x^s + x^u \) and \( x = x^s_k + x^u_k \). It suffices to show that
\[
x^s_k \in E^s_k, \quad x^u_k \in E^u_k. \quad (4.96)
\]
Write \( v^{(k)} = x^s \) and define elements \( v^{(n)} \in \mathcal{L} \) recursively by
\[
v^{(n+1)} = A v^{(n)}, \quad n \geq k.
\]
From (4.90) obtain
\[
\| v^{(k)} \|_* \geq \| v^{(k+1)} \|_* \geq \cdots \geq \| v^{(n)} \|_* \geq \cdots, \quad n \geq k,
\]
and the sequence with components \( v^{(n)} \in \mathcal{L}, n \geq k \), is bounded in the norm \( \| \cdot \|_* \) and thus, by (4.89), also in the norm \( \| \cdot \|_\infty \). Hence the sequence with elements \( v_n \in \mathbb{R}^d \) defined by
\[
v_n = v^{(n)}, \quad n \geq k,
\]
is also bounded. Clearly, the sequence \( \{ v_n \} \) satisfies (4.91) and \( v_k = x^s_k \). Hence, in view of the definition of the subspace \( E^s_k \), the first inclusion of (4.96) is
proved. The second inclusion of (4.96) is proved similarly. The second part of (4.93) is now proved.

Define now operators $P^s_k$ and $P^u_k$ by

$$P^s_k x = x^s_k, \quad P^u_k x = x^u_k.$$  

By definition of the points $x^s_k$ and $x^u_k$, these operators are linear and satisfy

$$P^s_k \mathbb{R}^d \subseteq E^s_k, \quad P^u_k \mathbb{R}^d \subseteq E^u_k, \quad P^s_k + P^u_k = I.$$  

From these and (4.93) it follows that the operators $P^s_k$ and $P^u_k$ are projections.

To estimate the norms $\|P^s_k\|$ and $\|P^u_k\|$ we use the chain of inequalities

$$\|P^s_k x\| = \|x^s_k\| \leq \|x^s\|_\infty = \|\mathcal{P}^s x\|_\infty \leq \|\mathcal{P}^s\|_\infty \|x\|_\infty = \|\mathcal{P}^s\|_\infty \|x\|$$

from which it follows that

$$\|P^s_k\| \leq \|\mathcal{P}^s\|_\infty \leq \max \{\|\mathcal{P}^s\|_\infty, \|\mathcal{P}^u\|_\infty\}.$$  

Similarly,

$$\|P^u_k\| \leq \|\mathcal{P}^u\|_\infty \leq \max \{\|\mathcal{P}^s\|_\infty, \|\mathcal{P}^u\|_\infty\},$$

so a uniform norm bound exists for the projections $P^s_k$ and $P^u_k$ with $k \in \mathbb{Z}$ (cf. (4.39)).

The equivariant identities (4.41) for the matrix sequence $\{A_n\}$ then follows immediately from the definition of subspaces $E^s_n$ and $E^u_n$ and from invertibility of the matrices $A_n$. It remains only to construct norms $\|\cdot\|_n$ which ensure that the inequalities (4.42) are valid. To do this, more detailed information about properties of the projections $\mathcal{P}^s$ and $\mathcal{P}^u$ is required.

Given a point $x \in \mathbb{R}^d$ construct points $x, x^s, x^u \in \mathcal{L}$ by formulae (4.94) and (4.95). It suffices to show that

$$x = x^s \quad \text{if} \quad x \in E^s_k, \quad (4.97)$$  

$$x = x^u \quad \text{if} \quad x \in E^u_k. \quad (4.98)$$

For definiteness, let $x \in E^s_k$. Then, in the same way as the inclusions (4.96) were obtained,

$$x^s_n \in E^s_n, \quad x^s_n \in E^u_n, \quad n \in \mathbb{Z}. \quad (4.99)$$

But since $x = x^s + x^u$, from the definition (4.94) of the point $x$, it follows immediately that

$$x^s_n = -x^u_n, \quad n \neq k,$$

which, in view of (4.99) and (4.93), can only be satisfied if

$$x^s_n = x^u_n = 0, \quad n \neq k.$$  

The required relations (4.97) and (4.98) follow from this.

Now define the norm $\|\cdot\|_k$ on $\mathbb{R}^d$ by
where \( x \in \mathcal{L} \) is the element defined by (4.94). From (4.89) and from the obvious equality \( \| x \| = \| x \|_\infty \) obtain
\[
Q^{-1} \| x \| = Q^{-1} \| x \|_\infty \leq \| x \|_k = \| x \|_* \leq Q \| x \|_\infty = Q \| x \|,
\]
and the uniform boundedness conditions (cf. (4.37)) are satisfied by the norms \( \| \cdot \|_k, k \in \mathbb{Z} \).

Now, given \( x \in E_k^s \), denote \( y = A_k x \). Then, as previously shown, \( y \in E_{k+1}^s \).
Define the element \( x \in \mathcal{L} \) by (4.94) and the element \( y \in \mathcal{L} \) in the same manner by
\[
y_{k+1} = y, \quad y_n = 0 \quad \text{for} \quad n \neq k + 1.
\]
Then clearly \( y = \mathcal{A} x \), and from the inclusions \( x \in E_k^s, y \in E_{k+1}^s \) and (4.97) it follows that
\[
x \in \mathcal{E}^s, \quad y = \mathcal{A} x \in \mathcal{E}^s,
\]
from which, by (4.90),
\[
\| A_k x \|_{k+1} = \| \mathcal{A} x \|_* \leq \lambda_s \| x \|_* = \lambda_s \| x \|_k, \quad x \in E_k^s.
\]
The inequality
\[
\| A_k x \|_{k+1} = \| \mathcal{A} x \|_* \geq \lambda_u \| x \|_* = \lambda_u \| x \|_k, \quad x \in E_k^u
\]
can be proved in similar fashion. The proof of the theorem is now complete. \( \square \)
Semi-Hyperbolicity and Hyperbolicity

In Chap. 4 it was shown that a semi-hyperbolic sequence of matrices is hyperbolic, that is semi-hyperbolicity implies hyperbolicity in the linear case. This is generally not true for nonlinear mappings, which is shown below by an example in Sect. 5.1, which demonstrates that a semi-hyperbolic mapping may not possess an invariant splitting and so cannot be a hyperbolic. Nevertheless, it is shown in Sect. 5.2 that a semi-hyperbolic mapping which is smooth and invertible in a neighborhood of a compact invariant set is hyperbolic on that set. The proofs depend substantially on background material and results presented in Chap. 4.

5.1 Perturbation of Anosov Endomorphisms

In this section an example is constructed which demonstrates that generally a semi-hyperbolic mapping may not possess an invariant splitting, and thus cannot be hyperbolic.

Briefly recall some properties of Anosov mappings from Sect. 3.2.2. Write the elements of $\mathbb{R}^d$ as vectors $x$ with coordinates $x_1, x_2, \ldots, x_d$ and let $T^d$ be the standard $d$-dimensional torus. This torus $T^d$ is a compact differentiable manifold in $\mathbb{R}^d$ with respect to the locally Euclidean metric

$$\rho(x, y) = \sqrt{|x_1 - y_1|^2_{\text{mod } 1} + |x_2 - y_2|^2_{\text{mod } 1} + \cdots + |x_d - y_d|^2_{\text{mod } 1}}$$
on T^d where

$$|t - s|_{\text{mod } 1} = \min \{|t - s + 2k| : k = 0, \pm 1\}, \quad 0 \leq t, s < 1.$$

The tangent space $T_x T^d$ can then be identified with $\mathbb{R}^d$ by an appropriate choice of natural coordinates generated by those of $\mathbb{R}^d$, so $T_x T^d = \mathbb{R}^d$ for each $x \in T^d$. Denote the natural projection from $\mathbb{R}^d$ onto $T^d$ by

$$\Pi(x) = (x_1 \text{ mod } 1, x_2 \text{ mod } 1, \ldots, x_d \text{ mod } 1), \quad x = (x_1, x_2, \ldots, x_d).$$
Let $A$ be a given $d \times d$ matrix with integer components $a_{ij}$ and define the mapping $f : \mathbb{T}^d \to \mathbb{T}^d$ defined by

$$f(x) = \Pi(Ax).$$

**Theorem 5.1.1.** Let $d \geq 3$. There exists a hyperbolic $d \times d$ matrix $A$ with integer components such that for every $\varepsilon > 0$ there can be found a differentiable semi-hyperbolic mapping $f_\varepsilon : \mathbb{T}^d \to \mathbb{T}^d$ which is $\varepsilon$-close in the $C^1$-norm to the mapping $f(x) = \Pi(Ax)$, but for which there is no continuous hyperbolic splitting.

**Proof.** First, let $d = 3$. Consider the $3 \times 3$ matrix

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 2 - \sqrt{3}$, $\lambda_2 = 2 + \sqrt{3}$ and $\lambda_3 = 2$ with corresponding eigenvectors $v_1 = (1, -1/\sqrt{3}, 0)$, $v_2 = (1, 1/\sqrt{3}, 0)$ and $v_3 = (0, 0, 1)$, respectively. Let $B$ be a fixed $3 \times 3$ matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

such that $v_2$ is not an eigenvector, but otherwise arbitrary.

Consider the sequence of points

$$x^{(0)} = (0, 0, 0), \quad x^{(n)} = (0, 0, 2^n) \quad \text{for} \quad n = -1, -2, \ldots,$$

and for $\varepsilon \geq 0$ denote by $f_\varepsilon : \mathbb{T}^3 \to \mathbb{T}^3$ a continuously differentiable mapping satisfying

(i) $f_\varepsilon$ is $\varepsilon$-close to $f$ in the $C^1$-norm;
(ii) $f_\varepsilon(x^{(-1)}) = f(x^{(-1)}) = 0$ and $(Tf_\varepsilon)_{x(-1)} = A + \varepsilon B$, where $T$ is the tangent mapping;
(iii) $f_\varepsilon(x) \equiv f(x)$ for all $x$ such that $\|x - x^{(-1)}\| \geq \frac{1}{8}$.

Such a mapping $f_\varepsilon$ exists for sufficiently small $\varepsilon > 0$. It satisfies

$$x^{(n)} = f_\varepsilon(x^{(n-1)}), \quad n = 0, -1, -2, \ldots \quad (5.1)$$

and is semi-hyperbolic by Lemma [3.2.5].

Since the mapping $f = f_0$ is an Anosov endomorphism, see Example [3.2.4], it has a hyperbolic splitting at every point $x$ independent of $x$, and can thus be written as

$$T_x \mathbb{T}^3 = E^s \oplus E^u, \quad x \in \mathbb{T}^3, \quad (5.2)$$
5.1 Perturbation of Anosov Endomorphisms

where $E^s$ is the eigenspace of the matrix $A$ corresponding to the eigenvalue $\lambda_1$ and $E^u$ is the eigenspace of the matrix $A$ corresponding to the eigenvalues $\lambda_2$ and $\lambda_3$.

Proof by contradiction is used to show that the mapping $f_\varepsilon$, for sufficiently small $\varepsilon > 0$, does not have a hyperbolic splitting.

Fix $\varepsilon > 0$ sufficiently small and suppose that

$$ T_x T^3 = E^s_{x,\varepsilon} \oplus E^u_{x,\varepsilon}, \quad x \in T^3, \quad (5.3) $$

is a hyperbolic splitting for $f_\varepsilon$. By Property (iii) $f_\varepsilon(x) \equiv f(x)$ in a neighborhood of the point $x = 0$, so $x = 0$ is a fixed point of $f_\varepsilon$ and by the Hartman–Grobman Theorem \[88\] the splitting (5.3) at the point $x = 0$ coincides with the splitting (5.2), that is

$$ E^s_{0,\varepsilon} \equiv E^s, \quad E^u_{0,\varepsilon} \equiv E^u. $$

In fact, to these equivalences can be added

$$ E^u_{x(n),\varepsilon} \equiv E^u, \quad n = -1, -2, \ldots. $$

Indeed, by Properties (ii) and (iii)

$$(Tf_\varepsilon)_{x(n)} = A \quad \text{for} \quad n = -2, -3, \ldots, \quad (5.4)$$

and since by supposition the subspaces $E^u_{x,\varepsilon}$ are equivariant for $Tf_\varepsilon$, then from (5.1) and (5.4) it follows that

$$ E^u_{x(n),\varepsilon} = (Tf_\varepsilon)^{n-k}_{x(k)} E^u_{x(k),\varepsilon} = (Tf_\varepsilon)^{n-1}_{x(n-2)} \cdots (Tf_\varepsilon)^{0}_{x(0)} E^u_{x(0),\varepsilon} = A^{n-k} E^u_{x(k),\varepsilon} \quad (5.5) $$

for all $k \leq n$. But, by the continuity of the splitting (5.3), the subspace $E^u_{x(k),\varepsilon}$ is close to $E^u$ for $k \to -\infty$ and thus does not contain vectors from $E^s$. Hence, taking the limit as $k \to -\infty$ in the right hand side of (5.5), we obtain

$$ E^u_{x(n),\varepsilon} = \lim_{k \to -\infty} A^{n-k} E^u_{x(k),\varepsilon} = E^u. $$

It remains to observe that

$$(Tf_\varepsilon)_{x(-1)} E^u_{x(-1),\varepsilon} = (Tf_\varepsilon)_{x(-1)} E^u = (A + \varepsilon B) E^u$$

$$ \neq E^u = E^u_{0,\varepsilon} = E^u_{x(0),\varepsilon} = E^u_{f_\varepsilon(x(-1)),\varepsilon} $$

since $(Tf_\varepsilon)_{x(-1)} = A + \varepsilon B$, and this contradicts the assumed equivariance of the splitting (5.3) with respect to $Tf_\varepsilon$. This contradiction completes the proof in the case when $d = 3$.

To prove the theorem in the case when $d > 3$ it suffices to consider the $d \times d$ block diagonal matrices

$$ \hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} $$

and repeat the argument above for the mapping $\hat{f}(x) = \Pi(\hat{A}x)$. \qed
5.2 Converse Case

Any hyperbolic mapping is semi-hyperbolic. Theorem 5.2.1 below shows that the converse is also valid for invertible smooth mappings, so establishing equivalence between semi-hyperbolicity and hyperbolicity for invertible mappings. The proof is valid for each of the various definitions of semi-hyperbolicity considered earlier, namely in the variable norm sense of Definitions 3.1.4, 4.2.1, or in the single norm sense of Definitions 3.1.5 and 4.2.16.

**Theorem 5.2.1.** Let \( f : X \to \mathbb{R}^d \) be a continuously differentiable mapping which is semi-hyperbolic either in the variable norm Definition 3.1.4 or the single norm Definition 3.1.5, on a bounded open set \( X \subseteq \mathbb{R}^d \). Further, suppose that \( f \) is invertible in a neighborhood of some compact set \( K \subset X \) and that \( K \) is an invariant set for both \( f \) and \( f^{-1} \), that is \( f(K) = f^{-1}(K) = K \). Then \( f \) is hyperbolic on \( K \).

**Proof.** First we consider an equivariant splitting of \( \mathbb{R}^d \) for the mapping \( f \) on \( K \). For a point \( x \in K \) define the sequence of linear operators

\[
A_{n,x} = (Df^{n+1})_x = (Df)_f^n(x),
\]

(5.6)

From the semi-hyperbolicity of \( f \), for any \( x \in K \) the sequence of linear operators \( \{A_{n,x}\} \) will be also semi-hyperbolic in the same sense. Then by Theorem 4.2.14, for the point \( x \) and for any integer \( n \) there exists a splitting of the space \( \mathbb{R}^N \) into the direct sum of subspaces \( \hat{E}_{n,x}^s \) and \( \hat{E}_{n,x}^u \) satisfying

\[
A_{n,x} \hat{E}_{n,x}^s = \hat{E}_{n+1,x}^s, \quad A_{n,x} \hat{E}_{n,x}^u = \hat{E}_{n+1,x}^u,
\]

that is,

\[
(Df^{n+1})_x \hat{E}_{n,x}^s = \hat{E}_{n+1,x}^s, \quad (Df^{n+1})_x \hat{E}_{n,x}^u = \hat{E}_{n+1,x}^u.
\]

(5.7)

By Lemma 4.2.12 and by (5.6) the subspace \( \hat{E}_{n,x}^s \) consists of the set of \( v \in \mathbb{R}^d \) for which the equation

\[
v_{k+1} = (Df^{k+1})_x v_k, \quad k \geq n,
\]

has a bounded solution satisfying \( v_n = v \), and the subspace \( \hat{E}_{n,x}^u \) is the set of those \( w \in \mathbb{R}^d \) for which the equation

\[
w_{k+1} = (Df^{k+1})_x w_k, \quad k < n,
\]

has a bounded solution satisfying \( w_n = w \). From these properties of the subspaces \( \hat{E}_{n,x}^s, \hat{E}_{n,x}^u \) and from invertibility of the linear operators \( Df_x \) at any point \( x \in K \) it follows that

\[
\hat{E}_{n,x}^s = \hat{E}_{0,f^n(x)}^s, \quad \hat{E}_{n,x}^u = \hat{E}_{0,f^n(x)}^u, \quad n \in \mathbb{Z}.
\]
So, denoting $E^s_x = \hat{E}^s_{0,x}$, $E^u_x = \hat{E}^u_{0,x}$ obtain a splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$, $x \in K$, which by (5.7) satisfies the equivariance property

$$Df_x E^s_x = E^s_{f(x)}, \quad Df_x E^u_x = E^u_{f(x)}, \quad x \in K.$$  

Thus $f$ satisfies Condition A1 of Definition 2.1.1.

Now from Theorem 4.2.14, in particular the inequalities (4.32) and (4.33), it follows that the mapping $f$ satisfies Condition A2 of Definition 2.1.1 and from the inequality (4.27) of Lemma 4.2.12 Condition A3 of Definition 2.1.1 also holds for $f$. The mapping $f$ thus satisfies all the conditions of an Anosov system in Definition 2.1.1 except that $f$ is defined not on a closed Riemann manifold, but on a neighborhood of a compact invariant set. This distinction formally excludes the use of Theorem 2.1.2 for proving continuous dependence of the splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ on $x \in K$. But a closer inspection reveals that $f$ being defined on a closed Riemann manifold is not essential to the proof of Theorem 2.1.2. The statement of that theorem remains valid under the supposition that $f$ is defined on a compact invariant set. Hence, by this reasoning, the splitting $\mathbb{R}^d = E^s_x \oplus E^u_x$ depends continuously on $x \in K$, and thus, by Definition 2.1.5, the mapping $f$ is hyperbolic on $K$. \hfill\Box

5.2.1 Alternative Proof

The investigation of the relationship between semi-hyperbolicity and hyperbolicity in Chap. 4 and in Sect. 5.2 was intentionally done in a more traditional analytical manner, via an explicit description of splitting subspaces, in order that their properties could be determined more fully. Modern techniques based on differential geometry and smooth dynamics are also applicable here and provide an alternative proof of Theorem 5.2.1, based on the Mather Projection Lemma, by adapting the proof of Hirsch and Pugh [60]. See that reference and similar works for an explanation of the terms used below.

Let $E$ be a vector bundle over a compact set $K \subset \mathbb{R}^d$ and let $f : K \to K$ be a $C^1$ diffeomorphism on a neighborhood of $K$. Denote by $C(E)$ the Banach space of bounded continuous sections of $E$ over $K$ with the sup-norm topology. Let $\tilde{f} : C(E) \to C(E)$ be the continuous linear mapping

$$\tilde{f}(\varphi) = Df \circ \varphi \circ f^{-1}. \quad (5.8)$$

Lemma 5.2.2 (Mather Projection Lemma). If $\tilde{f} - I$ is a hyperbolic isomorphism, then $f$ is hyperbolic on $K$.

Now let $C(T_K \mathbb{R}^d) = \mathcal{C}$ be the Banach space of bounded continuous sections of $T_K \mathbb{R}^d$ and let $\hat{f} : \mathcal{C} \to \mathcal{C}$ be the map (5.8) induced by $f$. Write $\mathcal{C} = \mathcal{C}(E_1|_K) \times \mathcal{C}(E_2|_K)$. Then, making use of Theorem 4.3.6, it can be shown that the conditions of Theorem 5.2.1 are satisfied by $\hat{f}$, so $\hat{f}$ is a hyperbolic linear operator. Hence, by the Mather Projection Lemma 5.2.2, $f$ is hyperbolic on the set $K$. 

\hfill\Box
Expansivity and Shadowing

Section 2.2 demonstrated that hyperbolic systems possess some rather strong and striking properties such as expansivity, see Definition 2.2.1 and Theorem 2.2.2, and shadowing, see Theorem 2.2.3. This chapter shows that these and other properties remain valid for semi-hyperbolic systems. Moreover, explicit values or sharp estimates of relevant parameters and intervals of validity will be obtained.

6.1 Expansivity

In this section, semi-hyperbolic dynamical systems are shown to be exponentially expansive, at least locally, and explicit rates of expansion are determined. Differentiability is not required, so the proof is given only for Lipschitz mappings.

6.1.1 Definitions

Let \((X, \rho)\) be a metric space.

**Definition 6.1.1.** A trajectory of a dynamical system generated by a mapping \(f : X \to X\) on a state space \(X\) is a sequence \(x = \{x_n\} \subset X\) satisfying

\[x_{n+1} = f(x_n)\]

for all \(n\) belonging to some interval \(I \subseteq \mathbb{Z}\). The trajectory is said to be finite when \(I\) is of finite length and infinite otherwise.

Recall from Definition 2.2.1 that the dynamical system generated by a homeomorphism \(f : X \to X\) is said to be \(\xi\)-expansive on \(X\) if the inequalities

\[\rho(f^n(x_0), f^n(y_0)) \leq \xi\]

for \(n = 0, \pm 1, \pm 2, \ldots\)
imply that \(x_0 = y_0\), that is any two bi-infinite trajectories \(\{x_n\} = \{f^n(x_0)\}\) and \(\{y_n\} = \{f^n(y_0)\}\) which always remain within a threshold \(\xi\) of each other are identical. The more interesting cases occur when the metric space \((X, \rho)\) is compact since then with expansivity complicated dynamical behavior arises.

An important characteristic of \(\xi\)-expansivity is the rate of divergence of different trajectories. Let \(\text{Tr}_{\pm k}(f, X)\) denote the set of all trajectories

\[
x = \{x_{-k}, \ldots, x_0, \ldots, x_k\}
\]

of the mapping \(f\) that are contained entirely in a set \(X\), and for any \(x, \tilde{x} \in \text{Tr}_{\pm k}(f, K)\) define

\[
\rho_k(x, \tilde{x}) = \max \|x_n - \tilde{x}_n\|.
\]

**Lemma 6.1.2.** Let \(K\) be a compact subset of \(\mathbb{R}^d\) and let \(f : K \to K\) be a continuous \(\xi\)-expansive mapping. Then for every \(0 < \varepsilon, \theta < \xi\) there exists a positive integer \(\kappa(\varepsilon, \theta)\) such that

\[
\rho_k(x, \tilde{x}) > \theta
\]

holds for all \(x, \tilde{x} \in \text{Tr}_{\pm k}(f, K)\) with \(\|x_0 - \tilde{x}_0\| \geq \varepsilon\) and \(k \geq \kappa(\varepsilon, \theta)\).

**Proof.** Suppose the contrary. Then for any positive integer \(k\) there exist trajectories \(x^{(k)}, \tilde{x}^{(k)} \in \text{Tr}_{\pm k}(f, K)\) satisfying

\[
\|x_0^{(k)} - \tilde{x}_0^{(k)}\| \geq \varepsilon \quad \text{and} \quad \rho_k(x^{(k)}, \tilde{x}^{(k)}) \leq \theta < \xi.
\]

By compactness of the set \(K\), the sequences \(\{x^{(k)}\}\) and \(\{\tilde{x}^{(k)}\}\) can be assumed to converge componentwise to trajectories \(x^*\) and \(\tilde{x}^* \in \text{Tr}_{\pm \infty}(f, K)\).

Then

\[
\|x_0^* - \tilde{x}_0^*\| \geq \varepsilon,
\]

but at the same time

\[
\|x_n^* - \tilde{x}_n^*\| \leq \theta < \xi, \quad n = 0, \pm 1, \pm 2, \ldots,
\]

which contradicts the \(\xi\)-expansivity of \(f\). \(\square\)

In expansive systems distinct trajectories often separate exponentially, at least locally, as is seen from the next Example.

**Example 6.1.3.** Let \((\Sigma, \rho)\) be the compact metric space of bi-infinite binary sequences \(x = \{x_i\}_{i=-\infty}^{+\infty}, x_i \in \{0, 1\}\), endowed with the metric

\[
\rho(x, y) = \sum_{i=-\infty}^{+\infty} 2^{-|i|} |x_i - y_i|,
\]

and consider the shift mapping \(\sigma : \Sigma \to \Sigma\) on \(\Sigma\) defined by
The inequality
\[ \rho(\sigma^n x, \sigma^n y) \leq \xi < 1 \] (6.1)
can be rewritten as \(|x_n - y_n| \leq \xi < 1\), which for binary sequences \(x\) and \(y\) can only be valid if
\[ x_n = y_n. \] (6.2)
By (6.2), (6.1) holds for any \(n\) and so the sequences \(x\) and \(y\) coincide. Hence the shift mapping \(\sigma\) is \(\xi\)-expansive for any \(\xi < 1\).

Indeed, the shift mapping \(\sigma\) possesses a stronger property than \(\xi\)-expansivity, namely, the inequalities
\[ \max \{ \rho(\sigma^n x, \sigma^n y), \rho(\sigma^{-n} x, \sigma^{-n} y) \} \geq 2^{n-1} \rho(x, y) \] (6.3)
are valid for \(n = -n_-, \ldots, 0, \ldots, n_+\) whenever
\[ \rho(x, y) \leq \xi < 1, \]
where \(n_-\) and \(n_+\) are the largest possible integers such that
\[ \rho(\sigma^n x, \sigma^n y) \leq \xi < 1 \quad \text{for} \quad n = -n_-, \ldots, 0, \ldots, n_. \] (6.4)

To see this, suppose that inequalities (6.4) hold with maximal \(n_-\) and \(n_+\). Then by (6.1) and (6.2)
\[ x_n = y_n, \quad n = -n_-, \ldots, 0, \ldots, n_+, \]
so \(\rho(x, y)\) can be represented as
\[ \rho(x, y) = \sum_{i < -n_-} 2^i |x_i - y_i| + \sum_{i > n_+} 2^{-i} |x_i - y_i|, \]
where at least one of summands on the right hand side must be greater than or equal to \(\frac{1}{2} \rho(x, y)\). For definiteness, suppose that
\[ \sum_{i > n_+} 2^{-i} |x_i - y_i| \geq \frac{1}{2} \rho(x, y). \] (6.5)

Similarly for any \(n = -n_-, \ldots, 0, \ldots, n_+\), obtain
\[ \rho(\sigma^n x, \sigma^n y) = \sum_{i \in \mathbb{Z}} 2^{-|i|} |x_{i+n} - y_{i+n}| = \sum_{i \in \mathbb{Z}} 2^{-|i-n|} |x_i - y_i| \]
\[ = \sum_{i < -n_-} 2^{-|i-n|} |x_i - y_i| + \sum_{i > n_+} 2^{-|i-n|} |x_i - y_i| \]
\[ = \sum_{i < -n_-} 2^{i-n} |x_i - y_i| + \sum_{i > n_+} 2^{n-i} |x_i - y_i| \]
\[ \geq \sum_{i > n_+} 2^{n-i} |x_i - y_i|. \]
Then by (6.5)

$$\rho(\sigma^n x, \sigma^n y) \geq \frac{1}{2} 2^n \rho(x, y),$$

and the inequalities (6.3) for \(n = -n_-, \ldots, 0, \ldots, n_+\) are proved.

Example 6.1.3 is of fundamental interest because the shift mapping \(\sigma\) is conjugate to the diffeomorphism of the Horseshoe which is one of the best known examples of a chaotic hyperbolic system. The example motivates the following definition of exponential expansivity.

Let \(O_\varepsilon(S)\) denote the open \(\varepsilon\)-neighborhood of a nonempty subset \(S \subset \mathbb{R}^d\).

**Definition 6.1.4.** Let \(X \subseteq \mathbb{R}^d\) be an open bounded set and let \(K\) be a compact subset of \(X\). A continuous mapping \(f : X \to \mathbb{R}^d\) is said to be exponentially expansive on \(K\) with exponent \(r > 1\) if there exist constants \(\xi\) and \(c > 0\) such that \(O_\xi(K) \subseteq X\) and for any (finite) trajectories

\[ x = \{x_{-n_-, \ldots, n_+}\}, \quad y = \{y_{-n_-, \ldots, n_+}\} \]

satisfying \(x \subseteq K\) and \(\|y_i - x_i\| \leq \xi\) for \(n = -n_-, \ldots, 0, \ldots, n_+\) at least one of the following groups of inequalities holds:

\[ \|x_n - y_n\| \geq cr^n \|x_0 - y_0\|, \quad n = 1, 2, \ldots, n_+, \quad (6.6) \]

or

\[ \|x_n - y_n\| \geq cr^{-n} \|x_0 - y_0\|, \quad n = -1, -2, \ldots, -n_. \quad (6.7) \]

The exponential expansivity of diffeomorphisms and homeomorphisms on a compact hyperbolic set is well known. It also holds under the weaker assumptions of semi-hyperbolicity, which is proved in the next section for Lipschitz mappings satisfying an even weaker version of the inequalities in Condition SH2(Lip). Explicit values of the exponential expansivity parameters can be determined in terms of the split coefficients and the other semi-hyperbolicity parameters.

**6.1.2 Lipschitz Mappings**

Conditions SH0(Lip), SH1(Lip) and SH2(Lip) of semi-hyperbolicity Definition 3.1.6 for a Lipschitz mapping are stronger than necessary for establishing expansivity. In particular, the inequalities of Condition SH2(Lip) can be weakened. As before, let \(s = (\lambda_s, \lambda_u, \mu_s, \mu_u)\) be a split which, without loss of generality, may be assumed positive. Consider a mapping \(f : X \to \mathbb{R}^d\), where \(X\) is an open subset of \(\mathbb{R}^d\), and let \(K\) be a nonempty compact subset of \(X\) such that \(K \cap f(K) \neq \emptyset\).

**Theorem 6.1.5.** Let \(f\) be a continuous mapping defined on \(X \subseteq \mathbb{R}^d\) and suppose that for each \(x \in K\) there exists a uniform splitting \(\mathbb{R}^d = E_s^x \oplus E_u^x\) satisfying Condition SH1(Lip) and the following modification of Condition SH2(Lip) of Definition 3.1.6:
6.1 Expansivity

SH2(Mod): For all \( x \) satisfying \( x, f(x) \in K \) and for all \( z \in \mathbb{R}^d \) satisfying \( \|P_x^s z\|, \|P_x^u z\| \leq \delta \), the inclusion \( x + z \in X \) and the inequalities

\[
\begin{align*}
\|P_f^s (x + z) - f(x)\| &\leq \lambda_s \|P_x^s z\| + \mu_s \|P_x^u z\|, \\
\|P_f^u (x + z) - f(x)\| &\geq \lambda_u \|P_x^u z\| - \mu_u \|P_x^s z\|,
\end{align*}
\]

(6.8) hold.

Then the mapping \( f \) is exponentially expansive in \( K \) with exponent

\[
r = \sigma(s)^{-1} = 2 \left( \frac{1}{\lambda_u} + \lambda_s + \sqrt{\left( \frac{1}{\lambda_u} - \lambda_s \right)^2 + \frac{4\mu_s\mu_u}{\lambda_u}} \right)^{-1},
\]

as in (4.3) and constants

\[
c = \frac{1}{2} h^{-1} \min \{ \gamma(s), \gamma(s)^{-1} \} \quad \text{and} \quad \xi = h^{-1} \delta,
\]

(6.9) where \( h \) as in Condition SH1(Lip) of Definition 3.1.6.

Proof. Two lemmas will form the basis of the proof. Since the split \( s \) is fixed write

\[
M = M(s), \quad \sigma = \sigma(s) \quad \text{and} \quad \gamma = \gamma(s),
\]

see equations (4.2), (4.3) and (4.5) in Sect. 4.1. In addition, the norm \( \| \cdot \|_* \) on \( \mathbb{R}^2 \) defined by

\[
\|(y_1, y_2)\|_* = \max \{ \gamma |y_1|, |y_2| \},
\]

is used. Since the split \( s \) is positive, \( \gamma > 0 \).

Lemma 6.1.6. The inequalities

\[
\eta \|z\| \leq \|(\|P_x^s z\|, \|P_x^u z\|)\|_* \leq 2 \eta h \max \{ \gamma, \gamma^{-1} \} \|z\|
\]

with \( \eta = \frac{1}{2} \min \{ \gamma, 1 \} \) hold for all \( z \in \mathbb{R}^d \) and all \( x \) from the semi-hyperbolicity set \( K \) of the mapping \( f \).

Proof. By Lemma 4.1.1

\[
\min \{ \gamma, 1 \} \max \{ \|P_x^s z\|, \|P_x^u z\| \} \leq \|(\|P_x^s z\|, \|P_x^u z\|)\|_* \leq \max \{ \gamma, 1 \} \max \{ \|P_x^s z\|, \|P_x^u z\| \},
\]

so it suffices to check that the inequalities

\[
\frac{1}{2} \|z\| \leq \max \{ \|P_x^s z\|, \|P_x^u z\| \} \leq h \|z\|
\]

hold for all \( z \in \mathbb{R}^d \) and all \( x \in K \). The inequality on the right follows from the semi-hyperbolicity Condition SH1(Lip), while that on the left is just

\[
\|z\| = \|P_x^s z + P_x^u z\| \leq \|P_x^s z\| + \|P_x^u z\| \leq 2 \max \{ \|P_x^s z\|, \|P_x^u z\| \}.
\]

The result now follows with \( \eta = \frac{1}{2} \min \{ \gamma, 1 \} \).
For the remaining lemma note that semi-hyperbolicity does not require the sets $K$ and $X$ to be invariant under the mapping $f$.

**Lemma 6.1.7.** Suppose that $x, f(x), f^2(x) \in K$ and $y, f(y), f^2(y) \in X$ with
\[
\|x - y\|, \|f(x) - f(y)\|, \|f^2(x) - f^2(y)\| < h^{-1}\delta. \tag{6.10}
\]
Then at least one of the following pair of inequalities
\[
\sigma \| (\| P_s^x r_0 \|, \| P_s^x r_0 \|)_* \| \geq \| (\| P_s^x f(x) r_1 \|, \| P_s^x f(x) r_1 \|)_* \|,
\]
\[
\sigma \| (\| P_s^{f^2(x)} r_2 \|, \| P_s^{f^2(x)} r_2 \|)_* \| \geq \| (\| P_s^x f(x) r_1 \|, \| P_s^x f(x) r_1 \|)_* \|,
\]
holds, where
\[
r_0 = x - y, \quad r_1 = f(x) - f(y), \quad r_2 = f^2(x) - f^2(y).
\]

**Proof.** By (6.10) and Condition SH1(Lip) of Definition 3.1.6,
\[
\| P_s^x r_0 \|, \| P_s^x f(x) r_1 \|, \| P_s^{f^2(x)} r_2 \| \leq \delta,
\]
\[
\| P_s^x r_0 \|, \| P_s^x f(x) r_1 \|, \| P_s^{f^2(x)} r_2 \| \leq \delta,
\]
from which, by (6.8),
\[
\| P_s^x f(x) r_1 \| \leq \lambda_s \| P_s^x r_0 \| + \mu_s \| P_s^x r_0 \|,
\]
\[
\| P_s^{f^2(x)} r_2 \| \geq \lambda_u \| P_s^x f(x) r_1 \| - \mu_u \| P_s^x f(x) r_1 \|.
\]
That is,
\[
\| P_s^x f(x) r_1 \| \leq \lambda_s \| P_s^x r_0 \| + \mu_s \| P_s^x r_0 \|
\]
\[
\| P_s^{f^2(x)} r_2 \| \leq \frac{\mu_u}{\lambda_u} \| P_s^x f(x) r_1 \| + \frac{1}{\lambda_u} \| P_s^{f^2(x)} r_2 \|.
\]
Here, by definition of the norm $\| \cdot \|_*$,
\[
\| P_s^x r_0 \| \leq \gamma^{-1} \| (\| P_s^x r_0 \|, \| P_s^x r_0 \|)_* \|,
\]
\[
\| P_s^x r_0 \| \leq \| (\| P_s^x r_0 \|, \| P_s^x r_0 \|)_* \|,
\]
\[
\| P_s^x f(x) r_1 \| \leq \gamma^{-1} \| (\| P_s^x r_1 \|, \| P_s^{f^2(x)} r_2 \|)_* \|,
\]
\[
\| P_s^{f^2(x)} r_2 \| \leq \| (\| P_s^x f(x) r_1 \|, \| P_s^{f^2(x)} r_2 \|)_* \|,
\]
and therefore
\[
\| P_s^x f(x) r_1 \| \leq \gamma^{-1} (\lambda_s + \mu_s) \| (\| P_s^x r_0 \|, \| P_s^x r_0 \|)_* \|,
\]
\[
\| P_s^{f^2(x)} r_2 \| \leq \gamma^{-1} \left( \frac{\mu_u}{\lambda_u} + \frac{1}{\lambda_u} \right) \| (\| P_s^x r_1 \|, \| P_s^{f^2(x)} r_2 \|)_* \|.
\]
From the definition of $\sigma = \sigma(s)$ and $\gamma = \gamma(s)$ in (4.6),
\[ \lambda_s + \mu_s \gamma = \sigma, \quad \frac{\mu_u}{\lambda_u} + \frac{1}{\lambda_u} \gamma = \sigma \gamma, \]

so

\[ \| P^s_{f(x)} r_1 \| \leq \sigma \gamma^{-1} \left( \| P^s_x r_0 \|, \| P^u_x r_0 \| \right) \|_*, \quad (6.11) \]

\[ \| P^u_{f(x)} r_1 \| \leq \sigma \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f^2(x)} r_2 \| \right) \|_*. \quad (6.12) \]

Suppose now that the lemma is invalid, that is both the inequalities

\[ \sigma \left( \| P^s_x r_0 \|, \| P^u_x r_0 \| \right) \|_* < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_*, \quad (6.13) \]

and

\[ \sigma \left( \| P^s_{f^2(x)} r_2 \|, \| P^u_{f^2(x)} r_2 \| \right) \|_* < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_*, \quad (6.14) \]

hold. Then inequalities (6.11) and (6.13) imply that the strict inequality

\[ \gamma \| P^s_{f(x)} r_1 \| < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_* \quad (6.15) \]

is valid from which, since \( \sigma = \sigma(s) < 1, \)

\[ \sigma \gamma \| P^s_{f(x)} r_1 \| < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_* \]

This last inequality together with (6.14) shows that

\[ \sigma \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f^2(x)} r_2 \| \right) \|_* < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_* \]

and by (6.12) the strict inequality

\[ \| P^u_{f(x)} r_1 \| < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_* \quad (6.16) \]

is valid.

Now, by definition of the norm \( \| \cdot \|_* \), the inequalities (6.15) and (6.16) imply that

\[ \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_* < \left( \| P^s_{f(x)} r_1 \|, \| P^u_{f(x)} r_1 \| \right) \|_* , \]

a contradiction and so the lemma is proved. \( \square \)

Now, to complete the proof of Theorem 6.1.5 consider two trajectories

\[ x = \{ x_{-n_-}, \ldots, x_0, \ldots, x_{n_+} \}, \quad y = \{ y_{-n_-}, \ldots, y_0, \ldots, y_{n_+} \} \]

satisfying \( x \subseteq K, y \subseteq X \) and \( \| y_n - x_n \| \leq \xi \) for \( n = -n_- , \ldots, n_+ \) and write

\[ r_n = x_n - y_n, \quad \nu_n = \left( \| P^s_{f(x_n)} r_n \|, \| P^u_{f(x_n)} r_n \| \right) \|_* \quad (6.17) \]

for \( n = -n_- , \ldots, n_+ \). From Lemma 6.1.7 it then follows that at least one of the inequalities
\[\nu_{-1} \geq \sigma^{-1}\nu_0, \quad \nu_1 \geq \sigma^{-1}\nu_0\]

holds.

For definiteness, suppose that \(\nu_{-1} \geq \sigma^{-1}\nu_0\). Then, by Lemma 6.1.7 again, at least one of the inequalities

\[\nu_{-2} \geq \sigma^{-1}\nu_{-1}, \quad \nu_0 \geq \sigma^{-1}\nu_{-1}\]

holds. But the second of these inequalities can not be valid at the same time as the inequality \(\nu_{-1} \geq \sigma^{-1}\nu_0\) because of \(\sigma < 1\). Hence, \(\nu_{-2} \geq \sigma^{-1}\nu_{-1}\). In the same way it can be shown that

\[\nu_{n-1} \geq \sigma^{-1}\nu_n, \quad \text{for} \quad n = -n_+ + 1, \ldots, -1, 0,
\]

and hence that

\[\nu_n \geq \sigma^{-n}\nu_0, \quad \text{for} \quad n = -n_+ + 1, \ldots, -1.\]

Recalling definition (6.17) of the \(\nu_n\) and using Lemma 6.1.6 to estimate the norms \(\|x_n - y_n\|\) via \(\nu_n\), obtain the inequalities (6.7) with constants \(c\) and \(\xi\) as in (6.9).

When \(\nu_1 \geq \sigma^{-1}\nu_0\), the proof follows along same lines.

\[\square\]

Remark 6.1.8. As mentioned above, condition (6.8) is a weakened version of Condition SH2(Lip) of Definition 3.1.6. Indeed, the inequalities (3.13)–(3.16) of Condition SH2(Lip), namely

\[
\begin{align*}
\|P_y^s (f(x + u + v) - f(x + \tilde{u} + \tilde{v}))\| &\leq \lambda_s \|u - \tilde{u}\|, \quad (3.13) \\
\|P_y^s (f(x + u + v) - f(x + u + \tilde{v}))\| &\leq \mu_s \|v - \tilde{v}\|, \quad (3.14) \\
\|P_y^u (f(x + u + v) - f(x + \tilde{u} + v))\| &\leq \mu_u \|u - \tilde{u}\|, \quad (3.15) \\
\|P_y^u (f(x + u + v) - f(x + u + \tilde{v}))\| &\geq \lambda_u \|v - \tilde{v}\|, \quad (3.16)
\end{align*}
\]

imply (6.8).

Putting \(y = f(x), u = P^s z, v = P^u z, \tilde{u} = 0\) in (3.13) obtain

\[
\|P_{f(x)}^s (f(x + z) - f(x + P^u z))\| \leq \lambda_s \|P^s z\|,
\]

while putting \(y = f(x), u = 0, v = P^u z, \tilde{v} = 0\) in (3.14) obtain

\[
\|P_{f(x)}^s (f(x + P^u z) - f(z))\| \leq \mu_s \|P^u z\|,
\]

and the first inequality of (6.8) follows immediately.

Similarly, the second inequality of (6.8) can be obtained by appropriate choice of elements \(u, v, \tilde{u}\) and \(\tilde{v}\) in (3.15) and (3.16).
6.2 Shadowing

The relationship between the behavior of a given dynamical system and perturbations of it are also very important for nonsmooth and nonhomeomorphic systems. For example, a computer simulation of a specific dynamical system is really only an approximation of that system because computer arithmetic is finite and there is subsequent round off error. In particular, computed trajectories are only pseudo-trajectories of the original system. This relationship is, however, difficult to express in terms of conjugacy, as is possible for diffeomorphic systems. A convenient and highly useful replacement of the conjugacy property in such a situation is provided by bi-shadowing, where trajectories and pseudo-trajectories are compared rather than the mappings themselves.

A typical problem can be roughly stated in the following general way: given a dynamical system

\[ x_{n+1} = f(x_n) \]

generated by a mapping \( f \) on a metric or topological space \( X \), suppose that the function \( f \) is perturbed at any iteration \( n \) so that another system

\[ y_{n+1} = \varphi(n, y_n) \]

is in fact observed, where \( \varphi \) is close to \( f \) and can possibly also vary with \( n \). Then the following Direct Shadowing Problem can be posed: for every trajectory of every small perturbation \( \varphi \) of a given mapping \( f \) does there exist a close trajectory of \( f \)? In considering non-autonomous perturbations \( \varphi \) of the mapping \( f \) we have considerably more latitude. The Direct Shadowing Problem is usually formulated in terms of relationships between the true trajectories and pseudo-trajectories of a system \( f \), as in Sect. 6.2.2.

The Inverse Shadowing Problem is also of interest namely can every trajectory of a mapping \( f \) be approximated by some trajectory of each perturbation \( \varphi \) of \( f \) from a predefined class \( \mathcal{T} \)? Here the class of perturbations \( \mathcal{T} \) is usually taken to be rather appropriate size to obtain meaningful and strong results. For example, the class \( \mathcal{T} \) may consist of all continuous autonomous mappings \( \varphi \) or of all, possibly discontinuous mappings resulting from some approximation technique or numerical method.

The asymmetrical roles of the classes of perturbations \( \mathcal{T} \) in each case should be emphasized, with the class \( \mathcal{T} \) being as broad as possible in Direct Shadowing and as meagre as possible in indirect or Inverse Shadowing for the sharpest results in each case. In the direct shadowing of the classical Shadowing Lemma, see Chap. 2 Theorem 2.2.3, \( \mathcal{T} \) consists of all possible non-autonomous perturbations of the given system, while in inverse shadowing a natural class \( \mathcal{T} \) consists of trajectories of all continuous mappings \( \varphi \) that are sufficiently close to \( f \).
6.2.1 Definitions

Consider a mapping \( f : X \to X \), where \( X \) is an open bounded subset of \( \mathbb{R}^d \), and let \( \| \cdot \| \) denote a fixed but otherwise arbitrary norm on \( \mathbb{R}^d \).

**Definition 6.2.1.** A \( \gamma \)-pseudo-trajectory of a dynamical system generated by a mapping \( f \) on \( X \) is a sequence \( y = \{ y_n \} \subset X \) with
\[
\| y_{n+1} - f(y_n) \| \leq \gamma, \quad \gamma > 0, \quad (6.18)
\]
for \( n = -n_-, \ldots, 0, \ldots, n_+ \) where \( n_\pm \leq \infty \). The term finite may be appended when \( n_\pm < \infty \), otherwise the pseudo-trajectory is infinite.

Clearly, any trajectory of a system is 0-pseudo-trajectory of itself, but not every pseudo-trajectory is a trajectory. Pseudo-trajectories arise naturally in a number of ways, such as the presence of roundoff error in computer calculations of trajectories, although accumulated roundoff error can rapidly destroy any meaningful connection between a computed pseudo-trajectory and an original trajectory. The concept of shadowing provides a way of comparing trajectories and pseudo-trajectories.

**Definition 6.2.2.** A trajectory \( x = \{ x_n \} \) is said to \( \varepsilon \)-shadow a \( \gamma \)-pseudo-trajectory \( y = \{ y_n \} \) on some finite or infinite interval \( I \subseteq \mathbb{Z} \) if
\[
\| x_n - y_n \| \leq \varepsilon, \quad n \in I.
\]

Without loss of generality, suppose that the set \( I \) is one of the following intervals: \( I = [0, N] \), \( I = (-\infty, 0] \), \( I = [0, \infty) \) or \( I = (-\infty, \infty) \).

The gist of a Shadowing Theorem like Theorem 2.2.3 is that, under certain assumptions on \( f \) such as hyperbolicity, for every \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that each \( \delta \)-pseudo-trajectory is \( \varepsilon \)-shadowed by a true trajectory. This is what is meant by specifying direct shadowing.

6.2.2 Direct Shadowing

The proof of the Shadowing Theorem for hyperbolic diffeomorphisms does not require the full structure of hyperbolicity. The theorem stated and proved below establishes the same result for differentiable semi-hyperbolic mappings. The proof is not only much shorter in the differentiable semi-hyperbolic case than for hyperbolic case, but is more general. Below, the continuity of \( Df_x \) for \( x \in \overline{X} \) is needed only to ensure uniform continuity of \( Df_x \).

**Theorem 6.2.3.** Let \( f : X \to X \) be differentiable and semi-hyperbolic (in the sense of Definition 3.1.5) on an open bounded set \( X \subset \mathbb{R}^d \) with \( Df_x \) continuous on \( \overline{X} \). Then for every sufficiently small \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that every \( \delta \)-pseudo-trajectory \( y = \{ y_n \} \) of \( f \) is \( \varepsilon \)-shadowed by a true trajectory \( x = \{ x_n \} \).
Proof. The proof adapts and slightly simplifies those of [30] and [32]. Let \( y = \{ y_n \} \) be a \( \delta \)-pseudo-trajectory of \( f \) defined over some integer interval \( \mathbb{I} \) of indices. Define the nonlinear mapping \( \mathcal{F} : \mathcal{X} \to \mathcal{Y} \) by

\[
(\mathcal{F}(x))_n = x_{n+1} - f(x_n), \quad n \in \mathbb{I},
\]

where \( x = \{ x_n \} \in \mathcal{X} \) and spaces \( \mathcal{X} \) and \( \mathcal{Y} \) are as follows

\[
\mathcal{X} = \ell^\infty([0, N + 1], \mathbb{R}^d), \quad \mathcal{Y} = \ell^\infty([1, N + 1], \mathbb{R}^d), \quad \text{if} \quad \mathbb{I} = [0, N],
\]
\[
\mathcal{X} = \ell^\infty((\infty, 1], \mathbb{R}^d), \quad \mathcal{Y} = \ell^\infty((\infty, 1], \mathbb{R}^d), \quad \text{if} \quad \mathbb{I} = (-\infty, 0],
\]
\[
\mathcal{X} = \ell^\infty([0, \infty), \mathbb{R}^d), \quad \mathcal{Y} = \ell^\infty([1, \infty), \mathbb{R}^d), \quad \text{if} \quad \mathbb{I} = [0, \infty),
\]
\[
\mathcal{X} = \ell^\infty((-\infty, \infty), \mathbb{R}^d), \quad \mathcal{Y} = \ell^\infty((-\infty, \infty), \mathbb{R}^d), \quad \text{if} \quad \mathbb{I} = (-\infty, \infty).
\]

Then \( \mathcal{F} \) is \( C^1 \) with Fréchet derivative

\[
(D\mathcal{F}(x)u)_n = u_{n+1} - Df_{x_n}u_n.
\]

Let \( \omega(\tau), \tau \geq 0 \), be the modulus of continuity of \( Df \) over \( \overline{X} \),

\[
\omega(\tau) = \sup \left\{ \| Df_y - Df_x \| : x, y \in \overline{X}, \| y - x \| \leq \tau \right\},
\]

and note that \( \omega(\tau) \to 0 \) as \( \tau \to 0^+ \) by the continuity of \( Df_x \) on the set \( \overline{X} \). Let \( \varepsilon_0 \) be the largest positive number less than or equal to \( \varepsilon \) such that \( \omega(\varepsilon_0) \leq 1/(2\alpha(s, h)) \), where

\[
\alpha(s, h) = \frac{h}{\nu(s)} = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} - h.
\]

Here \( \nu(s) \) is the constant (3.3). Note that the sequence of matrices \( \{ Df_{x_n} \} \) is semi-hyperbolic because of semi-hyperbolicity of the mapping \( f \). The derivative \( D\mathcal{F}(x) \) thus has a right inverse \( D\mathcal{F}(x)^{-1} \) which satisfies

\[
\| D\mathcal{F}(x)^{-1} \|_\infty \leq \frac{h}{\nu(s)} = \alpha(s, h).
\]

Choose \( \delta = \delta(\varepsilon) = \varepsilon_0/(2\alpha(s, h)) \). Now, if \( y \) is the \( \delta \)-pseudo-trajectory sequence and \( x \) is any \( \ell^\infty(\mathbb{I}, \mathbb{R}^d) \) sequence satisfying \( \| x - y \|_\infty \leq \varepsilon_0 \), then

\[
\| D\mathcal{F}(x) - D\mathcal{F}(y) \|_\infty \leq \sup_n \| Df_{x_n} - Df_{y_n} \|
\]
\[
\leq \omega(\varepsilon_0) \leq \frac{1}{2\alpha(s, h)} \leq \frac{1}{2\| D\mathcal{F}(y)^{-1} \|_\infty}.
\]

To complete the main proof, the following fixed point lemma due to Chow, Lin and Palmer [30] is used and a proof given for completeness. For simplicity of exposition, denote the norms in \( \mathcal{X}, \mathcal{Y} \) by the same symbol \( \| \cdot \| \).
Lemma 6.2.4. Let \( \mathcal{X}, \mathcal{Y} \) be Banach spaces and suppose that \( \mathcal{F} : \mathcal{X} \to \mathcal{Y} \) is \( C^1 \). Let \( y \in \mathcal{X} \) be a point such that \( D\mathcal{F}(y)^{-1} \) is a bounded linear right inverse of \( D\mathcal{F}(y) \) and let \( \varepsilon_0 > 0 \) be chosen so that

\[
|D\mathcal{F}(x) - D\mathcal{F}(y)| \leq \frac{1}{2|D\mathcal{F}(y)^{-1}|} \tag{6.20}
\]

for \( |x - y| \leq \varepsilon_0 \). If \( 0 < \varepsilon \leq \varepsilon_0 \) and

\[
|\mathcal{F}(y)| \leq \frac{\varepsilon}{2|D\mathcal{F}(y)^{-1}|}
\]

then the equation \( F(x) = 0 \) has a unique solution \( x \) such that \( |x - y| \leq \varepsilon \).

Proof. Write

\[ \mathcal{F}(x) = \mathcal{F}(y) + D\mathcal{F}(y)(x - y) + \eta(x). \]

For \( |x_1 - y|, |x_1 - y| \leq \varepsilon_0 \) by (6.20)

\[
|\eta(x_1) - \eta(x_2)| = |\mathcal{F}(x_1) - \mathcal{F}(x_2) - D\mathcal{F}(y)(x_1 - x_2)| \\
\leq \left| \int_0^1 (D\mathcal{F}(x_2 + \theta(x_1 - x_2)) - D\mathcal{F}(y)) \, d\theta \right| \cdot |x_1 - x_2| \\
\leq \frac{|x_1 - x_2|}{2|D\mathcal{F}(y)^{-1}|}. \tag{6.21}
\]

Rewrite the equation \( \mathcal{F}(x) = 0 \) as

\[ x = y - D\mathcal{F}(y)^{-1} \{ \mathcal{F}(y) + \eta(x) \} = T(x). \]

For \( 0 < \varepsilon \leq \varepsilon_0 \), define \( B(\varepsilon, y) = \{ x \in \mathcal{X} : |x - y| \leq \varepsilon \} \). If \( T \) is a contraction on \( B(\varepsilon, y) \), the lemma will then follow immediately.

To see this first note that if \( x \in B(\varepsilon, y) \) then

\[
|T(x) - y| = |D\mathcal{F}(y)^{-1} (\mathcal{F}(y) + \eta(x))| \\
\leq |D\mathcal{F}(y)^{-1}| \left( \frac{\varepsilon}{2|D\mathcal{F}(y)^{-1}|} + \frac{|x_1 - x_2|}{2|D\mathcal{F}(y)^{-1}|} \right) \\
= \frac{\varepsilon}{2} + \frac{|x_1 - x_2|}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]

where (6.20) and (6.21) are used with \( x_1 = x, x_2 = y \). Hence \( T \) maps \( B(\varepsilon, y) \) into itself. Moreover if \( x_1, x_2 \in B(\varepsilon, y) \) then, using (6.21),

\[
|T(x_1) - T(x_2)| = |D\mathcal{F}(y)^{-1} (\eta(x_1) - \eta(x_2))| \\
\leq |D\mathcal{F}(y)^{-1}| \frac{|x_1 - x_2|}{2|D\mathcal{F}(y)^{-1}|} = \frac{|x_1 - x_2|}{2}.
\]

Thus \( T \) is indeed a contraction on \( B(\varepsilon, y) \) and the proof of Lemma 6.2.4 follows.

\( \square \)
Now apply Lemma 6.2.4 to the mapping $F : \mathcal{X} \to \mathcal{Y}$ of (6.19), with the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ introduced above and endowed with the norm $\| \cdot \|_\infty$. Then the equation $F(x) = 0$ has a unique solution $x \in \mathcal{X}$ satisfying $\| x - y \|_\infty \leq \varepsilon$. That is, the $\delta$-pseudo-trajectory $y = \{ y_n \}$ is $\varepsilon$-shadowed by the true trajectory $x = \{ x_n \}$. This completes the proof of Theorem 6.2.3.

6.2.3 Inverse Shadowing

As a distance between the mappings $f, \varphi : \mathbb{R}^d \to \mathbb{R}^d$ consider the semi-norm

$$\| f - \varphi \|_\infty = \sup_{x \in \mathbb{R}^d} \| f(x) - \varphi(x) \|.$$ 

Note that this is not a norm because it may take infinite values.

**Definition 6.2.5.** A finite trajectory $x = \{ x_0, x_1, \ldots, x_N \}$ of a mapping $f$ is called $\alpha$-robust for some $\alpha > 0$, if there exists an $\varepsilon_0 > 0$ such that any continuous mapping $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ satisfying

$$\| f - \varphi \|_\infty \leq \varepsilon_0$$

has at least one trajectory $y = \{ y_0, y_1, \ldots, y_N \}$ such that

$$\| y_n - x_n \| \leq \alpha \| f - \varphi \|_\infty, \quad n = 0, 1, \ldots, N. \quad (6.22)$$

This is a form of inverse shadowing where the reference class $\mathcal{T}$ is the space of all continuous mappings on $X$.

**Remark 6.2.6.** The key condition in Definition 6.2.5 is the existence of $\alpha > 0$ independent of the length of the given trajectory. The estimates (6.22) can be obtained easily if $\alpha$ is allowed to depend on the length of the trajectory. For example, any trajectory $x$ is $(1 + L + \cdots + L^N)$-robust if the mapping $f$ is Lipschitz with Lipschitz constant $L$ in a neighborhood of the trajectory $x$. Indeed, in this case, for a given trajectory $x$ of the mapping $f$ and for a trajectory $y$ of a mapping $\varphi$ satisfying $y_0 = x_0$,

$$\| y_{n+1} - y_{n+1} \| = \| \varphi(y_n) - f(x_n) \|$$

$$\leq \| \varphi(y_n) - f(y_n) \| + \| f(y_n) - f(x_n) \|$$

$$\leq \| f - \varphi \|_\infty + L \| y_n - x_n \|,$$

from which

$$\| y_n - x_n \| \leq (1 + L + \cdots + L^n) \| f - \varphi \|_\infty, \quad n = 1, 2, \ldots, N.$$ 

Thus, any trajectory $x = \{ x_0, x_1, \ldots, x_N \}$ of the mapping $f$ is $\alpha_N$-robust with $\alpha_N = (1 + L + \cdots + L^N)$.
As the next theorem shows, semi-hyperbolicity allows the robustness constant \( \alpha \) to be chosen independently of the particular trajectory and its length \( N \), uniformly throughout the domain of semi-hyperbolicity \( X \).

**Theorem 6.2.7.** Let \( f : X \to X \) be differentiable and \((s,h)\)-semi-hyperbolic on an open set \( X \subseteq \mathbb{R}^d \). Then every finite trajectory \( x \subset X \) is \( \alpha \)-robust for any

\[
\alpha > \alpha(s,h) = \frac{\lambda_u - \lambda_s + \mu_u + \mu_s}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h,
\]

where \( \lambda_u, \lambda_s, \mu_u, \mu_s \) are the parameters of the split \( s = (\lambda_u, \lambda_s, \mu_u, \mu_s) \).

**Proof.** Denote by \( B_N \) the space of \((N+1)\)-sequences 

\[
z = \{z_0, z_1, \ldots, z_N\}, \quad z_n \in \mathbb{R}^d, \ n = 0, 1, \ldots, N,
\]
satisfying the boundary conditions

\[
P_s x_0 z_0 = P_u x_N z_N = 0.
\]

\( B_N \) is a subspace of the \((N+1)d\)-dimensional vector space \( \mathbb{R}^d \times \ldots \times \mathbb{R}^d \) \((N + 1 \text{ times})\) with norm

\[
\|z\|_\infty = \max_{0 \leq n \leq N} \|z_n\|.
\]

Let \( \varphi : \mathbb{R}^d \to \mathbb{R}^d \) be a given mapping and \( x = \{x_0, x_1, \ldots, x_N\} \) be a given trajectory of the mapping \( f \). Introduce the linear operator \( U_n : E^u_{x_{n-1}} \to E^u_{x_n} \) defined by \( U_n v = P^u_{x_n} Df_{x_{n-1}} v \). This operator is surjective from inequality (3.12) of Definition 3.1.5 and is thus invertible and \( U_n^{-1} \) is well defined.

Consider now the operator \( \mathcal{H} : B_N \to B_N \), which transforms each sequence \( z = \{z_0, z_1, \ldots, z_N\} \) into a sequence \( w = \{w_0, w_1, \ldots, w_N\} \) defined by the boundary conditions

\[
P_s x_0 w_0 = P^u_{x_N} w_N = 0,
\]

see (6.24), and by

\[
P^s_{x_n} w_n = P^s_{x_n} (\varphi(x_{n-1} + z_{n-1}) - x_n),
\]

\[
P^u_{x_{n-1}} w_{n-1} = U_n^{-1}(P^u_{x_n} z_n - P^u_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1} + P^u_{x_n} (-\varphi(x_{n-1} + z_{n-1}) + x_n + Df_{x_{n-1}} z_{n-1})),
\]

for \( n = 1, 2, \ldots, N \). The necessity of introducing the operator \( \mathcal{H} \) is explained by the next lemma, whose proof is obvious.

**Lemma 6.2.8.** The operator \( \mathcal{H} \) is continuous, and for any of its fixed point \( z = \{z_0, z_1, \ldots, z_N\} \) the sequence

\[
y = \{x_0 + z_0, x_1 + z_1, \ldots, x_N + z_N\}
\]
is a trajectory of the mapping \( \varphi \).
First, more notation and definitions are required. Given $\beta > 0$, let $\delta_{\beta}(\varepsilon)$ be the largest positive value of $\delta$ such that
\[
\|x_n + Df_{x_{n-1}} z - f(x_{n-1} + z)\| \leq \beta \varepsilon, \quad n = 1, 2, \ldots, N, \tag{6.25}
\]
for any $\|z\| \leq \delta$. For each $z \in \mathcal{B}$ define the pair of real numbers
\[
m^s(z) = \max_{0 \leq n \leq N} \|P^s_{x_n} z_n\|, \quad m^u(z) = \max_{0 \leq n \leq N} \|P^u_{x_n} z_n\|,
\]
and let $\mathbf{m}(z)$ be the 2-dimensional column vector with components $m^s(z)$, $m^u(z)$. Let $M = M(s)$ be the split matrix (4.2) and define the column vector
\[
\mathbf{h} = h(1, \lambda_u^{-1})^T.
\]

**Lemma 6.2.9.** Let $\beta > 0$. Then the inequality, component by component,
\[
\mathbf{m}(\mathcal{H}(z)) \leq M\mathbf{m}(z) + (1 + \beta)\|f - \varphi\|_{\infty} \mathbf{h}
\]
is valid for every continuous mapping $\varphi$ and every $z$ from the set
\[
\mathcal{W} = \{z \in \mathcal{B} : \|z\|_{\infty} \leq \delta_{\beta}(\|f - \varphi\|_{\infty})\}.
\]

**Proof.** First consider the component $m^s(\mathcal{H}(z))$. By definition
\[
m^s(\mathcal{H}(z)) = \max_{0 \leq n \leq N} \|v^s_n\|, \tag{6.26}
\]
where
\[
v^s_n = P^s_{x_n} (\varphi(x_{n-1} + z_{n-1}) - x_n).
\]
Rewrite this as
\[
v^s_n = I_1 + I_2 + I_3 + I_4, \tag{6.27}
\]
where
\[
I_1 = P^s_{x_n} Df_{x_{n-1}} P^s_{x_{n-1}} z_{n-1},
I_2 = P^s_{x_n} Df_{x_{n-1}} P^u_{x_{n-1}} z_{n-1},
I_3 = P^s_{x_n} (\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})),
I_4 = P^s_{x_n} (f(x_{n-1} + z_{n-1}) - (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1})).
\]
From (3.9),
\[
\|I_1\| \leq \lambda_s \|P^s_{x_{n-1}} z_{n-1}\|, \tag{6.28}
\]
while from (3.10),
\[
\|I_2\| \leq \mu_s \|P^u_{x_{n-1}} z_{n-1}\|.
\]
Condition SH1(Lip) of Definition 3.1.6 implies that
\[
\|I_3\| \leq h\|f - \varphi\|_{\infty}.
\]
Finally, Condition SH1(Lip) of Definition 3.1.6 and the definition of the function $\delta_\beta(\varepsilon)$ imply that
\[
\|I_4\| \leq h\beta\|f - \varphi\|_\infty. \tag{6.29}
\]
From (6.27) and (6.28)--(6.29) it then follows that
\[
\|v_n^s\| \leq \lambda_s\|P_{x_{n-1}}^s z_{n-1}\| + \mu_s\|P_{x_{n-1}}^u z_{n-1}\| + (1 + \beta)\|f - \varphi\|_\infty h,
\]
and by (6.26)
\[
m^s(\mathcal{H}(z)) \leq \lambda_s m^s(z) + \mu_s m^s(z) + (1 + \beta)\|f - \varphi\|_\infty h. \tag{6.30}
\]
Now consider the component $m^u(\mathcal{H}(z))$. By definition,
\[
m^u(\mathcal{H}(z)) = \max_{0 \leq n \leq N} \|v_n^u\|, \tag{6.31}
\]
where
\[
v_{n-1}^u = U_n^{-1} \left( P_{x_n}^u z_n - P_{x_n}^u Df_{x_{n-1}} P_{x_{n-1}}^s z_{n-1} + P_{x_n}^u (-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}) \right).
\]
Rewrite this as
\[
v_{n-1}^u = U_n^{-1} (J_1 + J_2 + J_3 + J_4), \tag{6.32}
\]
where
\[
J_1 = P_{x_n}^u z_n, \tag{6.33}
J_2 = -P_{x_n}^u Df_{x_{n-1}} P_{x_{n-1}}^s z_{n-1}, \tag{6.34}
J_3 = P_{x_n}^u (-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}), \tag{6.35}
J_4 = -f(x_{n-1} + z_{n-1}) + (f(x_{n-1}) + Df_{x_{n-1}} z_{n-1}). \tag{6.36}
\]
The relations (3.12) and (6.33) imply that
\[
\|U_n^{-1} J_1\| \leq \lambda_u^{-1} \|P_{x_n}^u z_n\|, \tag{6.37}
\]
while (3.11), (3.12) and (6.34) imply that
\[
\|U_n^{-1} J_2\| \leq \lambda_u^{-1} \mu_u \|P_{x_{n-1}}^s z_{n-1}\|.
\]
In addition, (3.12), (6.35) and Condition SH1(Lip) from Definition 3.1.6 give
\[
\|U_n^{-1} J_3\| \leq \lambda_u^{-1} h\|f - \varphi\|_\infty.
\]
Finally, (3.12), (6.36), Condition SH1(Lip) from Definition 3.1.6 and the definition of the function $\delta_\beta(\varepsilon)$ imply that
\[
\|U_n^{-1} J_4\| \leq \lambda_u^{-1} h\beta\|f - \varphi\|_\infty. \tag{6.38}
\]
From (6.32) and (6.37)–(6.38) it thus follows that
\[ \| v_{n-1}^u \| \leq \lambda_{u}^{-1} \left( \| P_{x_n}^u z_n \| + \mu_u \| P_{x_{n-1}}^s z_{n-1} \| + (1 + \beta) \| f - \varphi \|_{\infty} h \right), \]
so by (6.31)
\[ m^u(\mathcal{H}(z)) \leq \lambda_{u}^{-1} (m^u(z) + \mu_u m^s(z) + (1 + \beta) \| f - \varphi \|_{\infty} h). \] (6.39)

The Lemma follows from the inequalities (6.30) and (6.39). \( \square \)

Returning to the proof of Theorem 6.2.7, without loss of generality assume that the split \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) is positive. Recall from Sect. 4.1 that \( \sigma(M) < 1 \) is the spectral radius (4.3) of the split matrix \( M \) and that \( \| \cdot \|_* \) is the norm on \( \mathbb{R}^2 \) defined for a given positive split \( s \) by
\[ \|(y_1, y_2)\|_* = \max \{ |y_1|, |y_2| \}. \]
with \( \gamma = \gamma(s) > 0 \) given by (4.5).

Choose any fixed number \( \alpha > \alpha(s, h) \), with \( \alpha(s, h) \) defined by (6.23), write
\[ \beta = \frac{\alpha}{\alpha(s, h)} - 1, \]
and consider the convex set
\[ \mathcal{V} = \left\{ z : \| m(z) \|_* \leq \frac{1 + \beta}{1 - \sigma(M)} \| h \|_* \| f - \varphi \|_{\infty} \right\}, \]
which is well defined since \( \sigma(M) < 1 \).

Lemma 6.2.10. There exists an \( \varepsilon_0 > 0 \) such that \( \mathcal{V} \subseteq \mathcal{W} \) for
\[ \| f - \varphi \|_{\infty} \leq \varepsilon_0. \] (6.40)

Proof. First,
\[ \mathcal{V} \subseteq \mathcal{V}^+, \] (6.41)
where
\[ \mathcal{V}^+ = \left\{ z : \| z \|_{\infty} \leq 2 \frac{1 + \beta}{1 - \sigma(M)} \max \{ \gamma^{-1}, 1 \} \| h \|_* \| f - \varphi \|_{\infty} \right\}. \]
To see this, note that
\[ \| m(z) \|_* = \max \left\{ \gamma \max_{0 \leq n \leq N} \| P_{x_n}^s z_n \|, \max_{0 \leq n \leq N} \| P_{x_n}^u z_n \| \right\} \]
\[ = \max_{0 \leq n \leq N} \max \left\{ \gamma \| P_{x_n}^s z_n \|, \| P_{x_n}^u z_n \| \right\} \]
\[ = \max_{0 \leq n \leq N} \| (\| P_{x_n}^s z_n \|, \| P_{x_n}^u z_n \|) \|_* \]
and so by Lemma \[6.1.6\]
\[
\|m(z)\|_* \geq \frac{1}{2} \min \{\gamma, 1\} \max_{0 \leq n \leq N} \|z_n\| = \frac{1}{2} \min \{\gamma, 1\} \|z\|_\infty.
\]
The inclusion \((6.41)\) then follows from this last inequality.

Now, by definition of the differential of a mapping,
\[
\liminf_{\varepsilon \to 0^+} \frac{\delta_\beta(\varepsilon)}{\varepsilon} = \infty,
\]
where \(\delta_\beta(\cdot)\) is as in \[6.25\], and thus a number \(\varepsilon_0 > 0\) can be chosen such that
\[
2 \frac{1 + \beta}{1 - \sigma(M)} \max \{\gamma^{-1}, 1\} \|h\|_* \varepsilon \leq \delta_\beta(\varepsilon) \quad \text{for} \quad \varepsilon \leq \varepsilon_0.
\]
Hence, for \(f\) and \(\varphi\) satisfying \((6.40)\) the inclusion \(\mathcal{V}^+ \subseteq \mathcal{W}\) holds, and thus by \((6.41)\), \(\mathcal{V} \subseteq \mathcal{W}\). The lemma is proved. \(\square\)

Now turn to completion of the proof of Theorem \[6.2.7\]. Choose \(\varepsilon_0 > 0\) as in Lemma \[6.2.10\]. Then \(\mathcal{V} \subseteq \mathcal{W}\) and for any vector \(z \in \mathcal{V}\), by Lemma \[6.2.9\] the component by component inequality
\[
m(\mathcal{H}(z)) \leq Mm(z) + (1 + \beta)\|f - \varphi\|_\infty h
\]
holds. Hence
\[
\|m(\mathcal{H}(z))\|_* \leq \|Mm(z)\|_* + (1 + \beta)\|f - \varphi\|_\infty \|h\|_*.
\]
Recall from Sect. \[4.1\] that the corresponding matrix norm \(\|M\|_*\) of the matrix \(M\) coincides with the spectral radius \(\sigma(M) < 1\) of \(M\) and that \(\|My\|_* \leq \sigma(M)\|y\|_*\) for all \(y \in \mathbb{R}^2\). Hence
\[
\|m(\mathcal{H}(z))\|_* \leq \sigma(M)\|m(z)\|_* + (1 + \beta)\|f - \varphi\|_\infty \|h\|_*
\]
and by definition of the set \(\mathcal{V}\)
\[
\mathcal{H}(z) \in \mathcal{V} \quad \text{for} \quad z \in \mathcal{V},
\]
that is, the set \(\mathcal{V}\) is invariant under the operator \(\mathcal{H}\). Then, from continuity of \(\mathcal{H}\), see Lemma \[6.2.8\] and the Schauder Fixed Point Theorem, there exists a point \(z^*\) satisfying \(\mathcal{H}(z^*) = z^*\) such that
\[
z^* \in \mathcal{V} \subseteq \mathcal{W}.
\]
This and Lemma \[6.2.9\] imply that
\[
m(z^*) = m(\mathcal{H}(z^*)) \leq Mm(z^*) + (1 + \beta)\|f - \varphi\|_\infty h.
\]
Hence, by positivity of the components of the split matrix \(M\),
6.3 Bi-Shadowing

The concept of ε-shadowing as in Definition 6.2.2 and α-robustness as in Definition 6.2.5 can be applied to investigate perturbations of a given system via Theorems 6.2.3 and 6.2.7. Their practical utility is somewhat limited as the proofs are of existence rather than constructive. A unified approach for investigating both direct and inverse shadowing is provided by the concept of bi-shadowing which has the added advantage of providing explicit values of the parameters involved.

Let $\text{Tr}(f, K, \gamma)$ denote the totality of finite or infinite $\gamma$-pseudo-trajectories (6.18) of $f$ belonging entirely to the subset $K \subseteq X$. Since a true trajectory can be regarded as a $\gamma$-pseudo-trajectory for any $\gamma \geq 0$, in particular with $\gamma = 0$, it is natural to denote the corresponding set of true trajectories by $\text{Tr}(f, K, 0)$ or simply by $\text{Tr}(f, K)$. Obviously $\text{Tr}(f, K) \subseteq \text{Tr}(f, K, \gamma)$, with strict inclusion as there are $\gamma$-pseudo-trajectories which are not trajectories.

As a distance between mappings $\varphi$ and $f$ on $X$ use the semi-norm$$
\|\varphi - f\|_\infty = \sup_{x \in X} \|\varphi(x) - f(x)\|.
$$

**Definition 6.3.1.** A dynamical system generated by a mapping $f : X \to X$ is said to be bi-shadowing on a subset $K$ of $X$ with positive parameters $\alpha$ and $\beta$ if for any given finite pseudo-trajectory $x = \{x_n\} \in \text{Tr}(f, K, \gamma)$ with $0 \leq \gamma \leq \beta$ and any mapping $\varphi : X \to X$ satisfying

$$
\|\varphi - f\|_\infty \leq \beta - \gamma
$$

there exists a trajectory $y = \{y_n\} \in \text{Tr}(\varphi, X)$ such that

$$
\|x_n - y_n\| \leq \alpha(\gamma + \|\varphi - f\|_\infty)
$$

for all $n$ for which $y$ is defined.
Remark 6.3.2. Bi-shadowing conceptualizes the robustness of observed dynamical behavior of a dynamical system and perturbations such as those arising from computer simulations. It can also be interpreted as a form of dynamical structural stability when restricted to specific classes of mappings, such as continuous mappings. Moreover, it implies both direct shadowing and inverse shadowing properties, in the form of $\alpha$-robustness previously discussed. Taking $\varphi \equiv f$ in (6.42) and (6.43) gives $(\alpha \gamma)$-shadowing of any $\gamma$-pseudo-trajectory $x \in \text{Tr}(f, K, \gamma)$ by a true trajectory $y \in \text{Tr}(f, K)$, while $\alpha$-robustness of any trajectory $x$ of $f$ follows because a trajectory $y \in \text{Tr}(\varphi, X)$ can always be found which $(\alpha \parallel \varphi - f \parallel_\infty)$-shadows a given true trajectory $x \in \text{Tr}(f, K)$, considered here as the $\gamma$-pseudo-trajectory with $\gamma = 0$.

6.3.1 Bi-Shadowing of Finite Trajectories

The main result of this section is that semi-hyperbolicity is sufficient to ensure bi-shadowing of a dynamical system generated by a Lipschitz mapping with respect to a class of perturbed systems generated by continuous mappings. It not only generalizes existing versions of the Shadowing Lemma to a far broader class of systems, but includes inverse as well as direct shadowing and provides explicit values of the shadowing parameters.

Theorem 6.3.3. Let $f : X \to X$ be a Lipschitz mapping which is semi-hyperbolic on a compact subset $K \subset X$ with a split $s$ and constants $h, \delta$ as in Definition 3.1.6. Then it is bi-shadowing on $K$ with respect to continuous mappings $\varphi : X \to X$ with parameters $\alpha = \alpha(s, h)$ and $\beta = \beta(s, h, \delta)$ defined by

$$\alpha(s, h) = h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}$$

and

$$\beta(s, h, \delta) = \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}.$$  (6.45)

The proof proceeds through a number of lemmas. Denote by $B^u_x(r)$ the closed ball of the radius $r$ centered at 0 in the linear space $E^u_x$. For each $x, y \in K$ with $\|f(x) - y\| \leq \delta$ and each $z \in \mathbb{R}^d$ satisfying $\|P^s_x z\| \leq \delta$ define the mapping $F_{x,y,z} : B^u_x(\delta) \to E^u_y$ by

$$F_{x,y,z}(v) = P^u_y(f(x + P^s_x z + v) - f(x + P^s_x z)).$$

Lemma 6.3.4. Let $0 \leq r \leq \delta$. Then $B^u_y(\lambda_u r) \subseteq F_{x,y,z}(B^u_x(r))$.

Proof. The lemma is immediate for $r = 0$, so suppose that $r > 0$. Denote by $\partial B^u_x(r)$ and $\text{Int} B^u_x(r)$ the boundary and the interior of the ball $B^u_x(r)$. Clearly,

$$F_{x,y,z}(0) = P^u_y(f(x + P^s_x z) - f(x + P^s_x z)) = 0 \in \text{Int} B^u_y(\lambda_u r).$$  (6.46)
On the other hand, by inequality (3.16)
\[ \| F_{x,y,z}(v) - F_{x,y,z}(\tilde{v}) \| \geq \lambda_u \| v - \tilde{v} \| \quad \text{for} \quad v, \tilde{v} \in E^u_x, \| v \|, \| \tilde{v} \| \leq \delta. \] (6.47)
In particular, from (6.47)
\[ \| F_{x,y,z}(v) \| \geq \lambda_u r \quad \text{for} \quad v \in E^u_x, \| v \| = r \leq \delta, \]
and so
\[ F_{x,y,z}(\partial B^u_x(r)) \cap \text{Int } B^u_y(\lambda_u r) = \emptyset. \] (6.48)
By Condition SH0(Lip) of Definition 3.1.6, the dimensions of the linear spaces $E^u_x$ and $E^u_y$ are equal for any $x, y \in K$ with $\| f(x) - y \| \leq \delta$. Therefore, by (6.47), the mapping $F_{x,y,z}(v)$ is homeomorphic on the ball $B^u_x(r)$ with $0 < r \leq \delta$. Hence by the Principle of Domain Invariance [5, p. 396] the image of the boundary $\partial B^u_x(r)$ of the ball $B^u_x(r)$ under the map $F_{x,y,z}(v)$ is also the boundary $\partial F_{x,y,z}(B^u_x(r))$ of the set $F_{x,y,z}(B^u_x(r))$,
\[ F_{x,y,z}(\partial B^u_x(r)) = \partial F_{x,y,z}(B^u_x(r)). \]
By (6.48) this last implies that
\[ \partial F_{x,y,z}(B^u_x(r)) \cap \text{Int } B^u_y(\lambda_u r) = \emptyset. \]
This, together with (6.46), gives
\[ B^u_y(\lambda_u r) \subseteq F_{x,y,z}(B^u_x(r)) \]
and the lemma is proved. \(\square\)

**Lemma 6.3.5.** Let $x, y \in K$ and $\| f(x) - y \| \leq \delta$. Then the operator $Q_{x,y,z} = F^{-1}_{x,y,z}$ is well defined and continuous on $B^u_y(\lambda_u \delta)$ and satisfies $\| Q_{x,y,z}(v) \| \leq \lambda_u^{-1} \| v \|$.

As an immediate consequence of Lemma 6.3.4 and inequality (3.16),

**Lemma 6.3.4.** Let $x, y \in K$ and $\| f(x) - y \| \leq \delta$. Then the operator $Q_{x,y,z} = F^{-1}_{x,y,z}$ is well defined and continuous on $B^u_y(\lambda_u \delta)$ and satisfies $\| Q_{x,y,z}(v) \| \leq \lambda_u^{-1} \| v \|$.

As in the proof of Theorem 6.2.3 let $\mathcal{N}$ be the space of $(N+1)$-sequences
\[ z = \{ z_0, z_1, \ldots, z_N \}, \quad z_n \in \mathbb{R}^d, \quad n = 0, 1, \ldots, N. \]
The set $\mathcal{N}$ can be regarded as the $(N + 1)d$-dimensional vector space $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ ($N + 1$ times), with norm
\[ \| z \|_\infty = \max_{0 \leq n \leq N} \| z_n \|. \]
Let $\alpha = \{ x_0, x_1, \ldots, x_N \}$ be a given $\gamma$-pseudo-trajectory of the system $f$ and let $\varphi$ be a given continuous mapping such that
\[ \beta = \gamma + \| f - \varphi \|_\infty \] (6.49)
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satisfies the inequality

\[ \beta \leq \beta(s, h, \delta) \]  

(6.50)

with \( \beta(s, h, \delta) \) as in (6.45).

Introduce the operator \( \mathcal{H} : \mathcal{B}_N \to \mathcal{B}_N \) which transforms

\[ \mathbf{z} = \{z_0, z_1, \ldots, z_N\} \in \mathcal{B}_N \]

into a sequence \( \mathbf{w} = \{w_0, w_1, \ldots, w_N\} \in \mathcal{B}_N \) defined by

\[ P^s_{x_0} w_0 = 0, \]  

(6.51)

\[ P^s_{x_n} w_n = P^s_{x_n} (\varphi(x_{n-1} + z_{n-1}) - x_n) \]  

(6.52)

for \( n = 1, 2, \ldots, N \), and

\[ P^u_{x_N} w_N = 0, \]  

(6.53)

\[ P^u_{x_{n-1}} w_{n-1} = Q^u_{x_{n-1}, x_n, z_{n-1}} \left( P^u_{x_n} (-\varphi(x_{n-1} + z_{n-1}) \right) \]  

\[ \quad + f(x_{n-1} + z_{n-1}) + x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) + z_n) \]  

(6.54)

for \( n = 1, 2, \ldots, N \).

Define parameters \( a \) and \( b \) as the coordinates of the two-dimensional vector

\[ (a, b)^T = (I - M(s))^{-1} \mathbf{h} \quad \text{with} \quad \mathbf{h} = h(1, \lambda_u^{-1})^T, \]  

(6.55)

where \( M(s) \) is the \( 2 \times 2 \) split matrix given by (4.2). Explicitly,

\[ a = \frac{\lambda_u - 1 + \mu_s}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h, \]  

(6.56)

\[ b = \frac{1 - \lambda_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h, \]  

(6.57)

and so

\[ \alpha(s, h) = a + b, \quad \beta(s, h, \delta) = \delta \min\{a^{-1}, b^{-1}\}. \]  

(6.58)

Consider the set

\[ \mathcal{I}(\beta) = \{ \mathbf{z} \in \mathcal{B}_N : \|P^s_{x_n} z_n\| \leq a \beta, \|P^u_{x_n} z_n\| \leq b \beta, \ 0 \leq n \leq N \} . \]  

(6.59)

By (6.50) and (6.58)

\[ a \beta \leq a \beta(s, h, \delta) \leq \delta, \quad b \beta \leq b \beta(s, h, \delta) \leq \delta \]  

(6.60)

and since \( x + u + v \in X \) from Condition SH2(Lip) of Definition 3.1.6, trajectories from \( \mathcal{I}(\beta) \) belong to \( X \).

**Lemma 6.3.6.** The operator \( \mathcal{H} \) is well defined and continuous on \( \mathcal{I}(\beta) \) and for any of its fixed points \( \mathbf{z} = \{z_0, z_1, \ldots, z_N\} \in \mathcal{I}(\beta) \) the sequence

\[ \mathbf{y} = \{x_0 + z_0, x_1 + z_1, \ldots, x_N + z_N\} \]  

(6.61)

is a trajectory of the system \( \varphi \).
Proof. First the operator $H$ is well defined and continuous at any point $z \in \mathcal{S}(\beta)$. For, it is clear by (6.60) that the right hand side of (6.52) is well defined and depends continuously on $z \in \mathcal{S}(\beta)$. Hence, it suffices to prove that for any $n = 1, 2, \ldots, N$ the right hand side of the (6.54) is well defined and continuous for $z \in \mathcal{S}(\beta)$. By Lemma 6.3.5 it is sufficient to establish that

$$\|P_{x_n}^u (\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1}) + x_n - f(x_{n-1} + P_{x_{n-1}}^s z_{n-1}) + z_n)\| \leq \lambda u \delta.$$

Rewrite the last inequality in the form

$$\|J_1 + J_2 + J_3\| \leq \lambda u \delta,$$

where

$$J_1 = P_{x_n}^u (\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1})) + P_{x_n}^u (x_n - f(x_{n-1})),$$

$$J_2 = P_{x_n}^u (f(x_{n-1}) - f(x_{n-1} + P_{x_{n-1}}^s z_{n-1})),$$

$$J_3 = P_{x_n}^u z_n.$$

(6.62)

To estimate $\|J_1\|$ note that by (6.49)

$$\|\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})\| \leq \|\varphi - f\|_\infty = \beta - \gamma$$

and also that $\|x_n - f(x_{n-1})\| \leq \gamma$ since $x$ is a $\gamma$-pseudo-trajectory of the mapping $f$, so by Condition SH1(Lip)

$$\|J_1\| \leq \beta h. \tag{6.63}$$

By inequality (3.15),

$$\|J_2\| \leq \mu u \|P_{x_{n-1}}^s z_{n-1}\|. \tag{6.64}$$

Clearly

$$\|J_3\| = \|P_{x_n}^u z_n\|. \tag{6.65}$$

On the other hand, by definition (6.59) of the set $\mathcal{S}(\beta)$, $z \in \mathcal{S}(\beta)$ implies that

$$\|P_{x_{n-1}}^s z_{n-1}\| \leq \beta a, \quad \|P_{x_n}^u z_n\| \leq \beta b. \tag{6.66}$$

Hence by (6.63)–(6.66)

$$\|J_1 + J_2 + J_3\| \leq \|J_1\| + \|J_2\| + \|J_3\| \leq \beta (h + a \mu u + b)$$

so it suffices to establish that

$$\beta (h + a \mu u + b) \leq \lambda u \delta. \tag{6.67}$$
From (6.56) and (6.57) we have \( h + a\mu_\alpha + b = \lambda_\alpha b \), so (6.67) can be rewritten as
\[
\beta\lambda_\alpha b \leq \lambda_\alpha \delta,
\]
but this follows from (6.58). Hence the operator \( \mathcal{H} \) is well defined and continuous at any point \( z \in \mathcal{S}(\beta) \).

It remains to prove that the sequence (6.61) is a trajectory of the system \( \varphi \) provided that \( z = \{z_0, z_1, \ldots, z_N\} \in \mathcal{S}(\beta) \) is a fixed point of the operator \( \mathcal{H} \). By assumption \( z \) is a fixed point of the operator \( \mathcal{H} \), so equation (6.52) can be rewritten as
\[
P^s_{x_n} z_n = P^s_{x_n} (\varphi(x_{n-1} + z_{n-1}) - x_n)
\]
which can be rearranged to give
\[
P^s_{x_n} (x_n + z_n) = P^s_{x_n} \varphi(x_{n-1} + z_{n-1}). \tag{6.68}
\]
Similarly, equation (6.54) can be rewritten as
\[
P^u_{x_{n-1}} z_{n-1} = Q_{x_{n-1},x_n,z_{n-1}}^{-1} \left( P^u_{x_{n-1}} (-\varphi(x_{n-1} + z_{n-1}) \right.
+ f(x_{n-1} + z_{n-1}) + x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1} + z_n) \left.) \right).
\]
Applying the nonlinear operator \( F_{x_{n-1},x_n,z_{n-1}} \) to both sides of this last equation, obtain
\[
F_{x_{n-1},x_n,z_{n-1}} (P^u_{x_{n-1}} z_{n-1}) = P^u_{x_n} \left( f(x_{n-1} + z_{n-1}) \right.
- \varphi(x_{n-1} + z_{n-1}) + x_n - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1} + z_n) \left.) \right).
\]
By definition of \( F_{x,y,z}(v) \),
\[
F_{x_{n-1},x_n,z_{n-1}} (P^u_{x_{n-1}} z_{n-1}) = P^u_{x_n} \left( f(x_{n-1} + P^s_{x_{n-1}} z_{n-1} + P^u_{x_{n-1}} z_{n-1}) - f(x_{n-1} + P^s_{x_{n-1}} z_{n-1}) \right),
\]
so, by comparing the last two equations,
\[
0 = P^u_{x_n} (z_n - \varphi(x_{n-1} + z_{n-1}) + x_n)
\]
or on rearranging
\[
P^u_{x_n} (x_n + z_n) = P^u_{x_n} \varphi(x_{n-1} + z_{n-1}). \tag{6.69}
\]
From (6.68) and (6.69) it now follows that
\[
x_n + z_n = \varphi(x_{n-1} + z_{n-1}), \quad n = 1, 2, \ldots, N,
\]
which means that the sequence (6.61) is a trajectory of the system \( \varphi \). The lemma is proved.
Lemma 6.3.7. The set $\mathcal{S}(\beta)$ is invariant under the operator $\mathcal{H}$.

Proof. Let $z \in \mathcal{S}(\beta)$ and write $w = \mathcal{H}(z)$, see (6.52) and (6.54). Write (6.52) in the form

$$P_s^z w_n = I_1 + I_2$$

where

$$I_1 = P_s^z \left( (\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})) + (f(x_{n-1}) - x_n) \right),$$

$$I_2 = P_s^z \left( f(x_{n-1} + z_{n-1}) - f(x_{n-1}) \right),$$

and rewrite (6.54) in the form

$$P_u^{x_{n-1}} w_{n-1} = Q_{x_{n-1}, x_n, z_{n-1}} (J_1 + J_2 + J_3)$$

where $J_1, J_2, J_3$ are defined in (6.62).

To estimate $\|I_1\|$ note that by (6.49)

$$\|\varphi - f\|_\infty = \beta - \gamma \quad \text{and} \quad \|x_n - f(x_{n-1})\| \leq \gamma$$

since $x$ is a $\gamma$-pseudo-trajectory of the mapping $f$. Hence

$$\|\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})\| + \|f(x_{n-1}) - x_n\| \leq (\beta - \gamma) + \gamma = \beta$$

and by Condition SH1(Lip)

$$\|I_1\| \leq h\beta. \quad (6.70)$$

By (3.13), (3.14) and from Condition SH2(Lip)

$$\|I_2\| \leq \lambda_s \|P_s^z z_{n-1}\| + \mu_s \|P_u^{x_{n-1}} z_{n-1}\|. \quad (6.71)$$

Now for each $z \in \mathcal{B}_N$, define the pair of real nonnegative numbers

$$m^s(z) = \max_{0 \leq n \leq N} \|P_s^z z_n\|, \quad m^u(z) = \max_{0 \leq n \leq N} \|P_u^{x_{n-1}} z_n\|,$$

and denote by $m(z)$ the two-dimensional column vector with coordinates $m^s(z)$ and $m^u(z)$. From the estimates (6.70), (6.71) and definition (6.59) of the set $\mathcal{S}(\beta)$ it follows that

$$m^s(\mathcal{H} z) = m^s(w) \leq \beta h + \beta \lambda_s a + \beta \mu_s b. \quad (6.72)$$

Similarly, from (6.63)–(6.65), the definition (6.59) of the set $\mathcal{S}(\beta)$ and Lemma 6.3.5 it follows that

$$m^u(\mathcal{H} z) = m^u(w) \leq \lambda_u^{-1}(\beta \mu_u a + \beta b + \beta h). \quad (6.73)$$

Inequalities (6.72) and (6.73) are equivalent to the component by component inequality.
\[
m(\mathcal{H} z) = m(w) \leq \beta M(s)(a,b)^T + \beta h, \quad z \in \mathcal{I}(\beta),
\]
where, by (6.55),
\[
\beta M(s)(a,b)^T + \beta h = \beta (M(s)(I - M(s))^{-1} + I) h
\]
\[
= \beta (I - M(s))^{-1} h = \beta(a,b)^T.
\]
Hence, (6.74) is equivalent to the coordinate-wise estimate
\[
m(\mathcal{H} z) = m(w) \leq \beta(a,b)^T, \quad z \in \mathcal{I}(\beta),
\]
which, by the definition (6.59) of \(\mathcal{I}(\beta)\), means that the set \(\mathcal{I}(\beta)\) is invariant under the operator \(\mathcal{H}\).

**Proof of Theorem 6.3.3.** By Lemma 6.3.6 the operator \(\mathcal{H}\) is continuous on the convex set \(\mathcal{I}(\beta)\) and by Lemma 6.3.7 \(\mathcal{H}(\mathcal{I}(\beta)) \subseteq \mathcal{I}(\beta)\), so by the Brouwer Fixed Point Theorem \(\mathcal{H}\) has a fixed point
\[
z = \{z_0, z_1, \ldots, z_N\} \in \mathcal{I}(\beta).
\]
Hence, by Lemma 6.3.6 again, the sequence
\[
y = \{x_0 + z_0, x_1 + z_1, \ldots, x_N + z_N\}
\]
is a trajectory of the mapping \(\varphi\).

By definition (6.59) of the set \(\mathcal{I}(\beta)\),
\[
\|z_n\| \leq \|P^s_{x_n} z_n\| + \|P^u_{x_n} z_n\| \leq (a + b)\beta, \quad n = 0, 1, \ldots, N,
\]
holds for the fixed point \(z \in \mathcal{I}(\beta)\) of the operator \(\mathcal{H}\). Thus by the definitions of \(\beta\) and \(a, b\) in (6.49) and (6.58) respectively,
\[
\|z_n\| \leq \alpha(s, h)(\gamma + \|\varphi - f\|_\infty), \quad n = 0, 1, \ldots, N,
\]
that is
\[
\|x_n - y_n\| = \|z_n\| \leq \alpha(s, h)(\gamma + \|\varphi - f\|_\infty), \quad n = 0, 1, \ldots, N.
\]
The proof is complete. \(\square\)

**Remark 6.3.8.** From the Definition 6.3.1 of bi-shadowing, Theorem 6.3.3 ensures the existence of a trajectory \(y = \{y_n\} \in \text{Tr}(\varphi, X)\), for a given pseudo-trajectory \(x = \{x_n\} \in \text{Tr}(f, K, \gamma)\) with \(0 \leq \gamma \leq \beta(s, h, \delta)\) satisfying
\[
\|x_n - y_n\| \leq \alpha(s, h)(\gamma + \|\varphi - f\|_\infty).
\]
Often it is important that the inequalities
\[
\|P^s_{x_n} (x_n - y_n)\|, \quad \|P^u_{x_n} (x_n - y_n)\| \leq \delta,
\]
also hold. These follow from \(y_n - x_n = z_n \in \mathcal{I}(\beta)\), definition (6.59) of the set \(\mathcal{I}(\beta)\) and the inequalities (6.60) established in the proof of Theorem 6.3.3.
6.3.2 Bi-Shadowing of Infinite Trajectories

Results on bi-shadowing of infinite trajectories follow from expansivity of semi-hyperbolic Lipschitz mappings and Theorem 6.3.3.

**Theorem 6.3.9.** Let \( f : X \to X \) be a Lipschitz mapping which is semi-hyperbolic on a compact subset \( K \subset X \) with split \( s \) and constants \( h, \delta \) (see Definition 3.1.6), and let \( \alpha(s, h) \) and \( \beta(s, h) \) be as in (6.44) and (6.45). Then the following statements hold.

(i) For any infinite pseudo-trajectory \( x = \{x_n\} \in \text{Tr}(f, K, \gamma) \) and any continuous mapping \( \varphi : X \to X \) satisfying \( \|\varphi - f\|_\infty \leq \beta(s, h, \delta) - \gamma \) there exists an infinite trajectory \( y = \{y_n\} \in \text{Tr}(\varphi, X) \) such that

\[
\|x_n - y_n\| \leq \alpha(s, h)(\gamma + \|\varphi - f\|_\infty), \quad n \in \mathbb{Z}.
\]

(ii) For any given infinite pseudo-trajectory \( x = \{x_n\} \in \text{Tr}(f, K, \gamma) \) with

\[
\gamma \leq \min \{\delta/(2h\alpha(s, h)), \beta(s, h, \delta)\}
\]

there exists a unique infinite trajectory \( y = \{y_n\} \in \text{Tr}(f, X) \) such that

\[
\|x_n - y_n\| \leq \alpha(s, h)\gamma, \quad n \in \mathbb{Z}.
\]

**Proof.** To prove (i) for the given \( \gamma \)-pseudo-trajectory \( x = \{x_n\} \in \text{Tr}(f, K, \gamma) \), consider the sequence of finite \( \gamma \)-pseudo-trajectories

\[
x^{(k)} = \left\{ x^{(k)}_{-k}, \ldots, x^{(k)}_0, \ldots, x^{(k)}_k \right\} \in \text{Tr}(f, K, \gamma), \quad k = 1, 2, \ldots,
\]

with

\[
x^{(k)}_n = x_n, \quad n = 0, \pm 1, \pm 2, \ldots, \pm k.
\]

Then, for a given mapping \( \varphi \) satisfying \( \|\varphi - f\|_\infty \leq \beta(s, h, \delta) - \gamma \), by Theorem 6.3.3 there exists for any \( k = 1, 2, \ldots \) a finite trajectory

\[
y^{(k)} = \left\{ y^{(k)}_{-k}, \ldots, y^{(k)}_0, \ldots, y^{(k)}_k \right\} \subseteq \text{Tr}(\varphi, X)
\]

such that

\[
\|y^{(k)}_n - x_n\| = \|y^{(k)}_n - x^{(k)}_n\| \leq \alpha(s, h)(\gamma + \|\varphi - f\|_\infty) \quad (6.75)
\]

for \( n = 0, \pm 1, \pm 2, \ldots, \pm k \). Moreover, by Remark 6.3.8 the inequalities

\[
\|P^s_{x_n}(y^{(k)}_n - x_n)\| \leq \delta, \quad \|P^u_{x_n}(y^{(k)}_n - x_n)\| \leq \delta \quad (6.76)
\]

hold for \( n = 0, \pm 1, \pm 2, \ldots, \pm k \).

From (6.75), for any integer \( n \) the sequence \( \left\{ y^{(k)}_n \right\} \) is bounded and thus, without loss of generality, by taking an appropriate subsequence and relabeling, suppose that it converges to a limit \( y_n, y^{(k)}_n \to y_n \) as \( k \to \infty \).
Taking the limit in (6.76),
\[
\|P^s_{x_n}(y_n - x_n)\| \leq \delta, \quad \|P^u_{x_n}(y_n - x_n)\| \leq \delta,
\]
and so, since \(x_n \in K\) and Condition SH2(Lip) of Definition 3.1.6,
\[
y_n \in X, \quad n \in \mathbb{Z}. \tag{6.77}
\]
Taking the limit \(k \to \infty\) on both sides of
\[
y^{(k)}_{n+1} = \varphi(y^{(k)}_n)
\]
obtain
\[
y_{n+1} = \varphi(y_n).
\]
Together with (6.77) this means that
\[
y = \{y_n\}_{n=-\infty}^{\infty} \in \text{Tr}(\varphi, X).
\]
Finally, taking the limit \(k \to \infty\) in (6.75) the infinite trajectory \(y \in \text{Tr}(\varphi, X)\) of the system \(\varphi\) satisfies
\[
\|x_n - y_n\| \leq \alpha(s, h)(\gamma + \|\varphi - f\|_{\infty}), \quad n \in \mathbb{Z}.
\]
This completes the proof of (i).

To prove (ii), note first that the existence of an infinite trajectory \(y \in \text{Tr}(f, X)\) satisfying
\[
\|x_n - y_n\| \leq \alpha(s, h)\gamma, \quad n \in \mathbb{Z},
\]
follows from (i). To prove the uniqueness of such an infinite trajectory \(y \in \text{Tr}(f, X)\) the exponential expansivity of a semi-hyperbolic mapping established in Theorem 6.1.5 is used.

Suppose that the statement of (ii) is false. Then there exist trajectories \(y, \tilde{y} \in \text{Tr}(f, X)\) such that
\[
\|y_n - x_n\|, \|\tilde{y}_n - x_n\| \leq \alpha(s, h)\gamma, \quad n \in \mathbb{Z},
\]
and hence such that
\[
\|y_n - \tilde{y}_n\| \leq 2\alpha(s, h)\gamma \leq h^{-1}\delta, \quad n \in \mathbb{Z}.
\]
This contradicts Theorem 6.1.5 and so (ii) holds. \(\square\)

### 6.3.3 Cyclic Bi-Shadowing

Cyclic, or periodic, behavior is often of particular interest in dynamical systems. Here it is shown that bi-shadowing is preserved when attention is restricted to periodic trajectories, also often called cycles.
Definition 6.3.10. A trajectory $x = \{x_n\}_{n=0}^N \in \text{Tr}(f, K)$ is called a cycle, or periodic trajectory, of period $N$ if $x_N = x_0$ and a pseudo-trajectory $y = \{y_n\}_{n=0}^N \in \text{Tr}(f, K, \gamma)$ is called a $\gamma$-pseudo-cycle, or periodic $\gamma$-pseudo-trajectory, of period $N$ if $\|y_N - y_0\| \leq \gamma$.

Let $\text{Cyc}(f, K, \gamma) \subset \text{Tr}(f, K, \gamma)$ denote the totality of $\gamma$-pseudo-cycles of any period belonging entirely to the subset $K$ of $X$, with $\text{Cyc}(f, K, 0) \subset \text{Tr}(f, K)$ or $\text{Cyc}(f, K) \subset \text{Tr}(f, K)$ denoting the totality of proper cycles of any period which are contained entirely in $K$. Obviously $\text{Cyc}(f, K) \subset \text{Cyc}(f, K, \gamma)$ for every $\gamma > 0$.

Definition 6.3.11. A dynamical system generated by a mapping $f : X \to X$ is said to be cyclically bi-shadowing with positive parameters $\alpha$ and $\beta$ on a subset $K$ of $X$ if for any given pseudo-cycle $x \in \text{Cyc}(f, K, \gamma)$ with $0 \leq \gamma \leq \beta$ and any mapping $\varphi : X \to X$ satisfying (6.42) there exists a proper cycle $y \in \text{Cyc}(\varphi, X)$ of period $N$ equal to that of $x$ such that (6.43) holds for $n = 0, 1, \ldots, N$.

Note that the cycle $y$ in Definition 6.3.11 is required only to be in $X$ rather than in the subset $K$. Cyclic bi-shadowing is also a consequence of semi-hyperbolicity.

Theorem 6.3.12. Let $f : X \to X$ be a Lipschitz mapping which is semi-hyperbolic on a compact subset $K \subseteq X$ with a split $s$ and constants $h, \delta$. Then it is cyclically bi-shadowing on $K$ with respect to continuous mappings $\varphi : X \to X$ with parameters $\alpha(s, h)$ and $\beta(s, h, \delta)$ given by (6.44) and (6.45).

The proof repeats verbatim that of Theorem 6.3.3 with the following two minor modifications.

First, the boundary conditions
\[ P^s_{x_0}(w_0) = P^s_{x_N}(w_N), \quad P^u_{x_N}(w_N) = P^u_{x_0}(w_0) \]
should be used instead of (6.51) and (6.53) in the definition of the operator $H$.

Second, in the proof the following specifically cyclic version of Lemma 6.3.6 should be used.

Lemma 6.3.13. The operator $H$ is well defined and continuous on $\mathcal{I}(\beta)$, and for any of its fixed points $z = \{z_0, z_1, \ldots, z_N\} \in \mathcal{I}(\beta)$ the sequence
\[ y = \{x_0 + z_0, x_1 + z_1, \ldots, x_N + z_N\} \]
is a cycle of the system $\varphi$. 

7

Structural Stability

This chapter further explores some structural stability properties of semi-hyperbolic mappings. More precisely, it is shown that topological entropy is nondecreasing with respect to continuous perturbations of a Lipschitz semi-hyperbolic mapping. Conjugation and factorization properties of semi-hyperbolic mappings are also studied. Lastly, chaotic phenomena of semi-hyperbolic mappings are discussed. Throughout the chapter $\| \cdot \|$ will denote a fixed but otherwise arbitrary norm on $\mathbb{R}^d$.

7.1 Topological Entropy

Throughout this section, $X \subseteq \mathbb{R}^d$ is open and $K \subset X$ is a compact subset of $X$. Let $h_\varepsilon(f, K)$ and $h(f, K)$ denote the $\varepsilon$-entropy and entropy of the mapping $f$ as defined in Definition 2.2.6.

Recall from Chap. 6 that semi-hyperbolic mappings are expansive. The following lemma relating $\varepsilon$-entropy and entropy of expansive mappings, is well known [90], and is useful for providing a method of calculation for entropy.

**Lemma 7.1.1.** Let $K \subset \mathbb{R}^d$ be a compact set and $f : K \to K$ be a continuous $\xi$-expansive mapping with $f(K) = K$. Then

$$h(f, K) = h_\theta(f, K)$$

for every $0 \leq \theta < \xi$.

**Proof.** Fix $\theta$ with $0 < \theta < \xi$. As noted in footnote 2 in Chap. 2, $h_\varepsilon(f, K)$ is non-increasing in $\varepsilon$, so

$$h(f, K) = \sup_{\varepsilon > 0} h_\varepsilon(f, K)$$

and thus by Definition 2.2.6 it suffices to prove that

$$h_\varepsilon(f, K) \leq h_\theta(f, K) \tag{7.1}$$
for $\varepsilon > 0$ sufficiently small. From Lemma 6.1.2 there exists $\kappa(\varepsilon, \theta)$ for which the inequality
\[ \rho_N(x, \tilde{x}) \geq \varepsilon \quad \text{for} \quad x, \tilde{x} \in \text{Tr}_{\pm(N+\kappa(\varepsilon, \theta))}(f, K) \]
implies that $\rho_{N+\kappa(\varepsilon, \theta)}(x, \tilde{x}) \geq \theta$ for any positive integer $N$. Now, from equality $f(K) = K$ each trajectory $x \in \text{Tr}_N(f, K)$ can be extended to some trajectory in $\text{Tr}_{\pm(N+\kappa(\varepsilon, \theta))}(f, K)$. Hence
\[ C_\varepsilon(\text{Tr}_N(f, K)) \leq C_\theta(\text{Tr}_{\pm(N+\kappa(\varepsilon, \theta))}(f, K)), \]
where $C_\varepsilon(\text{Tr}_N(f, K))$ is defined in Sect. 2.2.4 as the binary logarithm of the maximal number of elements $x^{(1)}, \ldots, x^{(p)}$ in $\text{Tr}_N(f, K)$ such that $\rho_N(x^{(i)}, x^{(j)}) \geq \varepsilon$ for all $i \neq j$. By Definition 2.2.6 the inequality (7.1) immediately follows.

A rich theory of topological entropy has been developed for hyperbolic mappings (see [90] and the references therein) and many of the results remain valid for semi-hyperbolic mappings. The following theorem is illustrative of such generalizations.

**Theorem 7.1.2.** Let $X \subseteq \mathbb{R}^d$ be an open set and $f : X \to \mathbb{R}^d$ be a Lipschitz mapping which is continuously $s$-semi-hyperbolic with constants $h, \delta$ on a compact invariant set $f(K) = K \subset X$. Then
\[ h(g, \overline{O}_{\delta/2h}(K)) \geq h(f, K) \]
for each continuous mapping $g : X \to X$ satisfying
\[ \|g - f\|_C < \frac{\delta}{2\alpha(s, h)}, \]
where $\alpha(s, h)$ is defined by (6.44).

**Proof.** Given mappings $f$ and $g$, choose $\theta$,
\[ 2\alpha(s, h)\|g - f\|_C < \theta < h^{-1}\delta. \]

By Theorem 6.1.5 the mapping $f$ is $\xi$-expansive in $K$ with expansivity constant $\xi = h^{-1}\delta$, so by Lemma 7.1.1 $h(f, K) = h_\theta(f, K)$. By Definition 2.2.6 of topological entropy, $h_\sigma(g, \overline{O}_{\delta/2h}(K)) \leq h(g, \overline{O}_{\delta/2h}(K))$ for any $\sigma > 0$. Hence it remains only to prove the middle inequality in the following
\[ h(f, K) = h_\theta(f, K) \leq h_\sigma(g, \overline{O}_{\delta/2h}(K)) \leq h(g, \overline{O}_{\delta/2h}(K)), \]
with $\sigma = \theta - 2\alpha(s, h)\|g - f\|_C > 0$. This, in turn, by Definition 2.2.6 will follow from
\[ C_\theta(\text{Tr}_N(f, K)) \leq C_\sigma(\text{Tr}_N(g, \overline{O}_{\delta/2h}(K))), \quad N > 0. \]
To prove (7.2) let \( \{x^{(1)}, \ldots, x^{(p)}\} \) be the maximal subset of elements from the set \( \text{Tr}_{\pm N}(f, K) \) satisfying
\[
\rho_N(x^{(i)}, x^{(j)}) \geq \theta, \quad i \neq j.
\]
By Theorem 6.3.12, for each such \( x^{(i)} \in \text{Tr}_{\pm N}(f, K) \) there exists a trajectory \( y^{(i)} \in \text{Tr}_{\pm N}(g, X) \) for which, by the conditions of the theorem,
\[
\rho_N(y^{(i)}, x^{(i)}) \leq \alpha(s, h)\|g - f\|_C \leq \delta/2h.
\]
Hence \( y^{(i)} \in \text{Tr}_{\pm N}(g, \partial K/2h) \) for \( i = 1, \ldots, p \), and
\[
\rho_N(y^{(i)}, y^{(j)}) \geq \rho_N(x^{(i)}, x^{(j)}) - \rho_N(x^{(i)}, y^{(i)}) - \rho_N(x^{(j)}, y^{(j)})
\]
\[
\geq \theta - 2\alpha(s, h)\|g - f\|_C = \sigma
\]
for any \( j \neq i \), and (7.2) follows.

\[
\square
\]

### 7.2 Structural Stability Properties

Let \( X \subset \mathbb{R}^d \) be also bounded. Denote by \( \Sigma(X) \) the metric space of all bi-infinite sequences \( x = \{x_n\}_{n=-\infty}^{\infty} \) with \( x_n \in X \) for \( n = 0, \pm 1, \pm 2, \ldots \) with norm
\[
\rho(x, \hat{x}) = \sum_{n=-\infty}^{\infty} 2^{-|n|}\|x_n - \hat{x}_n\|.
\]  

(7.3)

Let \( \sigma \) denote the **shift operator** on \( \Sigma(X) \) defined as
\[
(\sigma x)_n = x_{n+1}, \quad x \in \Sigma(X), \quad n \in \mathbb{Z}.
\]

For a set \( Y \subseteq X \) and a mapping \( f \in C(X, \mathbb{R}^d) \) let \( \text{Tr}_{\pm \infty}(f, Y) \subseteq \Sigma(X) \) be the totality of bi-infinite trajectories \( y = \{y_n\} \subseteq Y \).

For completeness, note the following obvious

**Lemma 7.2.1.** If the set \( Y \subseteq X \subseteq \mathbb{R}^d \) is compact and \( f \in C(X, \mathbb{R}^d) \), then the set \( \text{Tr}_{\pm \infty}(f, Y) \subseteq \Sigma(X) \) is also compact in the metric space \((\Sigma(X), \rho)\).

**Definition 7.2.2.** Let \( f_1, f_2 \in C(X, \mathbb{R}^d) \) and let \( Y_1 \) and \( Y_2 \) be closed subsets of \( X \) such that \( f_1(Y_1) = Y_1 \) and \( f_2(Y_2) = Y_2 \). The restriction \( f_1|_{Y_1} \) is said to be a weak factorization of the restriction \( f_2|_{Y_2} \) if there exists a continuous surjection
\[
\Phi : \text{Tr}_{\pm \infty}(f_2, Y_2) \to \text{Tr}_{\pm \infty}(f_2, Y_2)
\]
which is shift invariant, \( \Phi \circ \sigma = \sigma \circ \Phi \).

A further closely connected concept is that of weak conjugacy.
**Definition 7.2.3.** Let \( f_1, f_2 \in C(X, \mathbb{R}^d) \) and let \( Y_1 \) and \( Y_2 \) be closed subsets of \( X \) such that \( f_1(Y_1) = Y_1 \) and \( f_2(Y_2) = Y_2 \). The restricted mappings \( f_1|_{Y_1} \) and \( f_2|_{Y_2} \) are said to be weakly conjugate if there exists a continuous injection

\[
\Psi : \text{Tr}_{\pm\infty}(f_1, Y_1) \to \text{Tr}_{\pm\infty}(f_2, Y_2)
\]

which is shift invariant.

In the above definitions \( Y_1 \) and \( Y_2 \) are closed subsets of the bounded set \( X \subseteq \mathbb{R}^d \). Hence they are compact, and so also are the metric spaces \((\text{Tr}_{\pm\infty}(f_1, Y_1), \rho)\) and \((\text{Tr}_{\pm\infty}(f_2, Y_2), \rho)\). Therefore the injective mapping \( \Psi \) in Definition 7.2.3 is a homeomorphism. Then, the restricted mappings \( f_1|_{Y_1} \) and \( f_2|_{Y_2} \) are weakly conjugate if the restrictions of the shift operator \( \sigma \) to the sets \( \text{Tr}_{\pm\infty}(f_1, Y_1) \) and \( \text{Tr}_{\pm\infty}(f_2, Y_2) \) are topologically conjugate, that is if \( \sigma_1 = \sigma|_{\text{Tr}_{\pm\infty}(f_1, Y_1)} \) and \( \sigma_2 = \sigma|_{\text{Tr}_{\pm\infty}(f_2, Y_2)} \) are topologically conjugate.

The notion of weak factorization extends an analogue of semi-conjugacy (see Sect. 2.2.2 for definitions) to semi-hyperbolic mappings. Weak conjugacy is a generalization of topological conjugacy of mappings and reduces to it in the case of invertible mappings. The suitability of such generalizations in the analysis of noninvertible mappings is well known, see, for example [129, Sect. 15.6]. In particular, topological entropy is an invariant with respect to weak conjugacy and is nonincreasing with respect to weak factorization.

The sets \( Y_1 \) and \( Y_2 \) above are often sets of chain recurrent points. Recall the definitions from Sect. 2.2.3 in the following form.

**Definition 7.2.4.** A point \( x \in Y \subseteq X \) is called \( \varepsilon \)-chain recurrent in \( Y \) for the mapping \( f \) defined on \( X \) if there exists an \( \varepsilon \)-pseudo-cycle \( \{x_n\} \subseteq Y \) of \( f \) with \( x_0 = x \). A point \( x \in Y \subseteq X \) is called chain recurrent for \( f \) defined if it is \( \varepsilon \)-chain recurrent in \( Y \) for any sufficiently small \( \varepsilon > 0 \).

Denote the set of \( \varepsilon \)-chain recurrent points of \( f \) in \( Y \) by \( \text{CR}(f, \varepsilon, Y) \). Then \( \text{CR}(f, Y) \), the set of chain recurrent points of \( f \), is

\[
\text{CR}(f, Y) = \bigcap_{\varepsilon > 0} \text{CR}(f, \varepsilon, Y).
\]

If \( \overline{Y} \subseteq X \) where \( X \) is bounded, then \( \text{CR}(f, \overline{Y}) \) is compact and \( f \)-invariant,

\[
f(\text{CR}(f, \overline{Y})) = \text{CR}(f, \overline{Y}).
\]

**Lemma 7.2.5.** Let \( X \) and \( Y \) be open bounded subsets of \( \mathbb{R}^d \), \( f \in C(X, \mathbb{R}^d) \) and \( \text{CR}(f, \overline{Y}) \subseteq Y \subseteq \overline{Y} \subseteq X \). Then there exists a nondecreasing function \( q(\varepsilon, f) \) of \( \varepsilon \geq 0 \) with

\[
\lim_{\varepsilon \to 0} q(\varepsilon, f) = q(0, f) = 0, \quad q(\varepsilon, f) > 0 \quad \text{for} \quad \varepsilon > 0,
\]

such that

\[
\text{CR}(f, \varepsilon, \overline{Y}) \subseteq \mathcal{C}_q(\varepsilon, f)(\text{CR}(f, \overline{Y})).
\]

In particular, there exists an \( \varepsilon = \varepsilon(f, Y) > 0 \) for which \( \text{CR}(f, \varepsilon, \overline{Y}) \subseteq Y \).
Proof. Suppose the contrary. Then, for some \( \varepsilon_0 > 0 \), there exists a sequence \( x^{(k)} = \{ x_n^{(k)} \} \) of \( 1/m \)-pseudo-cycles of \( f \) with
\[
x^{(k)} \subseteq \overline{Y}, \quad x_0^{(k)} \not\in \partial_{\varepsilon_0} (\text{CR}(f, \overline{Y})). \tag{7.4}
\]
Since \( Y \) is bounded, the set \( \overline{Y} \) is compact and the sequence \( \{ x^{(k)}_0 \} \) has a limit point \( y \in \overline{Y} \). By the first inclusion (7.4) and the definition of chain recurrence, \( y \in \text{CR}(f, \overline{Y}) \). But by the second relation (7.4), \( y \not\in \partial_{\varepsilon_0} (\text{CR}(f, \overline{Y})) \). A contradiction and the lemma follows. \( \square \)

7.2.1 Weak Factorization

Let \( X \) and \( Y \) be open bounded subsets of \( \mathbb{R}^d \) with \( \overline{Y} \subseteq X \) and let Lipschitz mapping \( f \in \text{Lip}(X, \mathbb{R}^d) \) be continuously semi-hyperbolic on \( \text{CR}(f, \overline{Y}) \subset Y \). By Lemma 3.1.12 there exists an \( \eta = \eta(f, Y) > 0 \) such that \( f \) is semi-hyperbolic in \( \partial_{\eta} (\text{CR}(f, Y)) \) for some split \( s \) and constants \( h, \delta \). Denote by \( \alpha, \beta \) the corresponding constants (6.44), (6.45),
\[
\alpha = \alpha(s, h) = h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u},
\]
\[
\beta = \beta(s, h, \delta) = \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max \{ \lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u \}}.
\]

Theorem 7.2.6. Let \( Y \) be an open set, \( \overline{Y} \subseteq X \), and let \( f \in \text{Lip}(X, \mathbb{R}^d) \) be continuously \( s \)-semi-hyperbolic in \( \text{CR}(f, \overline{Y}) \subset Y \) with constants \( h, \delta \). If \( g \in C(X, \mathbb{R}^d) \) with \( \| g - f \|_C < \varepsilon, \varepsilon < \delta/(2h\alpha), \ q(\varepsilon, f) < \eta, \ \partial_{q(\varepsilon, f) + \alpha \varepsilon} (\text{CR}(f, \overline{Y})) \subset Y \). \tag{7.5}

Then the restriction \( f|_{\text{CR}(f, \overline{Y})} \) is a weak factorization of \( g|_{\text{CR}(g, \overline{Y})} \).

Proof. Consider a mapping \( g \in C(X, \mathbb{R}^d) \), \( \| g - f \|_C < \varepsilon \), where \( \varepsilon \) satisfies (7.5). Denote the ball in \( \Sigma(X) \) of radius \( \varepsilon \) centered at \( x \) by
\[
B(x, \varepsilon) = \{ y \in \Sigma(X) : \| x_i - y_i \| \leq \varepsilon, \ i \in \mathbb{Z} \}
\]
and define the mapping \( \Phi \) for \( x \in \text{Tr}_{\pm \infty}(g, \text{CR}(g, \overline{Y})) \) by
\[
\Phi(x) = B(\alpha \varepsilon, x) \cap \text{Tr}_{\pm \infty}(f, \text{CR}(f, \overline{Y})). \tag{7.6}
\]
To prove the theorem it suffices to show that \( \Phi \) is well-defined, that is

(i) the set
\[
B(\alpha \varepsilon, x) \cap \text{Tr}_{\pm \infty}(f, \text{CR}(f, \overline{Y})) \tag{7.7}
\]
is non-empty;
(ii) the set (7.7) contains exactly one point;
(iii) the mapping $\Phi$ is a surjection of the set $\text{Tr}_{\pm\infty}(g, \text{CR}(g, Y))$ onto the set $\text{Tr}_{\pm\infty}(f, \text{CR}(f, Y))$;
(iv) the mapping $\Phi$ is continuous;
(v) the mapping $\Phi$ is shift-invariant.

(i) It suffices to show that for a fixed
\[ x = \{x_n\} \in \text{Tr}_{\pm\infty}(g, \text{CR}(g, Y)), \]  
(7.8)
for each positive integer $m$ there exists a $1/m$-pseudo-cycle
\[ x^{(m)}(n) = \{x_n^{(m)}\} \subset \text{CR}(g, 1/m, Y) \]
of the mapping $g$, satisfying
\[ \|x_n^{(m)} - x_n\| < 1/m, \quad n = -m, \ldots, 0, 1, \ldots, m. \]  
(7.9)

Note that by the continuity of $g$ for a given $m$ there can be found $\bar{\delta} = \bar{\delta}_m > 0$ such that each $\bar{\delta}$-pseudo-trajectory $y = y_{-m}, \ldots, y_0, y_1, \ldots, y_m$ of $g$ satisfying $y_0 = x_{-m}$, satisfies also the estimates
\[ \sup_{-m \leq n \leq m} \|y_n - x_n\| < 1/m. \]
Then, since $x_{-m} \in \text{CR}(g, Y)$ follows from (7.8), by chain recurrence such a $\bar{\delta}$-pseudo cycle $x^{(m)} \subseteq Y$ of the mapping $g$ can be found that $\|x_{-m}^{(m)} - x_{-m}\| < \bar{\delta}$. Hence, for this $\bar{\delta}$-pseudo cycle $x^{(m)}$ the inequalities (7.9) also hold. So, the existence of $x^{(m)}$ is proved.

Now each $x^{(m)} \in \text{Tr}_{\pm\infty}(g, \text{CR}(g, 1/m, Y)), m = 1, 2, \ldots,$ is simultaneously a $(1/m + \|f - g\|_C)$-pseudo-cycle of $f$,
\[ x^{(m)} \subseteq \text{CR}(f, 1/m + \|f - g\|_C, Y). \]

Consequently, by Lemma 7.2.5,
\[ x^{(m)} \subseteq \Theta_q(1/m + \|f - g\|_C, f)(\text{CR}(f, Y)) \]
and so, by $q(\varepsilon, f) < \eta(f, Y)$ (see (7.5)) and $\|f - g\|_C < \varepsilon$,
\[ x^{(m)} \subseteq \Theta_{q(\varepsilon, f)}(\text{CR}(f, Y)) \subseteq \Theta_{\eta(f, Y)}(\text{CR}(f, Y)) \]  
(7.10)
for all sufficiently large $m$. Hence, in view of the definition of $\eta(f, Y)$, the pseudo-cycles $x^{(m)}$ belong to the region of semi-hyperbolicity of $f$ for all sufficiently large $m$. From (6.44), (6.45) it is seen that $\delta/\alpha \leq \beta$, and then due to the first inequality of (7.5) $\varepsilon \leq \beta/2h \leq \beta$ holds. Therefore, by the cyclic
bi-shadowing Theorem 6.3.12, for each \( x^{(m)} \) satisfying (7.10) for sufficiently large \( m \) there exists a cycle \( y^{(m)} \) of \( f \) such that
\[
\| x^{(m)}_n - y^{(m)}_n \| < \alpha \varepsilon \quad \text{for all} \quad m. \tag{7.11}
\]
Hence
\[
y^{(m)} \subseteq \Theta_{\eta(f,Y) + \varepsilon} (\text{CR}(f,\overline{Y})),
\]
and \( y^{(m)} \subseteq Y \) by (7.5). Moreover, since the trajectory \( y^{(m)} \) is periodic, \( y^{(m)} \in \text{Tr}_{\pm\infty}(f,\text{CR}(f,\overline{Y})) \). Therefore, by Lemma 7.2.1 the sequence \( y^{(m)} \) has a limit point \( y \in \text{Tr}_{\pm\infty}(f,\text{CR}(f,\overline{Y})) \). By (7.9) and (7.11)
\[
\| x_n - y_n \| \leq \alpha \varepsilon \quad \text{for all} \quad n
\]
which means that \( y \in B(\alpha \varepsilon, x) \).

Hence the set (7.7) is non-empty for each
\[
x \in \text{Tr}_{\pm\infty}(g,\text{CR}(g,\overline{Y})).
\]

(ii) Now the set (7.7) consists of a single point. For, note that because of the first inequality of (7.5), for any two trajectories
\[
y, \tilde{y} \in B(\alpha \varepsilon, x) \cap \text{Tr}_{\pm\infty}(f,\text{CR}(f,\overline{Y})),
\]
the estimate \( \| y_i - \tilde{y}_i \| < 2 \alpha \varepsilon \leq \delta/h \) hold. By Theorem 6.1.5, the set (7.7) then contains no more than one element, and so the mapping \( \Phi \) is well-defined.

(iii) Further, \( \Phi \) is a surjection of \( \text{Tr}_{\pm\infty}(g,\text{CR}(g,\overline{Y})) \) onto
\[
\text{Tr}_{\pm\infty}(f,\text{CR}(f,\overline{Y})).
\]
For, by definition of the mapping \( \Phi \) it is sufficient, for each
\[
y \in \text{Tr}_{\pm\infty}(f,\text{CR}(f,\overline{Y})),
\]
to construct an element
\[
x \in \text{Tr}_{\pm\infty}(g,\text{CR}(g,\overline{Y}))
\]
with \( \| x_i - y_i \| < \alpha \varepsilon \). As above, for each positive integer \( m \) there exist a \( 1/m \)-pseudo-cycle \( y^{(m)} \subseteq \text{CR}(g,1/m,\overline{Y}) \) of the mapping \( f \) satisfying
\[
\| y^{(m)}_n - y_n \| < 1/m, \quad n = -m, \ldots, 0, 1, \ldots, m.
\]
By the definition of chain recurrence, \( y \) is a limit, in the metric (7.3), of a sequence \( \{ y^{(m)} \}, m = 1, 2, \ldots \) of \( 1/m \)-pseudo-cycles of the mapping \( f \). As was mentioned above, from the first inequality of (7.5) \( \varepsilon \leq \beta \). So, by the cyclic bi-shadowing Theorem 6.3.12 for sufficiently large \( m \) there exist cycles \( x^{(m)} \) of \( g \) satisfying
\[
\| x^{(m)}_n - y^{(m)}_n \| < \alpha \varepsilon \quad \text{for all} \quad n.
\]
It remains to define $x$ as a limit point of the sequence $\{x^{(m)}\}$ in the metric space $(\Sigma(X), \rho)$ and such a limit point exists by Lemma 7.2.1.

(iv) To prove the continuity of the mapping $\Phi$, suppose the contrary. Then there exist

$$x, x^{(m)} \in \text{Tr}_{\pm\infty}(g, \text{CR}(g, \overline{Y})), \quad m = 1, 2, \ldots,$$

such that

$$\rho(x, x^{(m)}) \to 0 \quad (7.12)$$

but $\rho(\Phi(x), \Phi(x^{(m)})) \geq \bar{\varepsilon}$ for some $\bar{\varepsilon} > 0$. In this case, without loss of generality, assume that

$$\| (\Phi(x))_0 - (\Phi(x^{(m)}))_0 \| \geq \bar{\varepsilon}, \quad m = 1, 2, \ldots. \quad (7.13)$$

Now, for any positive integer $N$, let $\rho_N(z, \tilde{z})$ be the semi-norm in $\Sigma(X)$ defined by

$$\rho_N(z, \tilde{z}) = \sup_{-N \leq n \leq N} \| z_n - \tilde{z}_n \|.$$

Choose $\theta$ satisfying $2\alpha\varepsilon < \theta < \delta/h$. Such a $\theta$ exists by (7.5). Since by Theorem 6.1.5 the mapping $g$ is $\xi$-expansive with $\xi = \delta/h$ on the set $\text{CR}(f, \overline{Y})$, then by Lemma 6.1.2 there exists a positive integer $\kappa(\bar{\varepsilon}, \theta)$, independent of $m$, such that

$$\rho_{\kappa(\bar{\varepsilon}, \theta)}(\Phi(x), \Phi(x^{(m)})) \geq \theta \quad (7.14)$$

holds whenever (7.13) is valid. But from the definition of the mapping $\Phi$ it follows that $\Phi(x) \in B(\alpha\varepsilon, x)$, and so $\rho_{\kappa(\bar{\varepsilon}, \theta)}(\Phi(x), x) \leq \alpha\varepsilon$. Hence

$$\rho_{\kappa(\bar{\varepsilon}, \theta)}(\Phi(x), \Phi(x^{(m)})) \leq \rho_{\kappa(\bar{\varepsilon}, \theta)}(\Phi(x), x) + \rho_{\kappa(\bar{\varepsilon}, \theta)}(x, x^{(m)}) + \rho_{\kappa(\bar{\varepsilon}, \theta)}(x^{(m)}, \Phi(x^{(m)})) \leq \alpha\varepsilon + \rho_{\kappa(\bar{\varepsilon}, \theta)}(x, x^{(m)}).$$

Here $2\alpha\varepsilon < \theta$ by the definition of $\theta$ and $\rho_{\kappa(\bar{\varepsilon}, \theta)}(x, x^{(m)}) \to 0$ in view of (7.12). Therefore, $\rho_{\kappa(\bar{\varepsilon}, \theta)}(\Phi(x), \Phi(x^{(m)})) < \theta$ for sufficiently large $m$, which contradicts (7.14) and so the mapping $\Phi$ is continuous.

(v) The shift invariance identity $\Phi \circ \sigma \equiv \sigma \circ \Phi$ follows directly from the definition of the mapping $\Phi$.

Theorem 7.2.6 is proved. \qed

7.2.2 Weak Conjugacy

Theorem 7.2.6 above is a weak form of semi-conjugacy for semi-hyperbolic mappings, while Theorem 7.2.7 below is a version of structural stability for such mappings. The explicit estimates of the radii of semi-conjugacy and structural stability in these theorems are particularly useful in applications. Note
that semi-conjugacy in Theorem 7.2.6 is a $C^0$-robust property, while conjugacy in Theorem 7.2.7 below is Lipschitz-robust.

Again suppose $X$ and $Y$ to be open bounded subsets of $\mathbb{R}^d$ with $\overline{Y} \subseteq X$ and let $f \in \text{Lip}(X, \mathbb{R}^d)$ be a continuously semi-hyperbolic Lipschitz mapping on $\text{CR}(f, \overline{Y})$ with $\text{CR}(f, \overline{Y}) \subseteq Y$. Let $\varepsilon(f, Y) > 0$ as defined in Lemma 7.2.5 such that $\text{CR}(f, \varepsilon, \overline{Y}) \subseteq Y$ for $\varepsilon \leq \varepsilon(f, Y)$. Then by Lemma 3.1.12 there exists a $\eta = \eta(f, Y) < \varepsilon(f, Y)$ such that every mapping $g \in \text{Lip}(X, \mathbb{R}^d)$ with $\|g - f\|_{\text{lip}} \leq \eta$ is semi-hyperbolic in $\mathcal{O}_\eta(\text{CR}(f, \overline{Y}))$.

**Theorem 7.2.7.** Let $Y$ be an open set with $\overline{Y} \subseteq X$ and let $f \in \text{Lip}(X, \mathbb{R}^d)$ be a Lipschitz mapping continuously $s$-semi-hyperbolic in $\text{CR}(f, \overline{Y}) \subseteq Y$ with constants $h, \delta$. If $g \in \text{Lip}(X, \mathbb{R}^d)$ with $\|g - f\|_{\text{lip}} < \eta(f, Y)$ then the restrictions $f|_{\text{CR}(f, \overline{Y})}$ and $g|_{\text{CR}(g, \overline{Y})}$ are weakly conjugate.

**Proof.** Consider $g \in \text{Lip}(X, \mathbb{R}^d)$ satisfying $\|g - f\|_{\text{lip}} < \eta(f, Y)$. Introduce the homotopy mapping $g_\lambda(x) = \lambda g(x) + (1 - \lambda)f(x)$, $0 \leq \lambda \leq 1$, $x \in X$. Clearly,

$$\|g_\lambda - f\|_{\text{lip}} \leq \|g - f\|_{\text{lip}} < \eta(f, Y). \quad (7.15)$$

To prove the theorem it suffices, by transitivity of the weak conjugacy relation, to find a finite set of parameters $\lambda_0 = 0, \lambda_1, \ldots, \lambda_k = 1$ such that the mappings $g_{\lambda_i}$ and $g_{\lambda_{i+1}}$ are weak conjugate for each $i = 0, 1, \ldots, k - 1$. Since the family $\{g_\lambda : 0 \leq \lambda \leq 1\}$ is a compact subset of $\text{Lip}(X, \mathbb{R}^d)$, the existence of parameters $\{\lambda_i\}$ follows from the result below. \hfill \Box

**Lemma 7.2.8.** There exists $\zeta > 0$ such that $g_\lambda|_{\text{CR}(g_\lambda, \overline{Y})}$ is weakly conjugate to $g_{\bar{\lambda}}|_{\text{CR}(g_{\bar{\lambda}}, \overline{Y})}$ for all $\lambda, \bar{\lambda} \in [0, 1]$ with $|\lambda - \bar{\lambda}| < \zeta$.

**Proof.** By (7.15) and the definition of $\eta(f, Y)$, for any $\lambda \in [0, 1]$ the mapping $g_\lambda$ is continuously semi-hyperbolic in $\text{CR}(f, \eta(f, Y), \overline{Y})$. Since $\eta(f, Y) < \varepsilon(f, Y)$, then by Lemma 7.2.5 $\text{CR}(f, \eta(f, Y), \overline{Y}) \subseteq Y$. Again, from (7.15) it follows that

$$\|g_\lambda - f\|_C \leq \|g - f\|_C < \eta(f, Y),$$

and so any chain recurrent point of $g_\lambda$ belonging to $\overline{Y}$ is an $\eta(f, Y)$-chain recurrent point of $f$. Hence, by definition of the value $\eta(f, Y)$,

$$\text{CR}(g_\lambda, \overline{Y}) \subseteq \text{CR}(f, \eta(f, Y), \overline{Y}) \subseteq Y, \quad 0 \leq \lambda \leq 1. \quad (7.16)$$

Then by Lemmata 3.1.12 and 7.2.5 for a given $\bar{\lambda}$ there exist $\eta > 0$ and $\zeta_1 > 0$ such that all mappings $g_\lambda(x)$ with $|\lambda - \bar{\lambda}| < \zeta_1$ are uniformly semi-hyperbolic in $\mathcal{O}_\eta(\text{CR}(g_{\bar{\lambda}}, \overline{Y}))$ with the same split $s$ and constants $h, \delta$.

By (7.16) $\text{CR}(g_{\bar{\lambda}}, \overline{Y}) \subseteq Y$ and Theorem 7.2.6 there exists a positive $\varepsilon$ satisfying

$$\varepsilon < \delta/(2h\alpha) \quad (7.17)$$

such that for each $g_\lambda$ with $\|g_\lambda - g_{\bar{\lambda}}\|_C < \varepsilon$ the following property is true:
(i) \( g_\lambda|_{\text{CR}(g_\lambda, Y)} \) is a weak factorization of \( g_\lambda|_{\text{CR}(g_\lambda, Y)} \) with the corresponding mapping

\[
\Phi_\lambda(x) = B(\alpha \varepsilon, x) \cap \text{Tr}_{\pm \infty}(g_\lambda, \text{CR}(g_\lambda, Y)),
\]

for \( x \in \text{Tr}_{\pm \infty}(g, \text{CR}(g, Y)) \).

On the other hand, take \( \varepsilon > 0 \) satisfying

\[
q(\varepsilon, g_\lambda) < \eta, \tag{7.18}
\]

where the function \( q(\varepsilon, f) \) is defined as in Lemma 7.2.5. Then for \( g_\lambda, g_\lambda^- \) with

\[
\|g_\lambda - g_\lambda^-\|_C < \varepsilon
\]

by Lemma 7.2.5

\[
\text{CR}(g_\lambda, Y) \subseteq \text{CR}(g_\lambda, \delta, Y) \subseteq \text{CR}(g_\lambda^-, \varepsilon, Y) \subseteq O_\eta(\text{CR}(g_\lambda, Y)),
\]

with any \( \delta < \varepsilon - \|g_\lambda - g_\lambda^-\|_C \). Thus,

(ii) \( \text{CR}(g_\lambda, Y) \subseteq O_\eta(\text{CR}(g_\lambda, Y)) \).

Choose \( \zeta > 0 \) and \( \lambda \in [0, 1] \) such that \( \zeta < \zeta_1 \) and \( |\lambda - \bar{\lambda}| < \zeta \) imply that \( \|g_\lambda - g_\lambda^-\| < \varepsilon \). From property (i) above, \( g_\lambda|_{\text{CR}(g_\lambda, Y)} \) is a weak factorization of \( g_\lambda|_{\text{CR}(g_\lambda, Y)} \). It remains to prove that \( \Phi_\lambda \) is an injection and that the inverse mapping \( \Phi_\lambda^{-1} \) is continuous.

To prove that \( \Phi_\lambda \) is injective choose arbitrary but fixed

\[
x, \tilde{x} \in \text{Tr}_{\pm \infty}(g_\lambda, \text{CR}(g_\lambda, Y))
\]

such that \( x \neq \tilde{x} \). By property (ii) and Theorem 6.1.5 each \( g_\lambda \) is \( \delta/h \)-expansive and so

\[
\sup_{i \in \mathbb{Z}} \|x_i - \tilde{x}_i\| \geq \delta/h. \tag{7.19}
\]

On the other hand, by property (i) and the inequality \( |\lambda - \bar{\lambda}| < \zeta \), the mapping \( \Phi_\lambda \) satisfies

\[
\sup_i \|\Phi_\lambda(x)_i - x_i\|, \sup_i \|\Phi_\lambda(\tilde{x})_i - \tilde{x}_i\| < \alpha \varepsilon
\]

from which, by (7.17),

\[
\sup_i \|\Phi_\lambda(x)_i - x_i\|, \sup_i \|\Phi_\lambda(\tilde{x})_i - \tilde{x}_i\| < \delta/2h. \tag{7.20}
\]

The latter inequality together with (7.19) implies that

\[
\sup_i \|\Phi_\lambda(x)_i - \Phi_\lambda(\tilde{x})_i\|
\geq \sup_i \|x_i - \tilde{x}_i\| - \sup_i \|\Phi_\lambda(x)_i - x_i\| - \sup_i \|\Phi_\lambda(\tilde{x})_i - \tilde{x}_i\| > 0,
\]
and the mapping $\Phi_\lambda$ is an injection.

Finally, to show continuity of the mapping $\Phi_\lambda^{-1}(x)$ in the metric (7.3) is similar to the proof of continuity of the mapping $\Phi(x)$ in Theorem 7.2.6. Suppose the contrary. Then there exists a sequence

$$y^{(m)} \in \text{Tr}_{\pm\infty}(g_\lambda, \text{CR}(g_\lambda, Y)),$$

converging in the metric (7.3) to some

$$y \in \text{Tr}_{\pm\infty}(g_\lambda, \text{CR}(g_\lambda, Y))$$

such that $x^{(m)} = \Phi_\lambda^{-1}(y^{(m)})$ does not converge to $x = \Phi_\lambda^{-1}(y)$. Then without loss of generality suppose that $\|x^{(m)}_0 - x_0\| \geq \eta$ for some positive $\eta$. Choose $\theta > 0$ such that $0 < 2\alpha \varepsilon < \theta < \delta/h$. Such a constant $\theta$ exists by (7.17). Then by Theorem 6.1.5 and Lemma 6.1.2 there exists a positive integer $\kappa(\eta, \theta)$ satisfying

$$\rho_{\kappa(\eta, \theta)}(x^{(m)}, x) = \max_{-\kappa(\eta, \theta) \leq i \leq \kappa(\eta, \theta)} \|x^{(m)}_i - x_i\| \geq \theta > 2\alpha \varepsilon.$$

From (7.19), (7.20) and this last inequality it follows that

$$\rho_{\kappa(\eta, \theta)}(y^{(m)}, y) \geq \rho_{\kappa(\eta, \theta)}(x^{(m)}, x) - \rho_{\kappa(\eta, \theta)}(y^{(m)}, x^{(m)}) - \rho_{\kappa(\eta, \theta)}(y, x) \geq \theta - 2\alpha \varepsilon > 0.$$

This contradicts

$$\lim_{m \to \infty} \rho(y^{(m)}, y) = 0,$$

and thus $\Phi_\lambda^{-1}(x)$ is continuous. The proofs of Lemma 7.2.8 and Theorem 7.2.7 are complete.

### 7.3 Chaos

#### 7.3.1 Definition of Chaotic Behavior

Let $X$ be an open bounded subset of $\mathbb{R}^d$. Consider a mapping $f : X \to X$ and the corresponding discrete-time dynamical system generated by $f$. A trajectory of this system is a sequence $x = \{x_n\}_{n=-n_-}^{\infty}$ satisfying

$$x_{n+1} = f(x_n), \quad n = -n_-, \ldots, 0, 1, 2, \ldots,$$

where $0 \leq n_- \leq \infty$, and $n_- = n_-(x)$ may depend on the particular trajectory $x$. Let $\text{Tr}(f)$ denote the totality of trajectories of the dynamical system generated by $f$ and let $\text{Tr}_\infty(f)$ be the subset of $x \in \text{Tr}(f)$ with $n_-(x) = \infty$. 
Important attributes of chaotic behavior include sensitive dependence on initial conditions, an abundance of unstable periodic trajectories and an irregular mixing effect characterized by the existence of a finite number of separated subsets $U_1, \ldots, U_m$ of $\mathbb{R}^d$ which can be visited by trajectories of some fixed iterate $f^k$ of $f$ in any prescribed order. Symbolic dynamics gives a more formal description of this last property. Let $\Sigma(m)$ denote the totality of all bi-infinite sequences $b = \{b_n\}_{n=\pm\infty}$ with $b_n \in \{1, \ldots, m\}$ for $n = 0, \pm 1, \pm 2, \ldots,$ and let

$$W = \{w_1, \ldots, w_m\}$$

be an unordered subset of distinct points in $X$. Sequences in $\Sigma(m)$ will be used to prescribe the order in which some disjoint balls of the form

$$U_i = \{z \in X : \|z - w_i\| < \varepsilon\}, \quad i = 1, \ldots, m,$$

are visited by trajectories.

Let $\varepsilon > 0$, $k \in \mathbb{N}$ and let $Y$ be a compact subset of $X$ for which

$$\max_{x, y \in Y} \|x - y\| \geq 2\varepsilon.$$

**Definition 7.3.1.** A continuous mapping $f$ is $(\varepsilon, k)$-chaotic in a neighborhood of $Y$ if for each finite subset $W = \{w_1, \ldots, w_m\}$ of $Y$ with

$$\min_{i \neq j} \|w_i - w_j\| \geq 2\varepsilon$$

there exists a mapping $Z_f : \Sigma(m) \to \text{Tr}_\infty(f)$ such that

S1: For each $b \in \Sigma(m)$ the trajectory $z = Z_f(b)$ of $f$ satisfies $z_{kj} \in U_{bj}$ for all integers $j$;

S2: The mapping $b \mapsto Z_f(b)$ is shift invariant in the sense that a 1-shift $\sigma$ of a sequence $b \in \Sigma(m)$ induces a $k$-shift $\sigma^k$ of the corresponding trajectory $Z_f(b)$;

S3: If a sequence $b \in \Sigma(m)$ is periodic with minimal period $p$, then the corresponding trajectory $z = Z_f(b)$ is periodic with minimal period $kp$;

S4: For each $\eta > 0$ there exists an uncountable subset $\Sigma_0(\eta)$ of $\Sigma(m)$ such that

$$\limsup_{n \to \infty} \|Z_f(a)_n - Z_f(b)_n\| \geq \frac{1}{2}\varepsilon$$

for all $a, b \in \Sigma_0(\eta)$, $a \neq b$, and

$$\liminf_{n \to \infty} \|Z_f(a)_n - Z_f(b)_n\| < \eta$$

for all $a, b \in \Sigma_0(\eta)$. 
Note that if \( f \) is \((\varepsilon,k)\)-chaotic on \( Y \) then the topological entropy \( \mathcal{E}^\text{top} \) of \( f \) in the \( \varepsilon \)-neighborhood of \( Y \) is positive and satisfies the inequality

\[
\mathcal{E}^\text{top} \geq k(\varepsilon)^{-1} \log C_{\varepsilon/4}(Y),
\]

where \( C_{\varepsilon}(Y) \) denotes \( \varepsilon \)-capacity of the compact set \( Y \) [47,65].

The above defining properties of chaotic behavior are similar to those in the Smale transverse homoclinic trajectory theorem [130, Theorem 16.2], with an important difference being that the existence of an invariant Cantor set is not required. In addition neither the uniqueness of the trajectory \( Z_f(b) \) for \( b \in \Sigma(m) \) nor the continuity of \( Z_f \) is assumed, so \( Z_f \) need not be a semi-conjugacy. Instead, the definition includes Condition S3, which is a corollary of uniqueness, and Condition S4 which is a form of sensitivity and irregular mixing as in the Li–Yorke definition of chaos [84], with the subset of trajectories \( Z_f(\Sigma_0(\eta)) \) corresponding to the Li–Yorke scrambled subset \( S_0 \) and the whole set \( Z_f(\Sigma(m)) \) to their whole set \( S \).

From a physical point of view the definition means that the trajectories of the system \( f \) appear to behave chaotically if the measurements are carried out with precision no better than \( \varepsilon \) at equal time intervals between subsequent measurements no shorter than \( k(\varepsilon) \).

**Definition 7.3.2.** The minimal \( \varepsilon_0 \geq 0 \), with the property that for each \( \varepsilon > \varepsilon_0 \) the system \( f \) is \((\varepsilon,k)\)-chaotic for an appropriate \( k \), is called the chaos threshold of the system \( f \).

The chaos threshold characterizes the accuracy of measurements for which the behavior of the system in the vicinity of the subset \( Y \) appears completely chaotic if the time lapse between subsequent measurements is sufficiently large. For instance, a chaotic diffeomorphism \( f \) which is topologically conjugate on \( Y \) to the shift operator \( \sigma \) on \( \Sigma(2) \) has zero chaos threshold.

### 7.3.2 Perturbations of Bi-Shadowing Systems

A bi-infinite trajectory \( \mathbf{x} = \{x_n\}_{-\infty}^{\infty} \) of a continuous bounded mapping \( f : X \to X \subset \mathbb{R}^d \) is called a homoclinic trajectory if its elements are not all identical and there exists a point \( x_* \in X \) such that \( \lim_{n \to \infty} x_{-n} = \lim_{n \to \infty} x_n = x_* \).

**Theorem 7.3.3.** Let \( \mathbf{x} \) be a homoclinic trajectory of a continuous bounded mapping \( f : X \to X \), \( f \) is both bi-shadowing and cyclically bi-shadowing on the unordered set \( \{\mathbf{x}\} \) with parameters \( \alpha \) and \( \beta \). Define

\[
\gamma(\varepsilon) = \frac{1}{2} \min \{\beta, \varepsilon/\alpha\} > 0.
\]

Then every mapping \( g \in C \) satisfying \( \|g - f\|_C < \gamma(\varepsilon) \) is \((\varepsilon,k)\)-chaotic on a neighborhood of \( \{\mathbf{x}\} \) for any positive integer \( k \geq k(\varepsilon) \), where

\[
k(\varepsilon) = \max \{m : \exists \text{ an integer } j_0 : \|x_j - x_*\| \geq \gamma(\varepsilon), \, j = j_0, \ldots, j_0 + m\}. \quad (7.21)
\]
Suppose that the strict inclusion $D$ bound, the graph of which is defined as the union $\bigcup\in\mathcal{D}$.

Proof. In the proof below there will be fixed $\varepsilon \leq \max_{x,y\in\mathcal{X}} \|x-y\|$, a positive integer $k > k(\varepsilon)$ and a finite set $\mathcal{W} = \{w_1, \ldots, w_m\}$, $m > 1$, satisfying

$$\min_{i\neq j} \|w_i - w_j\| \geq 2\varepsilon.$$ 

To prove the theorem it suffices to construct a mapping $Z_f$ which satisfies Conditions S1–S4. Let $\mathcal{O}(S)$ denote the open $\rho$-neighborhood of a subset $S$ of $\mathbb{R}^d$.

For each $w \in \mathcal{W}$ with $w \neq x_*$ there exist a positive integer $m_-(w)$, a non-negative integer $m_+(w)$ and a finite sequence

$$u = u(w) = \{u(w)_{-m_-(w)}, u(w)_{-m_-(w)+1}, \ldots, u(w)_{m_+(w)-1}\},$$

uniquely defined by

$$u(w)_0 = w, \quad u(w)_j = f(u(w)_{j-1}), \quad j = -m_-(w) + 1, \ldots, m_+(w) - 1$$

such that

$$u(w)_{-m_-(w)}, f(u(w)_{m_+(w)-1}) \in \mathcal{O}(\gamma(\varepsilon))(\{x_*\}), \quad u(w)_j \notin \mathcal{O}(\gamma(\varepsilon))(\{x_*\})$$

for $-m_-(w) < j < m_+(w)$.

Consider a given integer $k > k(\varepsilon)$ and a given sequence $b \in \Sigma(m)$. Define a sequence $v = \{v_j\}$ in $X$ by

$$v_{j+kn} = u(w_n)_j \quad \text{for} \quad -m_-(w_n) \leq j < m_+(w_n)$$

when $w_n \neq x_*$ and by

$$v_j = x_*$$

for all other $j$. This sequence is a $\gamma(\varepsilon)$-pseudo-trajectory of $f$. Hence, by bi-shadowing of $f$, for any mapping $g \in C$ with $\|g-f\|_C < \gamma(\varepsilon)$ the set $\mathcal{Z}_g(b)$ of all trajectories $z$ satisfying $\|z_{kn} - wb_n\| < \varepsilon$ for all $n$ is nonempty. Furthermore, by cyclically bi-shadowing of $f$, this set $\mathcal{Z}_g(b)$ contains a trajectory of minimal period $pk$ if $b$ is periodic with minimal period $p$.

Using standard constructions and Zorn’s lemma [66, pp. 31–36], a single-valued selector $Z_g$ of the multi-valued mapping $b \to \mathcal{Z}_g(b)$ can be chosen which satisfies Conditions S1–S3 in Definition 7.3.1. To see this, denote by $Z$ the totality of single-valued selectors $Z_g$ which are defined on subsets of $\mathcal{D}(Z) \subset \Sigma(m)$ and satisfy Conditions S1–S3 and consider this set as being partially ordered by inclusion of the corresponding graphs

$$\text{Gr}(Z_g) = \{(b, Z_g(b)) : b \in \mathcal{D}(Z)\}.$$ 

By construction every totally ordered subset, or chain, of $\hat{Z}$ of $Z$ has an upper bound, the graph of which is defined as the union $\bigcup_{Z_g \in \hat{Z}} \text{Gr}(Z_g)$.

Hence by Zorn’s lemma there exists a maximal element $Z_*$ in the set $Z$. Suppose that the strict inclusion $\mathcal{D}(Z_*) \subset \Sigma(m)$ holds. Then there exist an
element $b_* \in \Sigma(m) \setminus \mathcal{D}(Z_*)$. If for some positive integer $i$ the sequence $b_*$ is the $i$-th shift of a sequence $b_0 \in \mathcal{D}(Z_*)$ then the mapping

$$Z_0(b) = \begin{cases} 
Z_*(b), & b \in \mathcal{D}(Z_*), \\
\sigma^{-ik}Z_*(b_0), & b = b_*
\end{cases}$$

satisfies Conditions S1–S3 and strictly dominates $Z_*$, which contradicts the definition of $Z_*$. On the other hand, if the sequence $b_*$ cannot be represented as a shift of a sequence $b \in \mathcal{D}(Z_*)$ then define $Z_0(b)$ as an arbitrary element from the nonempty set $\mathcal{Z}_g(b)$ of all trajectories $z$ satisfying $\|z_{kn} - w_{b_n}\| < \varepsilon$ for all $n$; again the mapping $Z_0$ satisfies Conditions S1–S3 and strictly dominates $Z_*$, again a contradiction.

It remains to prove that the selected mapping $Z_g$ also satisfies Condition S4. This follows immediately from the next more general result.

**Lemma 7.3.4.** Let $(\Omega, d)$ be an uncountable compact metric space and let $\mathcal{I}$ be the set of sequences $s = \{s_j\}_{j \geq 1}$ with $s_j \in \Omega$ for all $j \geq 1$. Then for each $\eta > 0$ there exists an uncountable subset $\mathcal{I}(\eta)$ of $\mathcal{I}$ such that

$$\liminf_{j \to \infty} d(s_j, t_j) < \eta$$

for any $s, t \in \mathcal{I}(\eta)$.

**Proof.** By compactness of $(\Omega, d)$ there exists a finite partition $\mathcal{P}$ of $\Omega$ such that $\text{diam}(A) < \eta$ for each $A \in \mathcal{P}$. Consider the equivalence relation $E_\eta$ on $\mathcal{I}$ defined by: $E_\eta(s, t)$ if and only if $s_j$ and $t_j$ belong to the same subset from the partition for all sufficiently large $j$. Denote the set of equivalence classes by $\mathcal{E}$ and note that each equivalence class $\mathcal{E} \in \mathcal{I}$ contains no more than a denumerable set of elements, because each element from $\mathcal{E}$ differs from a chosen element $\{t_j\}_{j \geq 1}$ only for a finite number of indices $j$. On the other hand, the set $\mathcal{I}$ itself is uncountable by construction and so also is $\mathcal{E}$.

Choose a single element from each equivalence class in $\mathcal{E}$, denote the set of these sequences by $\mathcal{I}_*$ and say that two sequences $s, t \in \mathcal{I}_*$ are connected if there exist arbitrarily large $j$ for which $s_j$ and $t_j$ belong to the same subset from the partition. Since there are connected elements in every set which contains more than $\#(\mathcal{P})$ elements, there are connected elements in every denumerable set. The assertion of the Lemma will hold if an uncountable subset $\mathcal{I}(\eta) \subseteq \mathcal{I}$ of pairwise connected sequences can be constructed. That this can be done follows by an application of a transfinite analogue of the Ramsey Complete Graph Theorem ([52, p. 608, Theorem 5.23]; see also [53, pp. 427–428]): If $G$ is a graph of power $m$, where $m$ is a transfinite cardinal, and if every denumerable subset of $G$ contains two connected elements, then $G$ contains a complete graph of power $m$.

This completes the proof of Lemma 7.3.4 and hence the proof of Theorem 7.3.3. \qed
The positive integer $k(\varepsilon)$ defined by (7.21) represents the maximum number of iterations of an element of the unordered set $\{x\}$ that can remain outside of the $\gamma(\varepsilon)$ neighborhood of the homoclinic point $x_*$. 

Corollary 7.3.5. The chaos threshold of each continuous mapping $g$ satisfying $\|g - f\| < \gamma(\varepsilon)$ does not exceed $\varepsilon$.

The next theorem provides a simple means of locating homoclinic trajectories. Recall that a mapping $f : X \to X$ is said to be $\xi$-expansive in $X$ if for any infinite trajectories $x, y \in \text{Tr}(f)$ either $x = y$ or $\sup_{n \in \mathbb{Z}} \|x_n - y_n\| \geq \xi$.

Theorem 7.3.6. Let $w = \{w_0, \ldots, w_{p-1}\}$ be a $\gamma$-pseudo-cycle of a continuous mapping $f : \mathcal{X} \to \mathcal{X}$ which is bi-shadowing and cyclically bi-shadowing with parameters $\alpha, \beta$ on $\{w\}$ and $\xi$-expansive in $\mathcal{X}$. Suppose that $\gamma \leq \beta$, $\|f(w_0) - w_0\| \leq \beta$ and

$$2\alpha \gamma < \max_{i,j} \|w_i - w_j\|, \quad \alpha(\gamma + \|f(w_0) - w_0\|) < \xi. \quad (7.22)$$

Then $f$ has a homoclinic trajectory $x$ in an open $\alpha \gamma$ neighborhood of $\{w\}$.

Proof. The point $w_0$ is clearly an ($\|f(w_0) - w_0\|$)-pseudo-equilibrium of $f$. By the assumption that $\|f(w_0) - w_0\| \leq \beta$ and the cyclically bi-shadowing property there exists a proper equilibrium $x_*$ of $f$ satisfying

$$\|x_* - w_0\| \leq \alpha \|f(w_0) - w_0\|. \quad (7.23)$$

Now consider the bi-infinite sequence $\{y_j\}$ defined by

$$y_j = \begin{cases} w_0, & j < 0 \text{ or } j \geq p, \\ w_j, & \text{otherwise}, \end{cases}$$

which is obviously a $\gamma$-pseudo-trajectory of $f$. Since $\gamma \leq \beta$, there exists a proper trajectory $x = \{x_j\}$ in the $(\alpha \gamma)$-neighborhood of the pseudo-trajectory $\{y_j\}$. The elements of this trajectory are not all identical because of the first inequality in (7.22). To complete the proof it remains to show that the trajectory $x$ is homoclinic. That is,

$$\lim_{n \to \infty} x_{-n} = \lim_{n \to \infty} x_n = x_. \quad (7.24)$$

Suppose that

$$\lim_{n \to \infty} x_n = x_. \quad (7.25)$$

does not hold. Then there exists a sequence of indices $j_m \to \infty$ and an $\varepsilon_1 > 0$ such that

$$\|x_{j_m} - x_*\| > \varepsilon_1, \quad m = 1, 2, \ldots. \quad (7.26)$$

Consider a coordinate-wise limit point $x^* = \{x^*_n\}_{n > -\infty}$ of the sequence of shifted trajectories.
7.3 Chaos

\[ x^{(m)} = \left\{ x_{-j_m}^{(m)}, x_{-j_m+1}^{(m)}, \ldots \right\} \]
defined by \( x_{n-j_m}^{(m)} = x_n \) for \( n = 0, 1, 2, \ldots \). Then (7.26) implies

\[ \|x_0^* - x_*\| > \varepsilon_1. \] (7.27)

Now every sequence \( x^{(m)} \) is a trajectory of \( f \), so \( x^* \) is also a trajectory of \( f \) because \( f \) is continuous. Furthermore, \( x^* \) satisfies the inequalities

\[ \|x_n^* - x_*\| < \xi \] (7.28)

for all \( n \) because of (7.23) and the second inequality in (7.22). The inequalities (7.28) and (7.27) contradict the \( \xi \)-expansivity property, so the limit (7.25) must exist. The other limit follows similarly, which completes the proof. \( \square \)

Note that only the direct shadowing part of bi-shadowing has been used in the above proof.

7.3.3 Semi-Hyperbolic Lipschitz Mappings

From Theorems 7.3.3 and 6.3.12 we have the following corollary.

**Corollary 7.3.7.** Let \( f \) be \((s, h, \delta)\)-semi-hyperbolic on a compact subset \( Y \) of \( X \) and \( x \) be a homoclinic trajectory of \( f \) contained in \( Y \) and define \( k(\varepsilon) \) by (7.21) and \( \gamma(\varepsilon) \) by

\[ \gamma(\varepsilon) = \frac{1}{3} \min \{ \beta(s, h, \delta), \varepsilon/\alpha(s, h) \}. \] (7.29)

Then every mapping \( g \in C \) satisfying \( \|g - f\|_C < \gamma(\varepsilon) \) is \((\varepsilon, k)\)-chaotic on a neighborhood of \( \{x\} \) for any positive integer \( k \geq k(\varepsilon) \).

From this corollary the chaotic behavior of nonsmooth perturbations of a diffeomorphism with a hyperbolic homoclinic trajectory can be established, see \[7,65\]. In fact Theorems 7.3.3 and 6.3.12 can be used to consider homoclinic trajectories for much wider classes of systems, such as those with a Marotto snap-back repeller or generalizations thereof \[89,131,141\].

**Example 7.3.8.** Let 0 be a hyperbolic fixed point of a smooth mapping \( f \) on \( \mathbb{R}^d \) and let \( W^s \) and \( W^u \) denote the local stable and unstable manifolds of \( f \) at 0. Suppose that there is a point \( x_0 \in W^u \setminus \{0\} \) and a positive integer \( m \) with \( f^m(x_0) = 0 \) and that the linear space \( D_{x_0}T_{x_0}^u \) is transversal to \( T_0^s \), where \( D_x \) is the derivative of \( f^m \) at the point \( x \) and \( T_x^u \) and \( T_x^s \) are the tangent spaces to the stable and unstable manifolds. Since \( x_0 \in W^u \setminus \{0\} \), there exist points \( x_{-n}, n > 0, \) with \( \lim_{n \to \infty} x_{-n} = 0 \) and \( f(x_{-n}) = x_{-n+1} \). Let \( N \) be an arbitrary positive integer, write

\[ r(N) = \frac{\|x_{-N}\| + \|x_{-N+1}\|}{2} \quad \text{and} \quad U(N, \varepsilon) = \mathcal{O}_{r(N)}(0) \cup \mathcal{O}_{\varepsilon}(x_{-N}) \]
for arbitrary $\varepsilon > 0$. It can be shown that for suitable $\varepsilon > 0$ and integers $N, M > 0$ the mapping

$$F_{N, M, \varepsilon} = \begin{cases} f(x), & x \in O_r(N)(0), \\ f^{N+M}(x), & x \in O_\varepsilon(x-N) \end{cases}$$

is semi-hyperbolic on any compact subset of $U(N, \varepsilon) [47]$. Hence the mapping $f$ itself is bi-shadowing and cyclically bi-shadowing on its homoclinic trajectory

$$x = \{\ldots, x_{-n}, \ldots, x_{-1}, x_0, f(x_0), \ldots, f^{m-1}(x_0), 0, \ldots, 0, \ldots \}.$$ 

Consequently, for every $\varepsilon > 0$ there exists a $\gamma > 0$ such that the chaos threshold of every small continuous perturbation $g$ with $\|g - f\|_C < \gamma$ does not exceed $\varepsilon$.

### 7.3.4 Perturbations on Sets of Chain Recurrent Points

Another mechanism for generating robust chaotic behavior in nonsmooth dynamical systems involves its set of chain recurrent points.

Let $X$ be an open bounded subset of $\mathbb{R}^d$, let $Y$ be an open subset of $X$, and let $f : X \to X$. A point $x \in Y$ is called $(\varepsilon, Y)$-chain recurrent for $f$ if there exists a finite $\varepsilon$-pseudo-trajectory $\{x_0, x_1, \ldots, x_L\}$ of $f$ in $Y$ with $x_0 = x_L = x$, that is connecting $x$ with itself. Let $CR(f, \varepsilon, Y)$ denote the set of all $(\varepsilon, Y)$-chain recurrent points of $f$. The set of $Y$-chain recurrent points of $f$ is then defined as

$$CR(f, Y) = \cap_{\varepsilon > 0} CR(f, \varepsilon, Y).$$

Note that if $f$ is continuous and $Y \subseteq X$, then $CR(f, Y)$ is compact and

$$f(CR(f, Y)) = CR(f, Y).$$

The following lemma, which is analogous to Smale’s Spectral Decomposition Theorem for Axiom A diffeomorphisms [65], is an important technical step in formulating and proving the main result of this section.

**Lemma 7.3.9.** Let $Y$ be an open set with $Y \subseteq X$ and let $f \in \text{Lip}(X, \mathbb{R}^d)$ be bi-shadowing and expansive on $CR(f, Y)$ with $CR(f, Y) \subseteq Y$. Then

(i) the set $CR(f, Y)$ can be decomposed into a disjoint union of a finite number of closed sets $\Omega_1, \ldots, \Omega_K$ such that $f(\Omega_i) = \Omega_i$ for $i = 1, \ldots, K$. Moreover, for each $i = 1, \ldots, K$ there exists a bi-infinite trajectory $z = \{z_n\}_{n=-\infty}^{n=\infty}$ of $f$ belonging to $\Omega_i$ such that each of the two semi-trajectories $z^+ = \{z_n\}_{n=0}^{\infty}$ and $z^- = \{z_n\}_{n=-\infty}^{0}$ are dense in $\Omega_i$;

(ii) each set $\Omega_i$ can itself be decomposed into a disjoint union of a finite number of closed sets $\Omega_i^j$, $j = 1, \ldots, n_i$, which are cyclically permuted under $f$, such that $f^{n_i}|_{\Omega_i^j}$ is topologically mixing. That is, for any disjoint $U, V$ relatively open in $\Omega_i^1$ there exists an $N_i,U,V$ such that $(f^{n_i})^n(U) \cap V \neq \emptyset$ for every $n \geq N_i,U,V$. 

(iii) the periodic points of \( f_{\mid \text{CR}(f,Y)} \) are dense in \( \text{CR}(f,Y) \).

Proof of the first property. Points \( x,y \in \text{CR}(f,Y) \) are called chain connected in \( \text{CR}(f,Y) \) if for any \( \varepsilon > 0 \) there exist an \( \varepsilon \)-pseudo-trajectory \( x \) of \( f \) with \( x,y \in \{x\} \subseteq \text{CR}(f,Y) \). The set \( \text{CR}(f,X) \) is partitioned into equivalence classes of points connected pairwise with each other and every such class \( \mathcal{E} \) is a closed subset of \( \text{CR}(f,X) \) satisfying \( f(\mathcal{E}) = \mathcal{E} \). These equivalence classes are called the chain connected components of the restriction \( f_{\mid Y} \). Write \( E(x) \) for the chain connected component containing an element \( x \in \text{CR}(f,X) \).

The proofs of the next lemma and its corollary are straightforward, and are omitted.

**Lemma 7.3.10 (cf. [47, Lemma 4]).** For every \( x \in \text{CR}(f,Y) \) and every \( \varepsilon > 0 \) there exists a \( q = q(x,\varepsilon) > 0 \) such that each \( q \)-pseudo-cycle \( x \) of \( f \) with \( x \in \{x\} \subseteq Y \) satisfies \( \{x\} \subseteq O_{\varepsilon}(E(x)) \).

**Corollary 7.3.11.** For every \( x \in \text{CR}(f,Y) \) and \( q > 0 \) there exists a \( q \)-pseudo-cycle \( x \) of \( f \) with \( x \in \{x\} \) and \( \{x\} \subseteq \mathcal{E}(x) \).

A proof of the next lemma is given.

**Lemma 7.3.12.** Let \( \text{CR}(f,Y) \subset Y \) and \( f_{\mid Y} \) be \((s,h,\delta)\)-semi-hyperbolic on \( \text{CR}(f,Y) \) with respect to continuous mappings \( g : X \to X \). Let \( \alpha, \beta \) and \( \xi \) be the corresponding bi-shadowing and expansivity parameters in Theorem 6.3.12 and suppose that \( x \) and \( \hat{x} \) are members of distinct chain connected components \( \mathcal{E} \) and \( \hat{\mathcal{E}} \). Then \( \|x - \hat{x}\| \geq \min \{\beta, \xi/\alpha\} \).

**Proof.** Assume the contrary. In particular, suppose that

\[ \|x - \hat{x}\| = q < r = \min \{\beta, \xi/\alpha\} . \]

A contradiction to \( x \) and \( \hat{x} \) belonging to distinct chain connected components is obtained by showing that for each \( \varepsilon > 0 \) there exists an \( \varepsilon \)-pseudo-cycle \( x^* \subseteq Y \) which simultaneously contains both \( x \) and \( \hat{x} \). For this it suffices to construct two trajectories \( y \) and \( z \) satisfying, respectively,

\[
\begin{align*}
y & \subseteq Y, \quad x \in \{y_n : n > 0\}, \quad \hat{x} \in \{y_n : n > 0\}, \\
z & \subseteq Y, \quad \hat{x} \in \{z_n : n > 0\}, \quad x \in \{z_n : n > 0\}.
\end{align*}
\]  

To construct the sequence \( y \) fix a positive number \( \varepsilon < r - q \). By Corollary 7.3.11 there exist \( \varepsilon \)-pseudo-cycles \( x = \{x_0, \ldots, x_m\} \subseteq \mathcal{E}(x) \) and \( \hat{x} = \{\hat{x}_0, \ldots, \hat{x}_{\hat{m}}\} \subseteq \mathcal{E}(\hat{x}) \) satisfying \( x_0 = x \) and \( \hat{x}_0 = \hat{x} \). Denote by \( u \) and \( \hat{u} \) the infinite periodic sequences defined, respectively, by \( u_k = x_k \) for \( k = 0, \ldots, m \) and \( \hat{u}_k = \hat{x}_k \) for \( k = 0, \ldots, \hat{m} \). Then consider the infinite sequence \( w \) defined by

\[
w_k = \begin{cases} u_k, & k \leq 0, \\
\hat{u}_k, & k > 0. \end{cases}
\]  

(7.32)
Since \( u \) and \( \hat{u} \) are both \( \varepsilon \)-pseudo trajectories and \( \|u_0 - \hat{u}_0\| = \|x - \hat{x}\| = q \), the sequence \( w \) in (7.32) is a \((q + \varepsilon)\)-pseudo-trajectory of \( f \) with \( w \subseteq \text{CR}(f, \overline{Y}) \).

As \( f \) is bi-shadowing on \( \text{CR}(f, \overline{Y}) \) with the constants \( \alpha, \beta \) and \( q + \varepsilon < r \leq \beta \), there exists a true trajectory \( y \in \overline{Y} \) of \( f \) which is \( \alpha(s,h)q \) close to \( w \). It remains to prove that this trajectory satisfies the second and the third inclusions of (7.30). Because \( u - km = x, \hat{u}_k \hat{m} = \hat{x} \) for the positive integer \( k \), it suffices to establish the limit relations

\[
\lim_{n \to -\infty} \|y_n - u_n\| = \lim_{n \to \infty} \|y_n - \hat{u}_n\| = 0,
\]

which can be done in the same way as for the limits in (7.24) above using the \( \xi \)-expansivity.

The trajectory \( z \) is obtained similarly by bi-shadowing with respect to the pseudo-trajectory \( \hat{w} \) defined by \( \hat{w}_k = w_{-k} \) for all integers \( k \).

The chain connected components of \( \text{CR}(f, \overline{Y}) \) are the required sets \( \Omega_i \). Lemma 7.3.12 shows that there are only a finite number, while closedness and invariance follow from the definition.

It remains to show that each set \( \Omega_i \) contains a dense trajectory. Fix \( i \) and consider a countable subset of points \( \{x^0, \ldots, x^\nu, \ldots\} \) which is dense in \( \Omega_i \). Fix \( \varepsilon > 0 \) sufficiently small. By definition, for each \( \nu \) there exists \( \varepsilon/\nu \)-pseudo-cycles \( x^\nu = \{x^\nu_0, \ldots, x^\nu_{n(\nu)-1}\} \) with \( x^\nu_0 = x^\nu \). For each \( \nu \) adjoin \( \nu + 1 \) of the \( x^\nu \) to form a pseudo-cycle \( y^\nu \) of length \((\nu + 1)n(\nu)\). Then consider the bi-infinite sequence \( w \) of the form

\[
w = \{\ldots, y^\nu, \ldots, y^1, y^0, y^1, \ldots, y^\nu, \ldots\}
\]

By definition this is a bi-infinite \( \varepsilon \)-pseudo-trajectory of \( f \), so by the bi-shadowing property there exists a unique trajectory \( z \) in \( \Omega_i \) which is \( \alpha \varepsilon \) close to \( w \). It is then straightforward to check that \( z \) has the necessary properties.

**Proof of the second property.** The proof follows the usual manner, see [65] and the references therein, so will only be sketched here. Consider the set \( \Omega_1 \). The essential step is to construct the subset \( \Omega^1_1 \).

For each \( \varepsilon > 0 \) let \( P(\varepsilon) \) be the set of positive integers \( p \) with the property that there exists a \( \varepsilon \)-pseudo-cycle \( x \) belonging to \( \Omega_1 \) of the period \( p \). Define \( P = \cap_{\varepsilon > 0} P(\varepsilon) \), which is non-empty because of cyclic shadowing, and denote by \( n_1 \) the minimal common denominator of the integers in \( P \).

Let \( x \) be an arbitrary point in \( \Omega_1 \). For each \( \varepsilon > 0 \) let \( A(\varepsilon) \) denote the set of all \( \varepsilon \)-pseudo-cycles containing the point \( x \) as their first element. Then denote by \( A^*(\varepsilon) \) be totality of the elements of pseudo-cycles in \( A(\varepsilon) \) with indices of the form \( n_1 k \) for \( k = 1, 2, \ldots \). Finally, define \( \Omega^1_1 = \cap A^*_\varepsilon \). This set has all of the required properties.

**Proof of the third property.** This follows directly from Theorem 6.3.12.

This completes the proof of Lemma 7.3.9.
Using Lemma 7.3.9, Theorem 7.3.3 can be modified to establish the existence and robustness of chaotic behavior under continuous perturbation of a semi-hyperbolic mapping on components of its chain recurrent set.

Let $Y$ be an open set with $\overline{Y} \subseteq X$ and let $f \in \text{Lip}(X, \mathbb{R}^d)$ be bi-shadowing and expansive on $\text{CR}(f, \overline{Y})$ with $\text{CR}(f, \overline{Y}) \subset Y$ and with spectral decomposition $\Omega_i^j, j = 1, \ldots, n_i$ and $i = 1, \ldots, K$. Denote

$$\gamma(\varepsilon) = \frac{1}{3} \min \{\beta, \varepsilon/\alpha\}$$

where $\alpha, \beta$ are the bi-shadowing parameters. Property (ii) above implies the existence of a finite integer $k(\varepsilon)$ defined by

$$k(\varepsilon) = \max \{m : \exists v, w \in \Omega_1^1 \text{ with } (f^{n_1})^m(U^1_{\gamma(\varepsilon)}(v)) \cap U^1_{\gamma(\varepsilon)}(w) = \emptyset\} \quad (7.33)$$

where $U^1_{\gamma}(x) = U_{\gamma}(x) \cap \Omega_1^1$. Indeed, given $\varepsilon > 0$, consider a finite $\gamma(\varepsilon)/3$-net $a = \{a_1, \ldots, a_I\}$ in the compact set $\Omega_1^1$. By the topological mixing property, there then exists a positive integer $N$ such that

$$(f^{n_1})^n(U^1_{\gamma(\varepsilon)/3}(a_i)) \cap U^1_{\gamma(\varepsilon)/3}(a_j) \neq \emptyset, \quad i, j = 1, \ldots, I, \ n > N.$$ 

On the other hand, each set of the form $U^1_{\gamma(\varepsilon)}(x)$ for $x \in \Omega_1^1$ contains at least one set of the form $U^1_{\gamma(\varepsilon)/3}(a_i), i = 1, \ldots, I$, so this $N$ is a finite upper bound for $k(\varepsilon)$ in (7.33).

**Theorem 7.3.13.** Let $f, \Omega_1^1$ be as in Lemma 7.3.9. Then every mapping $g \in C$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ is $(\varepsilon, k)$-chaotic on a neighborhood of $\Omega_1^1$ for any positive integer $k \geq k(\varepsilon)$.

Theorems 7.3.3 and 7.3.13 have the following corollary.

**Corollary 7.3.14.** Let $\text{CR}(f, \overline{Y}) \subset Y$ and let $f$ be $(s, h, \delta)$-semi-hyperbolic on $\text{CR}(f, \overline{Y})$ with the spectral decomposition $\Omega_i^j, j = 1, \ldots, n_i$, and $i = 1, \ldots, K$, where $\Omega_1^i$ contains more than one element. Define $\gamma(\varepsilon)$ by (7.29) and $k(\varepsilon)$ by (7.33). Then every mapping $g \in C$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ is $(\varepsilon, k)$-chaotic on a neighborhood of $\Omega_1^i$ for any positive integer $k > k(\varepsilon)$. In particular, the chaos threshold of every continuous mapping $g$ satisfying $\|g - f\|_C < \gamma(\varepsilon)$ does not exceed $\varepsilon$.

**Example 7.3.15.** Let $f$ be the Smale horseshoe mapping on the square

$$Q = [-1, 1] \times [-1, 1]$$

with compression factor $1/5$ and expansion factor $5$ and let $h$ be a Lipschitz mapping on $Q$ with Lipschitz constant less than $2/3$ and with $\|h\|_C < 1/5$. Then the mapping $H = f + h$ is semi-hyperbolic on $\text{CR}(H, Q) \subset \text{Int } Q$. Note that $H$ need not be invertible on $\text{CR}(H, Q)$. Corollary 7.3.14 is applicable here, so for each $\varepsilon > 0$ the chaos threshold of every sufficiently small continuous perturbation of $f$ does not exceed $\varepsilon$. 
In this chapter various applications of semi-hyperbolicity and its consequences are considered, in particular in the context of delay differential equations, systems with hysteresis and the numerical approximation of chaotic attractors.

8.1 Semi-Hyperbolic Mappings in Banach Spaces

It will be shown here that semi-hyperbolicity of a Lipschitz mapping on a given set implies bi-shadowing for a wide class of dynamical systems in infinite dimensional Banach spaces. Definitions of shadowing and bi-shadowing are given in the next section and that of semi-hyperbolicity for Lipschitz mappings in Sect. 8.1. The main results are stated in Subsection 8.1.2, an example of its application to delay equation is introduced in Sect. 8.4. Note that finite dimensional perturbations of infinite-dimensional semi-hyperbolic mappings were also considered in [46].

8.1.1 Bi-Shadowing: Completely Continuous Perturbations

Let $E$ be a Banach space with the norm $\| \cdot \|$. Consider a mapping $f : X \rightarrow X$ where $X$ is a subset of $E$.

**Definition 8.1.1.** A dynamical system generated by a mapping $f : X \rightarrow X$ is said to be bi-shadowing with respect to completely continuous perturbations on a subset $K$ of $X$ with positive parameters $\alpha$ and $\beta$ if for any given finite pseudo-trajectory $x = \{x_n\} \in \text{Tr}(f, K, \gamma)$ with $0 \leq \gamma \leq \beta$ and any completely continuous mapping $\varphi : X \rightarrow X$ satisfying

$$\| \varphi - f \|_\infty \leq \beta - \gamma$$

(8.1)

there exists a trajectory $y = \{y_n\} \in \text{Tr}(\varphi, X)$ such that

$$\|x_n - y_n\| \leq \alpha(\gamma + \| \varphi - f \|_\infty)$$

(8.2)

for all $n$ for which $y$ is defined.
Definition 8.1.2. A dynamical system generated by a mapping \( f : X \to X \) is said to be cyclically bi-shadowing with respect to completely continuous perturbations on a subset \( K \) of \( X \) with positive parameters \( \alpha \) and \( \beta \) if for any given pseudo-cycle \( x \in \text{Per}(f, K, \gamma) \) with \( 0 \leq \gamma \leq \beta \) and any completely continuous mapping \( \varphi : X \to X \) satisfying (8.1) there exists a proper cycle \( y \in \text{Per}(\varphi, X) \) of period \( N \) equal to that of \( x \) such that (8.2) holds for \( n = 0, 1, \ldots, N \).

Note that the cycle \( y \) in Definition 8.1.2 is required only to be in \( X \) rather than in the subset \( K \).

Finally, a generalization of the notion of semi-hyperbolicity for Lipschitz mappings acting in a Banach space is required.

Definition 8.1.3. Let \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) be a split and \( K \) a subset of an open set \( X \subseteq \mathbb{R}^d \). A Lipschitz mapping \( f : X \to X \) is said to be \( s \)-semi-hyperbolic on \( K \) if there exist positive real numbers \( h, \delta \) such that for each \( x \in K \) there exists a splitting \( E = E^u_x \oplus E^s_x \) with corresponding projectors \( P^u_x \) and \( P^s_x \) satisfying Properties SH0(Lip)--SH2(Lip) from Definition 3.1.6:

\[
\text{SH0(Lip): The space } E^u_x \text{ is finite dimensional for all } x \text{ and dim } E^u_x = \text{dim } E^u_y \text{ if } x, y \in K \text{ with } \| f(x) - y \| \leq \delta; \]

\[
\text{SH1(Lip): } \sup_{x \in K} \{ \| P^u_x \|, \| P^u_x \| \} \leq h; \]

\[
\text{SH2(Lip): The inclusion } x + u + v \in X \quad (8.3) \]

and the inequalities

\[
\| P^u_y (f(x + u + v) - f(x + \tilde{u} + v)) \| \leq \lambda_s \| u - \tilde{u} \|, \quad (8.4)
\]

\[
\| P^s_y (f(x + u + v) - f(x + u + \tilde{v})) \| \leq \lambda_s \| v - \tilde{v} \|, \quad (8.5)
\]

\[
\| P^u_y (f(x + u + v) - f(x + \tilde{u} + v)) \| \leq \mu_s \| u - \tilde{u} \|, \quad (8.6)
\]

\[
\| P^u_y (f(x + u + v) - f(x + u + \tilde{v})) \| \geq \mu_u \| v - \tilde{v} \|, \quad (8.7)
\]

hold for all \( x, y \in K \) with \( \| f(x) - y \| \leq \delta \) and all \( u, \tilde{u} \in E^s_x \) and \( v, \tilde{v} \in E^u_x \) such that \( \| u \|, \| \tilde{u} \|, \| v \|, \| \tilde{v} \| \leq \delta \).

Note that continuity in \( x \) of the splitting subspaces \( E^s_x, E^u_x \) or of the projectors \( P^s_x, P^u_x \) is not assumed here, nor is invariance of the splitting subspaces, as is the case in the definition of hyperbolicity of a diffeomorphism.
8.1.2 Main Results

The main result of this section is that semi-hyperbolicity is sufficient to ensure bi-shadowing of a dynamical system generated by a Lipschitz mapping with respect to perturbed systems generated by completely continuous mappings.

**Theorem 8.1.4.** Let \( f : X \to X \) be a Lipschitz mapping which is \( s \)-semi-hyperbolic on a subset \( K \) of \( X \) with constants \( h, \delta \). Then it is bi-shadowing on \( K \), with respect to completely continuous mappings \( \varphi : X \to X \) with parameters

\[
\alpha(s, h) = h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} \quad (8.8)
\]

and

\[
\beta(s, h, \delta) = \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max\{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}. \quad (8.9)
\]

The proof of the theorem is not dissimilar to that of a finite-dimensional analogue, Theorem 6.3.3. Nevertheless, to make the presentation self-sufficient and to highlight some differences from the finite-dimensional case, a complete proof is given in the next section.

Cyclic bi-shadowing is also a consequence of semi-hyperbolicity.

**Theorem 8.1.5.** Let \( f : X \to X \) be a Lipschitz mapping which is \( s \)-semi-hyperbolic on a subset \( K \) of \( X \) with constants \( h, \delta \). Then it is cyclically bi-shadowing on \( K \) with parameters \( \alpha(s, h) \) and \( \beta(s, h) \) given by (8.8) and (8.9), with respect to completely continuous mappings \( \varphi : X \to X \).

8.1.3 Proofs

For a given split \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) and semi-hyperbolicity constant \( h \), define the matrix

\[
M(s) = \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u / \lambda_u & 1 / \lambda_u \end{bmatrix} \quad (8.10)
\]

and a two-dimensional vector \( a = (a, b)^T \) by

\[
a = (I - M(s))^{-1} h, \quad \text{where} \quad h = h(1, 1 / \lambda_u)^T. \quad (8.11)
\]

Then the bi-shadowing constants (8.8) and (8.9) satisfy

\[
\alpha(s, h) = a + b, \quad \beta(s, h, \delta) = \delta \min\{a^{-1}, b^{-1}\}. \quad (8.12)
\]

As in the finite-dimensional case, the proof proceeds through a series of Lemmas. Denote by \( B_x(r) \) the closed ball centered at 0 of the radius \( r \) in the linear space \( E_u^x \). For each \( x \in K \) and each \( z \in \mathbb{R}^d \) satisfying \( \|P_x^s z\| \leq \delta \) define the finite-dimensional mapping \( F_{x, z} : B_x(\delta) \to E_u^x \) by

\[
F_{x, z}(v) = P_u^x(f(x + P_x^s z + v) - f(x + P_x^s z)).
\]
Lemma 8.1.6. Let $0 \leq r \leq \delta$. Then

$$F_{x,z}(B_x(r)) \supseteq B_{f(x)}(\lambda_u r). \quad (8.13)$$

Proof. Since equality of singleton sets occurs when $r = 0$, let $r > 0$. Respectively, let $\partial B_x(r)$ and $B_x^o(r)$ denote the boundary and the interior of $B_x(r)$. Clearly,

$$F_{x,z}(0) = P^{u}_{f(x)}\left(f(x + P^s_xz) - f(x + P^s_xz)\right) = 0 \in B_{f(x)}^o(\lambda_u r); \quad (8.14)$$
on the other hand, by the inequality (8.6)

$$F_{x,z}(\partial B_x(r)) \cap B_{f(x)}^o(\lambda_u r) = \emptyset. \quad (8.15)$$

By Condition SH0(Lip) of Definition 8.1.3, and the Principle of Domain Invariance (see, [5, p. 396]), (8.15) implies that

$$\partial F_{x,z}(B_x(r)) \cap B_{f(x)}^o(\lambda_u r) = \emptyset. \quad (8.16)$$

This, together with (8.14), implies (8.13) and the lemma is proved. \qed

This lemma and inequality (8.6) immediately give

Lemma 8.1.7. The operator $Q_{x,z} = F_{x,z}^{-1}$ is defined and continuous on the ball $B_{f(x)}(\lambda_u \delta)$ and satisfies the estimate $\|Q_{x,z}(v)\| \leq \lambda_u^{-1}\|v\|$.

Denote by $\mathcal{Z}_N$ the space of $N + 1$-tuples

$$z = (z_0, z_1, \ldots, z_N), \quad z_n \in E, \ n = 0, 1, \ldots, N.$$ 

The set $\mathcal{Z}_N$ can be treated as the Banach space $E^{N+1}$ with norm

$$\|z\|_{E^{N+1}} = \max_{0 \leq n \leq N} \|z_n\|.$$

Let $x = \{x_0, x_1, \ldots, x_N\}$ be a given $\gamma$-pseudo-trajectory of the system $f$. Let $\varphi$ be a given completely continuous mapping. In what follows, suppose that the parameter

$$\beta = \gamma + \|f - \varphi\|_{\infty} \quad (8.16)$$
satisfies

$$\beta \leq \beta(s, h, \delta). \quad (8.17)$$

Define the operator $\mathcal{H} : \mathcal{Z}_N \to \mathcal{Z}_N$, which transforms

$$z = (z_0, z_1, \ldots, z_N) \in \mathcal{Z}_N$$

into

$$\mathcal{H}(z) = w = (w_0, w_1, \ldots, w_N) \in \mathcal{Z}_N.$$
defined by the relations

\[ P_{s}^{s}w_0 = 0 \]

and

\[ P_{x}^{s}w_n = P_{x}^{s}(\varphi(x_{n-1} + z_{n-1}) - x_n) \quad (8.18) \]

for \( n = 1, 2, \ldots, N \), and the relations

\[ P_{x}^{u}w_N = 0 \]

and

\[ P_{x_{n-1}}^{u}w_{n-1} = Q_{x_{n-1},z_{n-1}}(P_{x_{n-1}}^{u}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1})
\quad + x_n - f(x_{n-1} + P_{x_{n-1}}^{s}z_{n-1}) + z_n)) \quad (8.19) \]

for \( n = 1, 2, \ldots, N \).

Consider the set

\[ \mathcal{S}(\beta) = \{ z \in \mathbb{R}^N : \| P_{x}^{s}z_n \| \leq a\beta, \| P_{x}^{u}z_n \| \leq b\beta, \ n = 0, 1, \ldots, N \} \quad (8.20) \]

By (8.12) and (8.17)

\[ a\beta \leq a\beta(s,h,\delta) \leq \delta, \quad b\beta \leq b\beta(s,h,\delta) \leq \delta \]

and so by (8.3) trajectories from \( \mathcal{S}(\beta) \) belong to \( X \).

**Lemma 8.1.8.** The following statements are valid:

(i) The operator \( \mathcal{H} \) is defined and completely continuous for \( z \) belonging to the set \( \mathcal{S}(\beta) \).

(ii) For any fixed point \( z = (z_0, z_1, \ldots, z_N) \in \mathcal{S}(\beta) \) of \( \mathcal{H} \), the sequence

\[ y = \{ x_0 + z_0, x_1 + z_1, \ldots, x_N + z_N \} \]

is a trajectory of the system \( \varphi \).

**Proof.** (i) Clearly, by (8.17) the right hand side of (8.18) is defined and completely continuously depends on \( z \in \mathcal{S}(\beta) \). So we need only prove that for any \( n = 1, 2, \ldots, N \) the right hand side of (8.19), which is finite-dimensional, is defined and continuous for \( z \in \mathcal{S}(\beta) \). By Lemma 8.1.7 it is sufficient to establish that

\[ \| P_{x}^{u}(-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1})
\quad + x_n - f(x_{n-1} + P_{x_{n-1}}^{s}z_{n-1}) + z_n) \| \leq \lambda_u \delta. \]

Rewrite the last inequality in the form

\[ \| J_1 + J_2 + J_3 \| \leq \lambda_u \delta, \]
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where

\[ J_1 = P_x (\frac{\varphi(x_n - 1) + f(x_n - 1)}{x_n - 1} + x_n - f(x_n - 1)), \quad (8.21) \]
\[ J_2 = P_x (f(x_n - 1) - f(x_n - 1 + P_{x_n} z_n - 1)), \quad (8.22) \]
\[ J_3 = P_x z_n. \quad (8.23) \]

It suffices to find bounds for \( \|J_1\|, \|J_2\| \) and \( \|J_3\| \). For \( \|J_1\| \), note that by (8.16)
\[ \|\varphi - f\|_\infty \leq \beta - \gamma, \quad \|x_n - f(x_n - 1)\| \leq \gamma, \]
so by Condition SH1(Lip) of Definition 8.1.3
\[ \|J_1\| \leq \beta h. \quad (8.24) \]

From the inequality (8.6),
\[ \|J_2\| \leq \mu_u \|P_{x_n - 1} z_n - 1\|. \quad (8.25) \]

Clearly,
\[ \|J_3\| = \|P_z z_n\|. \quad (8.26) \]

On the other hand, \( z \in S(\beta) \) implies that
\[ \|P_{x_n - 1} z_n - 1\| \leq \beta a, \quad \|P_z z_n\| \leq \beta b. \quad (8.27) \]

From by (8.25)–(8.27)
\[ \|J_1 + J_2 + J_3\| \leq \|J_1\| + \|J_2\| + \|J_3\| \leq \beta(h + a\mu_u + b). \]

But
\[ \beta(h + a\mu_u + b) \leq \lambda_u \delta. \quad (8.28) \]

To see this, from (8.10), (8.11)
\[ a = h \frac{\lambda_u - 1 + \mu_s}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}, \]
\[ b = h \frac{1 - \lambda_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}. \]

and put \( h + a\mu_u + b = \lambda_u b \), rewriting (8.28) as
\[ \beta \lambda_u b \leq \lambda_u \delta \quad (8.29) \]

But then (8.29) follows from (8.12) and (i) is proved.

(ii) It is sufficient to establish that
\[ x_n + z_n = \varphi(x_n - 1 + z_n - 1), \quad n = 1, 2, \ldots, N. \quad (8.30) \]

Because \( z \) is a fixed point of \( \mathcal{H} \), equations (8.18) and (8.19) can be rewritten as
\[ P_{x_n}^s z_n = P_{x_n}^s (\varphi(x_{n-1} + z_{n-1}) - x_n), \quad (8.31) \]

and

\[ P_{x_{n-1}}^u z_{n-1} = Q_{x_{n-1}, z_{n-1}} (P_{x_{n-1}}^u (-\varphi(x_{n-1} + z_{n-1}) + f(x_{n-1} + z_{n-1})) + x_n - f(x_{n-1} + P_{x_{n-1}}^s z_n) + z_n). \quad (8.32) \]

From (8.31) it follows that

\[ P_{x_n}^s x_n + z_n) = P_{x_n}^s \varphi(x_{n-1} + z_{n-1}). \quad (8.33) \]

Apply the nonlinear, finite-dimensional operator \( F_{x_{n-1}, z_{n-1}} = Q_{x_{n-1}, z_{n-1}} \) to both sides of (8.32), to obtain

\[ P_{x_n}^u (f(x_{n-1} + z_{n-1}) - f(x_{n-1} + P_{x_{n-1}}^s z_{n-1})) = P_{x_n}^u ((x_n + z_n - \varphi(x_{n-1} + z_{n-1})) + (f(x_{n-1} + z_{n-1}) - f(x_{n-1} + P_{x_{n-1}}^s x_{n-1} z_{n-1})) \]

which, simplifying, gives

\[ 0 = P_{x_n}^u (z_n - \varphi(x_{n-1} + z_{n-1}) + x_n). \]

That is,

\[ P_{x_n}^u (x_n + z_n) = P_{x_n}^u \varphi(x_{n-1} + z_{n-1}) \quad (8.34) \]

and (8.30) follows from (8.33) and (8.34). The Lemma is proved.

**Lemma 8.1.9.** The set \( \mathcal{S}(\beta) \) is invariant for \( \mathcal{H} \).

**Proof.** First, rewrite (8.18) in the form

\[ P_{x_n}^s w_n = I_1 + I_2, \]

where

\[ I_1 = P_{x_n}^s ((\varphi(x_{n-1} + z_{n-1}) - f(x_{n-1} + z_{n-1})) + (f(x_{n-1}) - x_n)), \]
\[ I_2 = P_{x_n}^s (f(x_{n-1} + z_{n-1}) - f(x_{n-1})). \]

Similarly, rewrite (8.19) in the form

\[ P_{x_{n-1}}^u w_{n-1} = Q_{x_{n-1}, z_{n-1}} (J_1 + J_2 + J_3) \]

where \( J_1, J_2, J_3 \) are defined in (8.21).

To estimate \( \|I_1\| \) note that by (8.16)

\[ \|\varphi - f\|_{\infty} \leq \beta - \gamma, \quad \|x_n - f(x_{n-1})\| \leq \gamma, \]

so by Condition SH1(Lip) of Definition 8.1.3.
\[ \| I_1 \| \leq h\beta. \]  

(8.35)

By (8.4), (8.5)

\[ \| I_2 \| \leq \lambda_s \| P_{x_{n-1}}^s z_{n-1} \| + \mu_s \| P_{x_{n-1}}^u z_{n-1} \|. \]  

(8.36)

Now for each \( z \in \mathcal{Z}_N \) define the pair of real nonnegative numbers

\[ m^s(z) = \max_{0 \leq n \leq N} \| P_{x_n}^s x_n - x_{n-1} \|, \quad m^u(z) = \max_{0 \leq n \leq N} \| P_{x_n}^u x_n - x_{n-1} \|. \]

and denote by \( \mathbf{m}(z) \) the two-dimensional column vector with coordinates \( m^s(z), m^u(z) \). From the bounds (8.35), (8.36) and definition (8.20) of \( \mathcal{S}(\beta) \), it follows that

\[ m^s(\mathcal{H} z) = m^s(w) \leq \beta h + \beta \lambda_s a + \beta \mu_s b. \]  

(8.37)

Similarly, from (8.24)–(8.26), the definition (8.20) of \( \mathcal{S}(\beta) \) and Lemma 8.1.7, it follows that

\[ m^u(\mathcal{H} z) = m^u(w) \leq \lambda_u^{-1}(\beta \mu_u a + \beta b + \beta h). \]  

(8.38)

The inequalities (8.37), (8.38) are equivalent to the coordinate-wise bound

\[ \mathbf{m}(\mathcal{H} z) = \mathbf{m}(w) \leq \beta M(s) a + \beta k, \quad z \in \mathcal{S}(\beta). \]  

(8.39)

In view of (8.11)

\[ \beta M(s) a + \beta k = \beta (M(s) (I - M(s))^{-1} + I) k = \beta (I - M(s))^{-1} k = \beta a. \]

Consequently, (8.39) is equivalent to

\[ \mathbf{m}(\mathcal{H} z) = \mathbf{m}(w) \leq \beta a, \quad z \in \mathcal{S}(\beta), \]

which means that the set \( \mathcal{S}(\beta) \) is invariant for \( \mathcal{H} \).

The proof of Theorem 8.1.4 is completed as follows. From (i) of Lemma 8.1.8, the operator \( \mathcal{H} \) is completely continuous on the convex set \( \mathcal{S}(\beta) \) and, in view of Lemma 8.1.9, \( \mathcal{H}(\mathcal{S}(\beta)) \subseteq S(\beta) \). So by the Schauder Fixed Point Theorem, \( \mathcal{H} \) has a fixed point \( z = (z_0, z_1, \ldots, z_N) \in \mathcal{S}(\beta) \) and hence, by (ii) of Lemma 8.1.8, the sequence

\[ x^* = \{ x_0 + z_0, x_1 + z_1, \ldots, x_N + z_N \} \]

is a trajectory of the system \( \varphi \). By definition (8.20) of the set \( \mathcal{S}(\beta) \),

\[ \| z_n \| \leq \| P_{x_n}^s z_n \| + \| P_{x_n}^u z_n \| \leq (a + b)\beta, \quad n = 0, 1, \ldots, N, \]

and thus by (8.12), (8.16),

\[ \| z_n \| \leq \alpha(s,k)(\gamma + \| \varphi - f \|_\infty), \quad n = 0, 1, \ldots, N. \]

That is to say

\[ \| x_n - x^*_n \| \leq \alpha(s,k)(\gamma + \| \varphi - f \|_\infty), \quad n = 0, 1, \ldots, N, \]

and Theorem 8.1.4 is proved.
8.2 Nonautonomous Difference Equations

Let \( P \) be a nonempty space and let \( \theta y : P \to P \) be mapping, so that \( \{\theta^n : n \in \mathbb{Z}^+\} \) is a semi-group under composition. No assumption is made about the domain \( P \), nor properties of the mapping \( p \mapsto \theta p \), but in applications \( P \) is often either a metric space (deterministic systems) or a measure space (random systems), with \( p \mapsto \theta p \) being continuous in the former and measurable in the latter. In addition, let

\[
\mathfrak{X} = \{(X_p, \| \cdot \|_p) : p \in P\}
\]

be a family of Banach spaces (possibly subspaces of a common space) and let

\[
f = \{(f(p, \cdot) : p \in P\}
\]

be a family of mappings

\[
f(p, \cdot) : X_p \to X_{\theta p}, \quad p \in P,
\]

which are at least continuous in \( x \).

Consider the nonautonomous difference equation

\[
x_{n+1} = f(\theta^n p, x_n),
\]

(8.40)
on \( \mathfrak{X} \) with \( x_n \in X_{\theta^n p} \) for each \( n \in \mathbb{Z}^+ \). This generates a discrete-time nonautonomous semi-dynamical on \( \mathfrak{X} \) with the autonomous semi-dynamical system on \( P \) generated by \( \theta \) as its driving system and the associated cocycle mapping on \( \mathfrak{X} \) defined by

\[
\phi(0, p, x) := \{x\}
\]

and

\[
\phi(n, p, x) := f(\theta^n p, \cdot) \circ \cdots \circ f(p, x) \quad \text{for all } n \geq 1,
\]

where \( \phi(n, p, \cdot) : X_p \to X_{\theta^n p} \), see [70,71].

A simple example is a sequence of mappings \( f_n \) on \( \mathbb{R}^d \) with \( P = \mathbb{Z} \), \( f(n, \cdot) = f_n(\cdot) \) and \( \theta \) the left shift on \( \mathbb{Z}^+ \). A nontrivial example arises when the sequence of mappings \( f_n \) on \( \mathbb{R}^d \) is chosen periodically or, perhaps less regularly, from a finite family of mappings \( \{g_1, \ldots, g_r\} \), that is with \( f_n = g_{i_n} \) where the \( i_n \in \{1, \ldots, r\} \). The nonautonomous difference equation

\[
x_{n+1} = g_{i_n}(x_n)
\]

(8.41)
on \( \mathbb{R}^d \) can be rewritten in the form (8.40) with the set \( P = \{1, \ldots, r\}^{\mathbb{Z}^+} \) of infinite sequences \( p = \{i_n : n \in \mathbb{Z}^+\} \) with \( i_n \in \{1, \ldots, r\} \) and the left shift operator \( \theta \) defined on \( P \) by \( \theta(i_n) = i_{n+1} \). The space \( P \) here may be given the structure of a compact metric space with metric

\[
d(p, p') = \sum_{n=0}^{\infty} (r + 1)^{-|n|} |i_n - i'_n|.
\]

(8.42)
An important application is a variable time-step method for an autonomous differential equation $\dot{x} = f(x)$, such as the Euler method

$$x_{n+1} = x_n + h_n f(x_n),$$

where the variable time steps $h_n > 0$, in practice, come from a finite set, so $g_{i_n}(x) = x + h_n f(x)$ in (8.41) and the sequence space metric is as in (8.42). More generally, a variable parameter $q \in Q$ may be involved in a difference equation

$$x_{n+1} = f(x_n, q_n),$$

where $q_n \in Q$. For example, let

$$f(x, q) = \frac{|x| + q^2}{1 + q}, \quad x \in \mathbb{R}, \ Q = [0.5, 1].$$

Here, the sequence space is $P = [0.5, 1]^{\mathbb{Z}^+}$ of infinite sequences

$$p = \{q_n : n \in \mathbb{Z}^+\}, \quad q_n \in Q,$$

which is a compact metric space with the metric

$$d(p, p') = \sum_{n=0}^{\infty} 2^{-|n|} |q_n - q_n'|.$$

A simple example indicating the usefulness of different spaces $X_p$ is given in the next section.

### 8.2.1 Semi-Hyperbolicity

As in the autonomous case, a 4-tuple of real numbers called a split measures the expansion, contraction and other rates in the definition of a nonautonomous semi-hyperbolic system (8.40).

Let $\mathcal{f} = \{ (f(p, \cdot) : p \in P \}$ a family of mappings with $f(p, \cdot) : X_p \to X_{\theta p}$ for $p \in P$ and let $\mathcal{K} = \{ K_p : p \in P \}$ be a family of closed and bounded subsets with $K_p \subset X_p$ and $f(p, K_p) \cap K_{\theta p} \neq \emptyset$ for each $p \in P$.

**Definition 8.2.1.** A splitting $X_p = E^s_x(p) \oplus E^u_x(p)$ of a family of Banach spaces $X = \{(X_p, \|\cdot\|_p) : p \in P \}$ with projectors $P^s_x(p) : X_p \to E^s_x(p)$ and $P^u_x(p) : X_p \to E^u_x(p)$ for each $x \in X_p$ such that

- $P^s_x(p)X_p = E^s_x(p)$,
- $P^u_x(p)E^u_x(p) = 0$,
- $P^u_x(p)X_p = E^u_x(p)$,
- $P^u_x(p)E^s_x(p) = 0$,

is said to be uniform on the family $\mathcal{K} = \{ K_p : p \in P \}$ of closed and bounded subsets with respect to the family of mappings $\mathcal{f} = \{(f(p, \cdot) : p \in P \}$ on $X$ with $f(p, K_p) \cap K_{\theta p} \neq \emptyset$ for each $p \in P$ if there exist positive real numbers $\delta, h$ such that
SH0: \( \dim E^u_y(\theta p) = \dim E^u_x(p) \) for all \( x \in K_p \) and \( y \in K_{\theta p} \) such that \( \|f(p,x) - y\|_{\theta p} \leq \delta \) for every \( p \in P \); 

SH1: \( \sup_{p \in P} \sup_{x \in K_p} \{ \|P^u_x(p)\|_p, \|P^s_x(p)\|_p \} \leq h \) for every \( p \in P \).

The continuity in \( x \) of the splitting subspaces \( E^s_x(p), E^u_x(p) \) or of the projectors \( P^s_x(p), P^u_x(p) \) is not assumed in the above definition, nor is the invariance of the subsets \( K_p \in \mathcal{K} = \{ K_p : p \in P \} \) with respect to the corresponding mappings

\[
\mathfrak{f}(p, \cdot) \in \mathfrak{f} = \{ (f(p, \cdot) : p \in P) \}.
\]

That is, neither \( K_{\theta p} = f(p, K_p) \) nor \( K_{\theta p} \subseteq f(p, K_p) \) has to hold for any \( p \in P \). This allows considerably more flexibility in applications, see for example [43, 46].

**Definition 8.2.2.** A family of mappings \( \mathfrak{f} = \{ f(p, \cdot) : p \in P \} \) on a family of Banach spaces \( \mathfrak{X} = \{ X_p : p \in P \} \) is said to be uniformly \( s \)-semi-hyperbolic with a split \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) on a family \( \mathcal{K} = \{ K_p : p \in P \} \) of nonempty closed and bounded subsets of \( \mathfrak{X} \) such that \( f(p, K_p) \cap K_{\theta p} \neq \emptyset \) for all \( p \in P \), if there exists a uniform splitting \( X_p = E^s_x(p) \oplus E^u_x(p) \) of \( \mathfrak{X} \) with projectors \( P^s_x(p) \) and \( P^u_x(p) \) for each \( x \in K_p \) and \( p \in P \) (that is satisfying conditions SH0 and SH1) such that

**SH2:** for each \( p \in P \) the inclusion \( x + u + v \in X_p \) and the inequalities

\[
\begin{align*}
\|P^s_y(\theta p)[f(p,x+u+v) - f(p,x+\tilde{u}+v)]\|_{\theta p} &\leq \lambda_s \|u - \tilde{u}\|_p, \\
\|P^u_y(\theta p)[f(p,x+u+v) - f(p,x+\tilde{u}+v)]\|_{\theta p} &\leq \lambda_u \|v - \tilde{v}\|_p, \\
\|P^u_y(\theta p)[f(p,x+u+v) - f(p,x+\tilde{u}+v)]\|_{\theta p} &\leq \mu_u \|u - \tilde{u}\|_p, \\
\|P^u_y(\theta p)[f(p,x+u+v) - f(p,x+\tilde{u}+v)]\|_{\theta p} &\leq \mu_u \|v - \tilde{v}\|_p,
\end{align*}
\]

hold for all \( x \in K_p \) and \( y \in K_{\theta p} \) with \( \|f(p,x) - y\|_{\theta p} \leq \delta \) and all \( u, \tilde{u} \in E^s_x(p) \) and \( v, \tilde{v} \in E^u_x(p) \) with

\[
\begin{align*}
\|u\|_p, \|\tilde{u}\|_p, \|v\|_p, \|\tilde{v}\|_p &\leq \delta
\end{align*}
\]

A simple example can be constructed in terms of a sequence of linear mappings \( f_n(x) = A_n x \) in \( \mathbb{R}^d \) with hyperbolic matrices \( A_n \) for which the unstable linear spaces all have the same dimension. This can be put into formalism with \( P = \mathbb{Z} \) and \( \theta \) the left shift on \( \mathbb{Z}^+ \) with all \( X_n = \mathbb{R}^d \). If the unstable subspaces are each mapped onto their successor, a split with \( \mu_s = \mu_u = 0 \) can be used.

To see why it may be convenient to distinguish the elements \( X_p \) of \( \mathfrak{X} \), consider a real symmetric \( 2 \times 2 \) hyperbolic matrix \( A_0 \) and a \( 2 \times 2 \) rotation matrix \( R \) by \( \pi/2 \) and consider a nonautonomous linear difference equation with matrices \( A_n = R^n A_0 \) for \( n \in \mathbb{Z}^+ \), so \( P = \mathbb{Z}^+ \). Suppose that the space \( \mathbb{R}^2 \) has a fixed coordinate system and define \( X_n := R^n \mathbb{R}^2 \), so \( X_{n+1} = RX_n \). Note that the stable and unstable manifolds would be mapped onto the other if the spaces were not also rotated. Here a split with \( \mu_s = \mu_u = 0 \) is also appropriate. This example is, of course, trivial — one could avoid the rotations
altogether and consider simply the autonomous linear difference equation with matrix $A_0$. A less trivial example will be given in Sect. [8.2.3].

As above, let $\mathcal{K} = \{K_p : p \in P\}$ be a family of nonempty closed and bounded subsets with $K_p \subset X_p$ for each $p \in P$ such that $f(p, K_p) \cap K_{\theta p} \neq \emptyset$, $p \in P$, for a family of mappings $\mathcal{f} = \{f(p, \cdot) : p \in P\}$ with $f(p, \cdot) : X_p \to X_{\theta p}$, $p \in P$.

**Definition 8.2.3.** A trajectory of a family of mappings $\mathcal{f}$ on $\mathcal{X}$ is a sequence $x = \{x_n\}$ with $x_n \in X_{\theta^n p}$ satisfying

$$x_{n+1} = f(\theta^n p, x_n).$$

A $\gamma$-pseudo-trajectory of a family of mappings $\mathcal{f}$ on $\mathcal{X}$ is a sequence $y = \{y_n\}$ with $y_n \in X_{\theta^n p}$ satisfying

$$\|y_{n+1} - f(\theta^n p, y_n)\|_{\theta^{n+1}} \leq \gamma; \quad \gamma > 0,$$

for $n = -n_-, \ldots, 0, \ldots, n_+$ where $n_+ \leq \infty$. The qualifier finite may be used when $n_+ < \infty$ and infinite otherwise.

Given a finite or infinite interval $\mathbb{I}$ of integers $\mathbb{Z}$, let $\text{Tr}_1(\mathcal{f}, \mathcal{K}, \gamma)$ denote the totality of corresponding $\gamma$-pseudo-trajectories of $\mathcal{f}$ in $\mathcal{K}$. Since a true trajectory can be regarded as a $\gamma$-pseudo-trajectory for any $\gamma \geq 0$, in particular with $\gamma = 0$, denote the corresponding set of true trajectories by $\text{Tr}_1(\mathcal{f}, \mathcal{K}, 0)$ or simply by $\text{Tr}_1(\mathcal{f}, \mathcal{K})$. Obviously $\text{Tr}_1(\mathcal{f}, \mathcal{K}) \subset \text{Tr}_1(\mathcal{f}, \mathcal{K}, \gamma)$, with strict inclusion since there are $\gamma$-pseudo-trajectories which are not trajectories.

Let $\mathcal{g} = \{g(p, \cdot) : p \in P\}$ with $g(p, \cdot) : X_p \to X_{\theta p}$ for each $p \in P$. The semi-norm

$$\|\mathcal{g} - \mathcal{f}\|_{\infty} = \sup_{p \in P} \sup_{x \in X_p} \|g(p, x) - f(p, x)\|_{\theta p}$$

gives a notion of distance between families $\mathcal{g}$ and $\mathcal{f}$ on $\mathcal{X}$.

**Definition 8.2.4.** A nonautonomous semi-dynamical system generated by a family of Lipschitz mappings $\mathcal{f} = \{f(p, \cdot) : p \in P\}$ with $f(p, \cdot) : X_p \to X_{\theta p}$, such that $f(p, \cdot) : X_p \to X_{\theta p}$ for a family $\mathcal{K} = \{K_p : p \in P\}$ of nonempty closed and bounded subsets with $K_p \subset X_p$ and $f(p, K_p) \cap K_{\theta p} \neq \emptyset$, $p \in P$, is said to be bi-shadowing with respect to completely continuous perturbations on $\mathcal{K}$ with positive parameters $\alpha$ and $\beta$, if for any given finite pseudo-trajectory $x = \{x_n\} \in \text{Tr}_1(\mathcal{f}, \mathcal{K}, \gamma)$ with $0 \leq \gamma \leq \beta$ and any family of completely continuous mappings $\mathcal{g} = \{g(p, \cdot) : p \in P\}$ with $g(p, \cdot) : X_p \to X_{\theta p}$, $p \in P$, satisfying

$$\|\mathcal{g} - \mathcal{f}\|_{\infty} \leq \beta - \gamma,$$

there exists a trajectory $y = \{y_n\} \in \text{Tr}_1(\mathcal{g}, \mathcal{X})$ such that

$$\|x_n - y_n\|_p \leq \alpha(\gamma + \|\mathcal{g} - \mathcal{f}\|_{\infty}) = \alpha\beta$$

for all $n$ for which $y$ is defined.
If the spaces $X_p$ are finite-dimensional of like dimension, then the requirement that the sets $K_p$ are closed and bounded is equivalent to compactness. The condition that the perturbed mappings $g$ are completely continuous, appearing in the definition of bi-shadowing is then equivalent to continuity.

### 8.2.2 Semi-Hyperbolicity Implies Bi-Shadowing

The main result of this section is that semi-hyperbolicity is sufficient to ensure bi-shadowing of a nonautonomous semi-dynamical system generated by a family $f$ of Lipschitz mapping with respect to a class of perturbed systems generated by families $g$ of continuous mappings. As with the corresponding result for autonomous systems, which it generalizes, it also provides explicit values of the shadowing parameters.

**Theorem 8.2.5.** Let $\mathcal{R} = \{K_p : p \in P\}$ be a family of closed and bounded subsets of $\mathcal{X} = \{X_p : p \in P\}$ for a family $\mathcal{f} = \{f(p, \cdot) : p \in P\}$ of Lipschitz mappings with $f(p, \cdot) : X_p \to X_{\theta p}$, $p \in P$, which is semi-hyperbolic on $\mathcal{R}$ with a uniform split $s$ and constants $h, \delta$. Then, the nonautonomous semi-dynamical system generated by $\mathcal{f}$ is bi-shadowing on $\mathcal{R}$ with respect to any family $\mathcal{g} = \{g(p, \cdot) : p \in P\}$ of completely continuous mappings $g(p, \cdot) : X_p \to X_{\theta p}$, $p \in P$, with parameters $\alpha = \alpha(s, h)$ and $\beta = \beta(s, h, \delta)$ defined by

$$\alpha(s, h) := h \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}$$

and

$$\beta(s, h, \delta) := \delta h^{-1} \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\max \{\lambda_u - 1 + \mu_s, 1 - \lambda_s + \mu_u\}}.$$ 

Moreover, if each set $K_p$ is compact, then the bi-shadowing holds with respect to trajectories on infinite intervals.

The proof follows very closely those given in [43, 47] for the autonomous case, noting only that all estimates hold uniformly in $p \in P$, and will not be repeated here.

A particular strength of the result is that it does not require smooth perturbations and is thus applicable to, for example, perturbations due to hysteresis effects [46]. It also provides explicit error bounds. An important application is when a variable time-step numerical method such as the Euler method is applied to an autonomous differential equation. Due to the discreteness of computer arithmetic, which causes round-off error, an approximation of the Euler mapping is in fact computed [41, 43]. The bi-shadowing result ensures that the computed dynamics reflects that of the Euler method.

### 8.2.3 Twisted Horseshoe Mapping

The *twisted horseshoe mapping* $f = (f_1, f_2)$ on $I^2 = [0, 1] \times [0, 1]$ defined by
\[
\begin{align*}
    f_1(x, y) &= \begin{cases} 
    2x, & 0 \leq x \leq \frac{1}{2}, \\
    2 - 2x, & \frac{1}{2} < x \leq 1,
    \end{cases} \\
    f_2(x, y) &= \frac{x}{2} + \frac{y}{10} + \frac{1}{4}
\end{align*}
\]
has been investigated by Guckenheimer, Oster and Ipaktchi \[57\]. It has a fixed point
\[
(\bar{x}, \bar{y}) = \left( \frac{2}{3}, \frac{35}{54} \right),
\]
which is a saddle point with eigenvalues \(-2\) and \(\frac{1}{10}\). All the conditions of a theorem in \[71\] which generalizes Marotto’s snap-back repeller result to saddle points are thus satisfied, so the twisted horseshoe mapping behaves chaotically in a slight generalization of the sense of Li and Yorke (see \[68\]).

In fact, the twisted-horseshoe mapping also satisfies the Sharkovsky cycle co-existence ordering, due to a generalization in \[70\] of Sharkovsky’s theorem to triangular shaped mappings like the twisted horseshoe mapping.

The autonomous difference equation generated by the twisted horseshoe mapping can be interpreted as a nonautonomous difference equation
\[
  x_{n+1} = \frac{x_n}{10} + \frac{p_n}{2} + \frac{1}{4}
\]
driven by the semi-dynamical system on \([0, 1]\)
\[
  p_{n+1} = \begin{cases} 
    2p_n, & 0 \leq p_n \leq \frac{1}{2}, \\
    2 - 2p_n, & \frac{1}{2} < p_n \leq 1.
    \end{cases}
\]

Note that any trajectory of the driving system can be extended backwards indefinitely, though typically not uniquely. Let \(p = \{p_n : n \in \mathbb{Z}\}\) be such an entire trajectory and hold it fixed. Then the explicit solution of \(8.43\) given by
\[
  x_n = 10^{-n+n_0} x_{n_0} + \frac{1}{2} \sum_{j=n_0}^{n-1} 10^{j+1-n} p_j + \frac{1}{4} \sum_{j=n_0}^{n-1} 10^{j+1-n}.
\]
Let \(n\) fixed and let the \(x_{n_0}\) take values in a fixed bounded subset of \(\mathbb{R}\). Then, taking the pullback limit as \(n_0 \to -\infty\) (see \[72, 73\]), gives
\[
  \bar{x}_n = \frac{1}{2} \sum_{j=-\infty}^{n-1} 10^{j+1-n} p_j + \frac{1}{4} \sum_{j=-\infty}^{n-1} 10^{j+1-n},
\]
that is,
\[
  \bar{x}_n = \frac{1}{2} \sum_{j=-\infty}^{n-1} 10^{j+1-n} p_j + \frac{10}{36}.
\]
In particular, for two different initial values \(x_{n_0}\) and \(x'_{n_0}\), with the same driving sequence \(p\), it is easily shown that
|\[x_n - x'_{n}| = |x_{n0} - x'_{n0}|10^{-n+n_0} \to 0 \quad \text{as} \quad n \to \infty,\]

and hence, with \(x'_{n} \equiv \bar{x}_{n},\)

|\[x_n - \bar{x}_{n}| \to 0 \quad \text{as} \quad n \to \infty.\]

Thus all solutions of (8.43) with different starting points converge to the same limit if they correspond to the same driving sequence.

Define \(\bar{x}(p) := \bar{x}_0\) and let \(\theta\) be the shift operator on the space of sequences \(P = [0, 1]^\mathbb{Z}^+\). Then \(\bar{x}_n = \bar{x}(\theta np)\) and the singleton subsets \(A_p := \{\bar{x}(p)\}\) for the pullback attractor (which is also forwards attracting) of the nonautonomous dynamical system generated by the twisted horseshoe system. Let \(X_p\) be the space \(\mathbb{R}\) with the origin shifted to \(\bar{x}(p)\). With the change of coordinates \(z_n := x_n - \bar{x}(p)\) the mapping (8.43) simplifies to

\[z_{n+1} = \frac{1}{10} z_n,\]

with \(z_n \in X_{\theta np}\) and \(z_{n+1} \in X_{\theta np+1,p}\). Semi-hyperbolicity on \(K_p \equiv X_p\) for the split \(s = (10^{-1}, 0, 0, 0)\) and trivial unstable subspaces follows immediately. The bi-shadowing theorem 8.2.5 is applicable to this NDS with continuous perturbations of the linear mapping (8.43).

The bi-shadowing theorem 8.2.5 is also applicable to the twisted horseshoe mapping in both variables on the product space \(\mathbb{R}^2\) with \(P = \mathbb{Z}^+\). This allows for shadowing with perturbations of both the linear mapping (8.43) and the tent-mapping (8.44).

8.3 Chaotic Attractors under Discretization

The solutions \(\phi(t; x_0)\) with initial values \(\phi(0; x_0) = x_0\) of an ordinary differential equation

\[\frac{dx}{dt} = f(x)\quad (8.45)\]

on \(\mathbb{R}^d\) generate a continuous-time dynamical system on \(\mathbb{R}^d\), which is a flow \(\{\phi(t; \cdot)\}\) of diffeomorphisms on \(\mathbb{R}^d\) if \(f\) is smooth enough. It is well-known (cf. [74,135]) that if this system has a compact attractor \(A_0\), then the discrete-time dynamical system generated by a one-step numerical scheme

\[x_{n+1} = F_h(x_n) := x_n + hf(h; x_n)\quad (8.46)\]

with constant time-step \(h > 0\), such as an Euler or Runge–Kutta scheme, also has a compact attractor \(A_h\) such that

\[\text{dist}(A_h, A_0) \to 0 \quad \text{as} \quad h \to 0+,\quad (8.47)\]

where dist is the Hausdorff semi-distance of nonempty compact subsets of \(\mathbb{R}^d\) defined by
\[ \text{dist}(A, B) = \max_{a \in A} \min_{b \in B} \|a - b\|. \]

Convergence (8.47) is often described as the upper semi-continuity of the numerical attractor \(A_h\) at \(h = 0\).

In general, a numerical attractor \(A_h\) need not be lower semi-continuity under time-discretization at \(h = 0\), that is

\[ \text{dist}(A_0, A_h) \to 0 \quad \text{as} \quad h \to 0^+, \] (8.48)

unless additional assumptions are made about the dynamics inside the attractor \(A_0\). Here it will be assumed that \(A_0\) is a hyperbolic invariant set for the flow generated by the differential equation (8.45) containing a point \(\bar{x}_0 \in A_0\) for which the \(\omega\)-limit set \(\omega^+(\bar{x}_0) = A_0\), which is typical of many chaotic attractors, where

\[ \omega^+(\bar{x}_0) = \bigcap_{T \geq 0} \bigcup_{s \geq T} \{\phi(s; x_0)\}. \] (8.49)

The main technical tool to be used in the proof is the bi-shadowing property satisfied by the time-1 map \(\phi(1 : \cdot)\) of the continuous-time system on the hyperbolic set \(A_0\).

Recall that a one-step numerical scheme (8.46) with time-step \(h > 0\) is of \(p\)th order if for every bounded subset \(D\) of \(\mathbb{R}^d\) there exists a constant \(C_D\) such that its one-step discretization error satisfies the inequality

\[ \|\phi(h; x_0) - F_h(x_0)\| \leq C_D h^{p+1} \]

for all \(x_0 \in D\) and \(h > 0\). There then exists a constant \(\gamma > 0\) such that the global discretization error satisfies the inequality

\[ \|\phi(nh; x_0) - F_h^n(x_0)\| \leq C_D e^{\gamma nh} h^p \] (8.50)

for all \(x_0 \in D\), \(h > 0\) and \(n = 0, 1, \ldots, N_{D,h}\) such that \(\phi(t; x_0) \in D\) for \(0 \leq t \leq hN_{D,h}\).

**Theorem 8.3.1.** Suppose that the mapping \(f : \mathbb{R}^d \to \mathbb{R}^d\) in the differential equation (8.45) is \(p + 1\) times continuously differentiable and that the one-step numerical scheme \(F_h : \mathbb{R}^d \to \mathbb{R}^d\) (8.46) is of \(p\)th order. If the differential equation (8.45) has a hyperbolic attractor \(A_0\) which contains a dense recurrent trajectory, then the numerical scheme (8.46) has an attractor \(A_h\) which is continuous in \(h\) at \(h = 0\), that is

\[ H(A_h, A_0) = \max\{\text{dist}(A_h, A_0), \text{dist}(A_0, A_h)\} \to 0 \quad \text{as} \quad h \to 0^+. \]

**Proof.** By the assumed existence of the attractor \(A_0\) there is an open bounded subset \(\mathcal{X}\) of \(\mathbb{R}^d\) which is positively invariant for the flow \(\phi\) generated by the differential equation (8.45) and attention can be restricted to this set without loss of generality. The results of [58, 74, 135] then establish the existence of a
numerical attractor $A_h$ of the numerical scheme (8.46) for sufficiently small $h > 0$ which are upper semi-continuous in $h$ at $h = 0$ in the sense of convergence (8.47). Hence it remains only to show the lower semi-continuity of the $A_h$ at $h = 0$.

Write $\Phi(x) := \phi(1; x)$ for the time-1 mapping of the differential equation (8.45), so $\Phi : \mathcal{X} \to \mathcal{X}$. Note that $\Phi$ is a diffeomorphism on $\Phi(\mathcal{X})$ and that the original attractor $A_0 \subseteq \Phi(\mathcal{X})$ is also an attractor for the discrete-time dynamical system generated by $\Phi$, and hence $A_0$ is invariant under $\Phi$. Moreover, the diffeomorphism $\Phi$ inherits the hyperbolicity of the flow $\phi$ on $A_0$. Hence by Theorem 6.3.1 there exist positive constants $\alpha$ and $\beta$ such that the diffeomorphism $\Phi$ is bi-shadowing with parameters $\alpha$ and $\beta$ on $A_0$ with respect to the space of continuous functions $\mathcal{C}(\mathcal{X})$.

It can be assumed without loss of generality that mapping $F_h$ in the numerical scheme (8.46) satisfies $F_h(\mathcal{X}) \subseteq \mathcal{X}$ for sufficiently small $h$, in particular for $h \leq 1$. The $F_h^n \in \mathcal{C}(\mathcal{X})$ for $n = 1, 2, \ldots$ and by (8.50)

$$\sup_{x \in \mathcal{X}} \|\phi(nh; x) − F_h^n(x)\| \leq C_\mathcal{X} e^{\gamma nh} h^p$$

$n = 1, 2, \ldots$. Suppose that $C_\mathcal{X} e^{\gamma} h^p \leq \beta$ and that there is an integer $N_h$ such that $h = 1/N_h$, that is $N_h h = 1$. Then

$$\left\| \Phi − F_h^{N_h} \right\|_{\infty} := \sup_{x \in \mathcal{X}} \left\| \Phi(x) − F_h^{N_h}(x) \right\| \leq C_\mathcal{X} e^{\gamma} h^p \leq \beta. \quad (8.51)$$

Now let $\bar{x}_0 \in A_0$ be any point such that $\omega^+(\bar{x}_0) = A_0$ and define $\bar{x}_0(t) = \phi(t, \bar{x}_0)$ for each $t \in [0, 1]$. For each such $t$ consider the trajectory $\{\Phi^n(\bar{x}_0(t))\}_{n \in \mathbb{N}}$ of the mapping $\Phi$. By the bi-shadowing property of $\Phi$ in $A_0$ with $\delta = 0$ there then exists a $\bar{y}_0(t) \in \mathcal{X}$ such that

$$\left\| \Phi^n(\bar{x}_0(t)) − F_h^{N_h}(\bar{y}_0(t)) \right\| \leq \alpha \left\| \Phi − F_h^{N_h} \right\|_{\infty} \leq \alpha C_\mathcal{X} e^{\gamma} h^p \quad (8.52)$$

for all $n = 0, 1, \ldots$. In particular, for $n = 0$

$$\left\| \bar{x}_0(t) − \bar{y}_0(t) \right\| \leq \alpha C_\mathcal{X} e^{\gamma} h^p,$$

that is

$$\bar{y}_0(t) \in \mathcal{C}_\alpha C_\mathcal{X} e^{\gamma} h^p (\{\bar{x}_0(t)\}) \subseteq \mathcal{C}_\alpha C_\mathcal{X} e^{\gamma} h^p (A_0)$$

where $\mathcal{C}_\rho(S)$ is the closed ball of radius $\rho$ about a subset $S$ of $\mathbb{R}^d$.

Since the points $\bar{y}_0(t)$ for $0 \leq t \leq 1$ belong to a common bounded subset of $\mathbb{R}^d$, by the absorbing property of the (maximal) numerical attractor $A_h$ for each $\varepsilon > 0$ there exists an integer $N(\varepsilon, h)$ such that

$$y_j(t) := F_h^j(\bar{y}_0(t)) \in \mathcal{C}_{\varepsilon/2} (A_h) \quad \text{for all } j \geq N(\varepsilon, h), 0 \leq t \leq 1.$$

But, from (8.52), $\left\| \Phi^n(\bar{x}_0(t)) − y_n N_h(t) \right\| \leq \alpha C_\mathcal{X} e^{\gamma} h^p$, that is
\( \Phi^n(x_0(t)) \in C_\alpha C_{\mathcal{X} e^\gamma h^p} \{y_{nN_h}(t)\} \) for all \( n \geq 0, \ 0 \leq t \leq 1 \).

Hence

\[ \Phi^n(x_0(t)) \in C_\alpha C_{\mathcal{X} e^\gamma h^p + \varepsilon/2} (A_h) \] for all \( n \geq N(\varepsilon, h)/N_h, \ 0 \leq t \leq 1 \),

or equivalently

\[ \bigcup_{n \geq N(\varepsilon, h)/N_h, \ 0 \leq t \leq 1} \{\Phi^n(x_0(t))\} \subseteq C_\alpha C_{\mathcal{X} e^\gamma h^p + \varepsilon/2} (A_h). \quad (8.53) \]

Now the group property of the flow \( \phi \) gives

\[ \Phi^n(x_0(t)) = \phi(n, x_0(t)) = \phi(n, \phi(t, x_0(t))) = \phi(n + t, x_0), \]

so

\[ \bigcup_{n \geq N(\varepsilon, h)/N_h} \{\Phi^n(x_0(t))\} = \bigcup_{s \geq N(\varepsilon, h)/N_h} \{\phi(s, x_0)\} \]

and hence from (8.53)

\[ \bigcup_{s \geq N(\varepsilon, h)/N_h} \{\phi(s, x_0)\} \subseteq C_\alpha C_{\mathcal{X} e^\gamma h^p + \varepsilon/2} (A_h). \]

But \( \omega^+(x_0) = A_0 \), so by 8.49

\[ A_0 \subseteq \bigcup_{s \geq N(\varepsilon, h)/N_h} \{\phi(s, x_0)\} \subseteq C_\alpha C_{\mathcal{X} e^\gamma h^p + \varepsilon/2} (A_h) \]

or equivalently

\[ \text{dist}(A_0, A_h) \leq \alpha C_{\mathcal{X} e^\gamma h^p + \varepsilon/2}. \]

Restricting \( h \) further to \( 0 < h < h_0(\varepsilon) \) so that \( \alpha C_{\mathcal{X} e^\gamma h^p} \leq \varepsilon/2 \) also holds and so

\[ \text{dist}(A_0, A_h) \leq \varepsilon \quad (8.54) \]

for all \( 0 < h < h_0(\varepsilon) \) with \( h^{-1} \) an integer.

Finally, the case that \( h^{-1} \) is not an integer needs to be considered. Let \( N_h \) be the integer such that \( hN_h < 1 < h(N_h + 1) \). Then

\[ \|F_h^{N_h}(x) - \Phi(x)\| = \|F_h^{N_h}(x) - \phi(1; x)\| \]

\[ \leq \|F_h^{N_h}(x) - \phi(hN_h; x)\| + \|\phi(hN_h; x) - \phi(1; x)\| \]

\[ \leq C_{\mathcal{X} e^\gamma hN_h} h^p + \left| \int_{hN_h}^1 f(\phi(s; x)) \, ds \right| \]

\[ \leq C_{\mathcal{X} e^\gamma h^p} + M(1 - hN_h) \leq C_{\mathcal{X} e^\gamma h^p} + Mh \leq Kh \]
where $M = \max_{x \in \mathcal{X}} \|f(x)\|$ and $K = 2 \max \{C \mathcal{X} e^\gamma, M\}$. Repeating the above argument with

$$
\|F^N_h - \Phi\|_\infty \leq Kh
$$

instead of the global discretization estimate (8.51), that is with $C \mathcal{X} e^\gamma h^p$ replaced by $Kh$. With an appropriate modification to $h_0(\varepsilon)$, the inequality (8.54) then holds for all $h$ sufficiently small. Convergence (8.48), that is the lower semi-continuity of the numerical attractor at $h = 0$, then holds.

This completes the proof of Theorem 8.3.1. □

8.4 Delay Differential Equations

Consider the linear delay equation

$$
x'(t) = Ax(t) + Bx(t - h).
$$

(8.55)

Here $x(t) \in \mathbb{R}^d$, $A$ and $B$ are real $d$-matrices and $h$ is a positive constant. We shall call this equation hyperbolic if

$$
\det (wI - A - e^{wB}) = 0
$$

(8.56)

does not have a purely imaginary solution $w = ip$.

To each solution $w$ of this equation there corresponds a solution of the delay equation (8.55) of the form

$$
e^{wt}a, \quad -\infty < t < \infty,
$$

(8.57)

where $a$ is an eigenvector of the matrix $wI - A - e^{wB}$ with the eigenvalue $w$.

We will also consider nonlinear delay equations of the form

$$
y'(t) = Ay(t) + By(t - h) + F(y(t), y(t - h)).
$$

(8.58)

Here $F(y, v)$ is a continuous $\mathbb{R}^d$-valued function, which is locally Lipschitz in $y$ and $A$, $B$ are as before. Denote by $L(F)$ the set of all continuous function $y(t)$, $t \geq -h$, satisfying the equation (8.58) for $t > 0$. In particular, $L(0)$ denotes the set of all continuous function $x(t)$, $t \geq -h$, satisfying the equation (8.55) for $t > 0$.

**Theorem 8.4.1.** Let the equation (8.55) be hyperbolic. Then there exists a constant $\gamma > 0$ with the following properties.

(i) For each $x(t) \in L(0)$ and for each uniformly bounded $F(x, u)$ there exists a continuous function $y(t) \in L(F)$ satisfying the inequality

$$
|y(t) - x(t)| < \gamma \sup_{y, v} |F(y, v)|, \quad t \geq -h.
$$

(8.59)
(ii) Let $F(y,v)$ be a uniformly bounded and $y(t) \in L(F)$. Then there exists a function $x(t) \in L(0)$ satisfying \((8.59)\).

This demonstrates robustness of solutions of a hyperbolic delay equation with respect to arbitrary continuous perturbations of small amplitude. In particular, any nonlinear perturbation \((8.58)\) of a linear equation \((8.55)\) has bounded at $t \to \infty$ solutions which shadow a given bounded at $t \to \infty$ solution of the linear equation. The main steps in the proof of this theorem are summarized as follows.

\textbf{Step 1.} For each continuous function $\xi(s)$, $s \in [-h,0]$, the equation \((8.58)\) has a unique solution $y(t;\xi,F)$, $t \geq -h$, which is continuous and satisfies

$$y(s;\xi,F) = \xi(s), \quad s \in [-h,0],$$

because $F(y,v)$ is supposed continuous, uniformly bounded and satisfy local Lipschitz condition in $y$. Introduce the shift operator $S_F$ for equation \((8.58)\) by

$$(S_F \xi)(\tau) = y(h - \tau;\xi,F), \quad -h \leq \tau \leq 0.$$ 

The operator $S_F$ is completely continuous as an operator in the space $C([-h,0],\mathbb{R}^d)$.

In particular, denote by $S$ the shift operator for the linear equation \((8.55)\). The operator $S$ is a linear completely continuous operator in $C$.

\textbf{Step 2.} Let $\xi(\tau), \tau \in [-h,0]$, be an eigenfunction of the complexification of the operator $S$ with a complex eigenvalue $w$. Then by the definition $\xi(s)$ satisfies the equation

$$w\xi'(\tau) = wA\xi(\tau) + B\xi(\tau), \quad -h \leq \tau \leq 0.$$ 

Thus the set of nonzero eigenvalues of the linear operator $S$ coincides with the set of complex number $z = e^{hw}$ where $w$ is a solution of the equation \((8.56)\) (The corresponding complex eigenfunction are restrictions of functions \((8.57)\) on $[-h,0]$.)

Since $S$ is completely continuous, the spectrum of $S$ consists of zero and all complex numbers $e^{wh}$ where $w$ is a solution of the equation \((8.56)\).

\textbf{Step 3.} By the previous step and the hyperbolicity of the linear equation \((8.55)\) the spectrum $\sigma(S)$ of the linear operator $S$ consists of two disjoint parts

$$\sigma(S) = \sigma^s(S) \cup \sigma^u(S),$$

such that $\sigma^s$ is located strictly inside the unit disc of a the complex plane and $\sigma^u$ is located strictly outside the unit disc. By the Decomposition Theorem \cite[p. 421]{128}, it means that the space $C$ can be decomposed into a direct sum

$$C = E^s \oplus E^u$$
so that both $E^s$ and $E^u$ are invariant for $S$ with

$$
\sigma(S|E^s) = \sigma^s, \quad \sigma(S|E^u) = \sigma^u.
$$

Further, since $S$ is completely continuous, the subspace $E^u$ is finite-dimensional. Note that the parallel projection $P^s$ of $C$ onto $E^s$ in the direction of $E^u$ can be written in an explicit form as

$$
P^s = -\frac{1}{2\pi i} \int_{|z|=1} (S-zI)^{-1} \, dz.
$$

(8.60)

**Step 4.** Introduce an auxiliary norm $\| \cdot \|_s$ onto the subspace $E^s$ by

$$
\|x\|_s = \sum_{n=0}^{\infty} \| S^n x \|_C.
$$

Clearly this norm is equivalent to the norm $\| \cdot \|$ and the restriction of the operator $S$ onto $E^s$ contracts in this norm with some constant $\lambda_s < 1$. Analogously, introduce an auxiliary norm $\| \cdot \|_u$ onto the subspace $E^u$ by

$$
\|x\|_u = \sum_{n=0}^{\infty} \| S^{-n} x \|_C.
$$

This norm is also equivalent to the $C$-norm and the restriction of the operator $S$ onto $E^u$ expands in this norm with some constant $\lambda_u > 1$. Introduce in $C$ an auxiliary norm $\| \cdot \|_*$ by

$$
\|\xi\|_* = \max \{ \| P^s \xi \|_s, \| P^u \xi \|_u \}
$$

where $P^s$ is defined by (8.60) and $P^u = I - P^s$. Denote by $s$ the split $(\lambda_s, \lambda_u, 0, 0)$. By construction, the linear operator $S$ is $s$-semi-hyperbolic with constants $h, \delta$ where

$$
k = \max \{ \| P^s \|_*, \| P^u \|_* \}
$$

and $\delta$ is an arbitrary positive number.

**Step 5.** In Step 1 the shift operator $S_F$ of the nonlinear equation (8.58) is completely continuous. This operator also satisfies the estimate

$$
\| S_F \xi - S \xi \|_* < \gamma_1 \sup_{y,v} |F(y,v)|, \quad t \geq -h,
$$

for some positive $\gamma_1$. Thus Theorem 8.1.4 is applicable and, taking into account the equivalence of norms $\| \cdot \|_C$ and $\| \cdot \|_*$, as a corollary to that theorem it follows that:

**Corollary 8.4.2.** There exist a constant $\gamma > 0$ with the following properties.
(i) For each trajectory
\[ \eta = \eta_0, \eta_1, \ldots \] (8.61)
of the shift operator \( S \) there exists a trajectory
\[ \eta^F = \eta^F(0), \eta^F_1, \ldots \] (8.62)
of the operator \( S_F \) with
\[ \| \eta_n - \eta^F_n \|_C \leq \gamma \sup_{y,v} |F(y,v)|, \] (8.63)

(ii) For each trajectory (8.62) of the shift operator \( S_F \) there exists a trajectory (8.61) of the operator \( S \) satisfying (8.63).

Theorem 8.4.1 then follows.

Note that the construction of the last section can be carried out also for some systems described by parabolic equations. Also note that some hysteresis perturbations, like Prandtl, Besseling and Ishlinskii models in plasticity or Preisach, Giltay and Madelung models in magnetism [22,77,138] can be taken into account both in analysis of delay and parabolic equations.

8.5 Systems with Hysteresis

Consider a smooth mapping \( f : \mathbb{R}^d \to \mathbb{R}^d \). The dynamical system generated by a difference equation of the form
\[ x_n = f(x_{n-1}), \quad n = 1, 2, \ldots \] (8.64)
is often used in technical, physical or mechanical applications, where \( f \) usually occurs via a Poincaré section. Throughout, this will be referred to as the system \( f \). Realistically, a system (8.64) can describe the actual underlying system only approximately. Thus an important mathematical problem is the robustness of the system to perturbations. Classical results in this direction state that a \( C^r \) dynamical system preserves some of its structural properties under a small smooth perturbation [56,111,132]. However, there are some kinds of nonsmooth perturbations which are very important. In this section we analyze a specific class of perturbations which arise in systems with weak hysteresis nonlinearities. An important feature of such models is that hysteresis nonlinearities are treated as continuous but nonsmooth dynamical systems \( W \), often with an infinite dimensional set \( \Omega = \Omega(W) \) of internal states \( \omega \). This includes such nonlinearities as play, stop, the Besseling–Ishlinskii and Preisach–Giltay models and so on. Further details may be found in [77,91].

In such situations the natural description of state space of a perturbed system (8.64) is \( Q = \mathbb{R}^d \times \Omega \). So it is more realistic to describe the dynamics of the perturbed system \( W \) by relations of the form
\[(x_n, \omega_n) = W(x_{n-1}, \omega_{n-1}) = (\varphi(x_{n-1}, \omega_{n-1}), \psi(x_{n-1}, \omega_{n-1})).\]

Here \(\varphi: \mathbb{R}^d \times \Omega \to \mathbb{R}^d\) and \(\psi: \mathbb{R}^d \times \Omega \to \Omega\) are continuous mappings. Some concrete examples of systems which arise in the theory of hysteresis nonlinearities are given in Sects. 8.5.1 and 8.5.2.

We are concerned with the relationship between the trajectories of a smooth system \(f\) and those of systems \(W\) which are close to \(f\) in some sense. An appropriate measure of the distance between the two types of system was introduced and investigated in Sect. 6.2.3. It is important to note that, without extra assumptions, the system (8.64) is not structurally stable in general.

A natural, additional assumption is hyperbolicity. In these circumstances, estimates of the distance between trajectories of \(f\) and its perturbation \(W\) should not depend explicitly upon the time interval over which the trajectories are considered. Instead, it is preferable that any estimate should be uniform so long as the trajectories remain in the region in question. This is the principal question that we address in this section.

### 8.5.1 Transducer Stop

Recall that the nonlinearity stop with threshold value \(h\) or transducer stop [77, p. 23–24] is a system \(U_h\) with the state space \([-h,h]\), scalar inputs \(u(t)\) and outputs \(\omega(t)\). For a smooth input \(u(t), t \geq 0\), and initial state \(\omega_0 \in [-h,h]\) the corresponding output

\[\omega(t) = (U_h[\omega_0]u)(t), \quad t \geq 0,\]

is defined as a unique absolutely continuous solution of the problem

\[\omega' = q(\omega, u'(t)), \quad \omega(0) = \omega_0,\]

where

\[q(\omega, u) = \begin{cases} \min \{u, 0\}, & \omega \geq h, \\ u, & |\omega| < h, \\ \max \{u, 0\}, & \omega \leq -h. \end{cases}\]

Consider the system described by the equations

\[x' = G(x, \omega), \quad \omega(t) = (U_h[\omega_0](c, x))(t). \quad (8.65)\]

Here \(x \in \mathbb{R}^d; h > 0\) and \(\omega \in [-h,h]\) are parameters, \(c\) is a fixed vector from \(\mathbb{R}^d\) and \(U_h\) is the stop nonlinearity with threshold value \(h\). Equations of such type arise as description of mechanical systems with elastic-plastic Prager elements, technical systems with plays or stops and many control systems.

Suppose that the function \(G\) satisfies a global Lipschitz condition. Then the equation (8.65) has a unique solution for any initial condition \(x(0) = x_0\) and each initial state \(\omega_0\) of the hysteresis nonlinearity \(U_h\). Let the shift operator
\(S_h(x_0, \omega_0)\) denote the image of the initial value \((x_0, \omega_0)\) after unit time along the trajectories of the system (8.65). Suppose that \(F(x) = G(x, 0)\) is a smooth function, satisfying \(F(0) = 0\) and the matrix \(DF_0\) does not have eigenvalues with zero real part. Similarly, let \(S_0(x_0)\) be the image of the initial value \(x_0\) after unit time along the trajectories of equation

\[
x' = F(x).
\]

(8.66)

The mappings \(W(x, \omega) = S_h(x, \omega)\) and \(f(x) = S_0(x)\) generate dynamical systems \(W\) and \(f\) respectively, where the state space of the system \(W\) is the product \(\mathbb{R}^d \times [-h, h]\).

Clearly, the system \(f\) is semi-hyperbolic in some open ball centered at the origin. From Theorem 6.2.7 it follows immediately

Theorem 8.5.1. There exist \(\alpha > 0\) and \(h_0 > 0\) with the following property: for any trajectory \(x(t) \in \mathcal{B}, 0 \leq t < t_* \leq \infty\), of the equation (8.66) and any \(h \leq h_0\) there exists a trajectory \((x_h(t), \omega_h(t))\), \(0 \leq t < t_*\), of (8.65) satisfying

\[
|x(t) - x_h(t)| \leq \alpha h, \quad 0 \leq t < t_*.
\]

Corollary 8.5.2. There exist \(\alpha > 0\) and \(h_0 > 0\) with the following property: for any \(x_0 \in \mathcal{B}\) belonging to the stable manifold of the equation (8.66) there exists a trajectory \((x_h(t), \omega_h(t))\), \(t \geq 0\), of (8.65) satisfying

\[
|x_0(t) - x_h(t)| \leq \alpha h, \quad t > 0.
\]

This result can be treated as a kind of ‘the stable manifold robustness theorem’ with respect to hysteresis perturbations of a system.

Analogues of Theorem 8.5.1 are valid for equations with such nonlinearities as play or generalized play, with multi-dimensional plays and stops, with Mises and Treska models [77], and so on.

8.5.2 Ishlinskii and Besseling Systems

Let \(U_h\) be the stop nonlinearity with threshold \(h\), as in Subsection 8.5.1. Consider \(h\) as a parameter, \(0 \leq h \leq \infty\), and let \(\mu\) be a Borel measure on \([0, \infty]\) satisfying

\[
\int_0^\infty h \, d\mu(h) < \infty.
\]

Denote by \(\mathcal{H}\) the totality of continuous functions \(z(h), h \geq 0\), satisfying \(|z(h)| \leq h\). Now introduce a system \(W_{\mu}\), with scalar inputs and outputs and with state space \(\mathcal{H}\), as follows. For a given smooth input \(u(t), t \geq 0\), and an initial state \(z_0 \in \mathcal{H}\), the corresponding output

\[
z(t) = (W_{\mu}[z_0]u)(t), \quad t \geq 0,
\]

is defined as
\[ z(t) = \int_{0}^{\infty} (U[z_0(h)]u)(t) \, d\mu(h). \]

A model of this type includes fundamental mechanical models such as the Ishlinskii and Besseling systems [77, p. 342–346]. It might be thought of as describing a continuum of linked transducers.

Suppose that the function \( G \) is globally Lipschitz, as in previous section. Consider the system described by equations

\[ x' = G(x, z), \quad z(t) = (W_\mu[z_0](c, x))(t). \quad (8.67) \]

This extends the system (8.65). Again, (8.67) has a unique solution \( x(t), t \geq 0 \), for each initial condition \( x(0) = x_0 \). Define the corresponding shift operator \( S_\mu(x_0) \).

From Theorem 6.2.7 it follows that

**Theorem 8.5.3.** There exist \( \alpha > 0 \) and \( \epsilon_0 > 0 \) with the following property: for any trajectory \( x(t), 0 \leq t < t_* \leq \infty \), of the equation (8.66), for any measure \( \mu \) satisfying

\[ r(\mu) = \int_{0}^{\infty} h \, d\mu(h) \leq \epsilon_0 \]

and any \( z(h) \in \mathcal{H} \), there exists a trajectory \( (x_\mu(t), z_\mu(t)), 0 \leq t < t_*, \) of (8.67) satisfying

\[ |x(t) - x_\mu(t)| \leq \alpha r(\mu), \quad 0 \leq t < t_* \]

Analogues of Theorem 8.5.3 are valid for models such as the multi-dimensional Ishlinskii system, the Preisach–Giltay model [77] and its multi-dimensional analogue [76].
Extensions and Further Applications

Two extensions of semi-hyperbolicity and their applications are considered in this chapter. The first extension, split-hyperbolicity [121], is a modification of the semi-hyperbolicity concept that is applicable to analysis of mappings defined in metric spaces; it appeared also useful in rigorous computer aided analysis of chaotic behavior. The second extension, topological hyperbolicity [121], is a tool to prove shadowing for various classes of dynamical systems. Split-hyperbolicity is briefly discussed in the next section, and a simplified version of the topological hyperbolicity concept is discussed in Sect. 9.2. In the last three sections some recent applications of split-hyperbolicity and topological in various subject areas are briefly sketched.

9.1 Split-Hyperbolicity

9.1.1 Split-Hyperbolicity in Product-Spaces

Let \( J \in \mathbb{Z} \) be a set, possibly infinite, of consecutive integers. Let \( M_n^s, M_n^u \) where \( n \in J \) be complete metric spaces with metrics \( \rho_n^s, \rho_n^u \). Suppose that the non-empty balls in \( M_n^s \) and in \( M_n^u \) are connected, that is they cannot be represented as a disjoint union of two nonempty sets each of which is both relatively open and relatively closed.

Elements from the Cartesian product \( M_n = M_n^s \times M_n^u \) are treated as pairs \( x = (x^s, x^u) \). The spaces \( M_n \) are endowed with the usual metric

\[
\rho_n(x, y) := \max\{\rho_n^s(x^s, y^s), \rho_n^u(x^u, y^u)\}.
\]

Let \( f \) denote the bi-infinite sequence \( \{f_i\}_{i \in \mathbb{Z}} \) of continuous maps, which may be partially defined, from \( M_n \) to \( M_{n+1} \):

\[
f_n(x^s, x^u) = (f_n^s(x^s, x^u), f_n^u(x^s, x^u))
\]

where \( f_n^s : M_n^s \times M_n^u \to M_{n+1}^s \) and \( f_n^u : M_n^s \times M_n^u \to M_{n+1}^u \).
Let the sequence \( \mathbf{x} = \{x_n\}_{n \in J} \) of the elements \( x_n \in M_n \) be fixed, such that the images \( f_n(x_n) \) are defined for all \( n \in J \). Denote by \( B^s_n[r] \) the closed \( r \)-ball in \( M^s_n \) centered at \( x^s_n \in M^s_n \); the balls \( B^u_n[r] \) for \( x^u_n \in M^u_n \), \( r > 0 \), are defined in a similar way. Let \( \delta^s, \delta^u \) be some positive constants. Denote

\[
U_n := B^s_n[\delta^s] \times B^u_n[\delta^u].
\]

Let \( \mathcal{D}_n \) be the set of those \( y \in U_n \) which satisfy \( f_n(y) \in U_{n+1} \).

Recall that a four-tuple \( s = (\lambda^s, \mu^u, \mu^s, \mu^u) \) of non-negative real numbers is called split if

\[
\lambda^s < 1 < \mu^u \quad \text{and} \quad \Delta(s) := (1 - \lambda^s)(\lambda^u - 1) - \mu^s\mu^u > 0.
\]

**Definition 9.1.1.** The sequence \( \mathbf{f} \) of the maps \( f_n \) is said to be split (\( s \)-) hyperbolic in the \((\delta^s, \delta^u)\)-neighborhood of the sequence \( \mathbf{x} \) if it satisfies the following three conditions:

C0: \( \mathcal{D}_n \) is closed for all \( n \in J \), and, for each boundary point \( y \) of \( \mathcal{D}_n \), either \( y \) belongs to the boundary of \( U_n \) or \( f(y) \) belongs to the boundary of \( U_{n+1} \) (whenever \( n+1 \in J \)).

C1: For all \( n, n+1 \in J \) and all \( y, z \in \mathcal{D}_n \) the following inequalities hold

\[
\rho_{n+1}^s(f_n(y), f_n(z)) \leq \lambda^s \rho_n^s(y^s, z^s) + \mu^s \rho_n^u(y^u, z^u),
\]

\[
\rho_{n+1}^u(f_n(y), f_n(z)) \geq -\mu^u \rho_n^s(y^s, z^s) + \lambda^u \rho_n^u(y^u, z^u).
\]

C2: The map \( w \mapsto f_n^u(v, w) \) is open as a map from \( B^u_n[\delta^u] \) to \( B^u_{n+1}[\delta^u] \) for each \( v \in B^s_n[\delta^s] \), in the sense that the image \( f_n^u(v, U) \) of an open subset \( U \) of \( B^u_n[\delta^u] \) is relatively open in \( B^u_{n+1}[\delta^u] \) (whenever \( n, n+1 \in J \)).

Conditions C0 and C2 are technical and intended for use in some complicated situations. The following simple lemma is stated without proof.

**Lemma 9.1.2.** Let \( f_n \) be defined on \( U_n \) and let \( M_n^u = \mathbb{R}^d_u \). Then conditions C0 and C2 from the definition of split-hyperbolicity hold.

For a given sequence \( \mathbf{x} = \{x_n\}_{n \in J} \) and a given \( \mathbf{f} \) the discrepancy \( D(\mathbf{x}; \mathbf{f}) \) is defined by the formula:

\[
D(\mathbf{x}; \mathbf{f}) := \max_{n, n+1 \in J} \rho_{n+1}(f_n(x_n), x_{n+1}).
\]

Introduce the numbers

\[
a^s := \frac{\lambda^u - 1 + \mu^s}{\Delta(s)}, \quad a^u := \frac{1 - \lambda^s + \mu^u}{\Delta(s)}.
\]

One of the main tools in the investigation of split-hyperbolic systems is the following Shadowing Theorem taken from [121].
Theorem 9.1.3 (Shadowing Theorem). Let the sequence $f$ be split-hyperbolic in the $(\delta^s, \delta^u)$-neighborhood of the sequence $\bar{x} = \{\bar{x}_n\}_{n \in J}$ and let the discrepancy $D(\bar{x}; f)$ satisfy the inequality

$$D(\bar{x}; f) < \min \left\{ \frac{\delta^s}{a^s}, \frac{\delta^u}{a^u}, \delta^u \right\}.$$  

Then there exists a trajectory $x = \{x_n\}_{n \in J}$ of $f$ satisfying

$$\rho^s_n(x^s_n, \bar{x}^s_n) \leq a^s D(\bar{x}; f) \quad \text{and} \quad \rho^u_n(x^u_n, \bar{x}^u_n) \leq a^u D(\bar{x}, f).$$

In the case when $J$ is the set of all integers the trajectory $x$ is unique.

9.1.2 An Application to Hysteresis

In this section, following [118], it is shown the split-hyperbolicity can be used to analyze systems with hysteresis.

Let $f(t, x) : \mathbb{R}_+ \times \mathbb{R}^d$ be $T$-periodic in time $t$ and Lipschitz continuous in $x$:

$$|f(t, x) - f(t, y)| < \ell_f |x - y|.$$  

Consider an ordinary differential equation

$$x' = f(t, x),$$

with $x \in \mathbb{R}^d$. Suppose that this equation has a $T$-periodic solution with $x(0) = x_T$. Let $F$ be a shift operator along the trajectories for the time $T$. Thus the operator $F$ has a fixed point $x_T$. It will be assumed that the linearization of $F$ in a small neighborhood of $x_T$ has no eigenvalues with absolute values equal to 1. Then it is possible to change coordinates to split the space into two invariant subspaces $\mathbb{R}^d = \mathbb{R}^d_s \times \mathbb{R}^d_u$ corresponding to eigenvalues inside and outside the unit circle, respectively.

Now consider an extended system that involves the output of a hysteresis nonlinearity:

$$x' = f(t, x) + \varepsilon g(x, z(t)), \quad z(t) = (\Gamma[z_0]Lx)(t). \quad (9.1)$$

Here $L : \mathbb{R}^d \to \mathbb{R}^m$ is a linear map with the Euclidean norm estimate $\|L\| \leq \ell$; $\Gamma[z_0]$ is an operator which transforms functions $u : \mathbb{R}_+ \to \mathbb{R}^m$, $\mathbb{R}_+ = [0, +\infty)$, to functions $z : \mathbb{R}_+ \to \mathcal{Z}$, where $\mathcal{Z}$ is a complete metric space equipped with a metric $\rho_z$; the argument $z_0$ represents initial memory; and, finally, $g : \mathbb{R}^d \times \mathcal{Z} \to \mathbb{R}^d$ is a global Lipschitz continuous function. As usual, the notation $(\Gamma[z_0]u)(t)$ refers to the value of the function $z = \Gamma[z_0]u$ at the time $t$. 
Let $W^{1,1}_t$ be the Banach space of absolutely continuous functions $u : [0, t] \rightarrow \mathbb{R}^m$ equipped with the standard norm

$$\|u\|_{W^{1,1}_T} = |u(0)| + \int_0^t |u'(s)| \, ds$$

and define

$$W^{1,1}_{loc} = \left\{ u : \mathbb{R}^+ \rightarrow \mathbb{R}^m, \, u_{|[0,t]} \in W^{1,1}_t \right\}.$$

**Definition 9.1.4.** A nonlinearity $\Gamma : W^{1,1}_{loc} \times Z \rightarrow C(\mathbb{R}^+; \mathbb{R}^m)$ is called a normal nonlinearity \([23]\) with threshold $h > 0$ if it satisfies the Volterra property:

$$u(s) = v(s), \quad 0 \leq s \leq t \quad \text{implies} \quad (\Gamma[z_0]u)(t) = (\Gamma[z_0]v)(t) \quad \text{for all} \quad t \geq 0,$$

the semigroup property:

$$(\Gamma[(\Gamma[z_0]u)(t_1)]v)(t_2 - t_1) \equiv (\Gamma[z_0]u)(t_2) \quad \text{where} \quad v(t) = u(t - t_1),$$

the Lipschitz condition N1 and the contraction property N2 below:

**N1:** There exists a constant $\gamma_u > 0$ such that for every $z_0 \in Z$, every $t \geq s \geq 0$ and every $u, v \in W^{1,1}_t$ the inequality

$$\rho_z((\Gamma[z_0]u)(s), (\Gamma[z_0]v)(s)) \leq \gamma_u \|u - v\|_{W^{1,1}_T}$$

holds;

**N2:** There exists a continuous and bounded function $q : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $q(\alpha) < 1$ for $\alpha > h$ such that

$$\rho_z((\Gamma[z_0]u)(t), (\Gamma[z_1]u)(t)) \leq q(\text{osc}_t u)\rho_z(z_0, z_1)$$

holds for all $t \geq 0$, all $z_0, z_1 \in Z$ and all $u \in W^{1,1}_t$. Here $\text{osc}_t u$ is defined as follows

$$\text{osc}_t u := \sup_{0 \leq \tau, \sigma \leq t} |u(\tau) - u(\sigma)|.$$

Although the definition seems to be complicated, numerous hysteresis nonlinearities are in fact normal nonlinearities. Examples are the so-called stop or von Mises nonlinearity \([77, 79]\), and certain modifications of the Preisach nonlinearity \([126]\).

**Theorem 9.1.5.** For sufficiently small $\varepsilon$ it is possible to introduce metrics $\rho^s$ in $\mathbb{R}^d_s \times Z$ and $\rho^u$ in $\mathbb{R}^d_u$ such that the shift operator $G$ along trajectories of (9.1) is split-hyperbolic in $U \times Z$, where $U$ is a small neighborhood of $x_T$.

**Corollary 9.1.6.** There exists $\varepsilon_1$ such that for all $0 < \varepsilon < \varepsilon_1$, the shift operator $G$ has a unique fixed point in $U \times Z$. 

9.1.3 Split-Hyperbolicity and Strong Compatibility

A rectification of Theorem 9.1.5, which has been used in computer aided analysis of chaotic behavior, is formulated here, see [118] and Sect. 9.3.1 below.

For any positive integer \( m \), let
\[
\Sigma(m) = \Sigma(\{1, \ldots, m\})
\]
denote the totality of all bi-infinite sequences
\[
\omega = \{\omega_i\}_{i=-\infty}^{\infty} \quad \text{with} \quad \omega_i \in \{1, \ldots, m\} \quad \text{for} \quad i = 0, \pm 1, \pm 2, \ldots,
\]
and let \( \sigma = \sigma_m \) denote the (left) shift on \( \Sigma(m) \) given by
\[
\sigma_m(\omega) = \omega' = (\ldots, \omega_{i-1}', \omega_0', \omega_1', \ldots)
\]
where \( \omega_i' = \omega_{i+1} \). In addition, let
\[
A = (a_{i,j}), \quad i, j = 1, \ldots, m,
\]
be a square \( m \times m \)-matrix, the components of which are either zeros or ones, and introduce the set
\[
\Sigma_A = \{ \omega \in \Sigma(m) : a_{\omega_i,\omega_{i+1}} = 1, \ i = 0, \pm 1, \pm 2, \ldots \}.
\]
The set \( \Sigma_A \) is shift invariant and the restriction \( \sigma_A \) of \( \sigma_m \) to \( \Sigma_A \) is a topological Markov chain with the matrix \( A \).

Let \( f : M \to M \) be a continuous map in a complete metric space \( M \) with the metric \( \rho \). A trajectory of \( f \) (or, to be more precise, of the dynamical system generated by \( f \)) is a sequence
\[
x = \{x_i\}_{i=-i_-}^{\infty}
\]
satisfying
\[
x_{i+1} = f(x_i), \quad i = -i_-, \ldots, 0, 1, 2, \ldots,
\]
where \( 0 \leq i_- < \infty \) (note that \( i_- = i_-(x) \) depends on the particular trajectory \( x \)). Let \( \sigma_f \) be the left shift map naturally defined on the set \( \operatorname{Tr}(f) \) of bi-infinite trajectories of \( f \). It is convenient to endow the sets \( \Sigma_A \) and \( \operatorname{Tr}(f) \) with the usual metrics:
\[
d(\omega, \omega') = \sum_{i=-\infty}^{\infty} \frac{|\omega_i - \omega_i'|}{2|i|}, \quad d(x, x') = \sum_{i=-\infty}^{\infty} \frac{\rho(x_i, x_i')}{2|i|}.
\]

**Definition 9.1.7.** Let \( \mathcal{X} = (X_1, \ldots, X_m) \) be a finite family of compact connected subsets of \( M \). A continuous map \( f \) is said to be strongly \( (\mathcal{X}, \sigma_A) \)-compatible if for any \( \omega \in \Sigma_A \) there exists a unique trajectory \( x = \varphi(\omega) \in \operatorname{Tr}(f) \) satisfying \( x_i \in X_{\omega_i} \).
The following assertion is well known.

**Lemma 9.1.8.** Let $f$ be strongly $(\mathcal{X}, \sigma_A)$-compatible. Then the map $\varphi$ has the following properties

(i) A shift of $\omega \in \Sigma_A$ induces a shift of the trajectory $\varphi(\omega)$: $\varphi \sigma_A = \sigma_f \varphi$;
(ii) if $\omega \in \Sigma_A$ is $p$-periodic, then the trajectory $x = \varphi(\omega)$ is also $p$-periodic;
(iii) the map $\varphi$ is continuous.

**Theorem 9.1.9.** Suppose that the nonempty balls in the metric spaces $M_i^s$, $M_i^u$ are connected, that is cannot be represented as a disjoint union of two nonempty sets each of which is both relatively open and relatively closed. Suppose that $g_{i,j} = h_j^{-1} f h_i$ is $s$-hyperbolic on $B_i[\delta^u, \delta^s], B_j[\delta^u, \delta^s]$ whenever $a_{i,j} = 1$. Suppose also that

$$\rho_j(g_{i,j}(\tilde{y}_i), \tilde{y}_j) < \min \left\{ \frac{\delta^s}{a^s}, \frac{\delta^u}{a^u}, \delta^u \right\}$$

whenever $a_{i,j} = 1$. Then $f$ is strongly $(\mathcal{X}, \sigma_A)$-compatible, where $\mathcal{X} = \{ X_i = h_i(B_i[\delta^u, \delta^s]) : i = 1, \ldots, m \}$.

This follows from Theorem 9.1.3. Various properties of split-hyperbolicity were investigated in [121].

### 9.2 Topological Hyperbolicity

Topological hyperbolicity is a tool that can be used in the computer-aided analysis of long periodic trajectories and chaotic behavior. It has been applied for analysis various interesting phenomena such as piece-wise systems [119, 144], hysteresis in the trade cycle [10, 96], oscillators with hysteresis [22], complicated behavior in a truncated Lang–Kobayashi model [127], chaotic Wave Patterns in KdVB-type equation [36], emerging avian influenza virus [106] and Sect. 9.4 below. Note also recent applications to the investigation of chaos in singularly perturbed systems, see [120, 122] and Sect. 9.5 below, are not computer aided. Other toolboxes have been developed for computer-aided analysis of chaotic behavior: we mention the Tacker’s method [136, 137], and Michaikov-Mrozek method [97, 98] (see also some further developments in [54, 99]) and Zgliczynski method [142, 143]. Each of the aforementioned methods have some advantages and disadvantages. In particular, the semi-hyperbolicity approach seems to be more robust with respect to higher-dimensional and hysteresis type perturbations, and it may be more convenient in analysis of multi-rate systems.

Informally, a parallelogram $P$ is said to be **topologically hyperbolic** if the image of $P$ under a continuous mapping $F$, $F(P)$, intersects $P$ in such a way
that they form a distorted ‘cross-shape’ with one another. A topologically hyperbolic parallelogram always contains a fixed point of the mapping $F$. The mapping $F$ ‘expands’ along the unstable (dashed) direction and ‘contracts’ along the stable (solid) direction, see Fig. 9.1. Note that a ‘stretching’ behavior is observed in one direction (the unstable direction) and a ‘compression’ is observed in the other (stable) direction. If $P$ and its image, $F(P)$ intersect in this manner, then, from the two dimensional version of topological degree theory [78] or [38], there exists a fixed point of the mapping $F$ in the intersection of $P$ and $F(P)$.

![Fig. 9.1. A topologically hyperbolic parallelogram $P$ and its image $F(P)$.](image1)

Next, suppose that there is a chain of parallelograms $P_1, \ldots, P_q$, where the image of the preceding parallelogram in the chain intersects the next parallelogram in the manner outlined above. Figure 9.2 shows three such parallelograms, $P_1$, $P_2$ and $P_3$, and their images under the mapping $F$. In this figure, the image of $P_1$, $F(P_1)$, crosses $P_2$, $F(P_2)$ crosses $P_3$, and $F(P_3)$ crosses $P_1$. Note that the set $F(P_1)$ intersects $P_1$ in a ‘cross’ shape, in addition to intersecting $P_2$. Therefore, there exist periodic orbits with any given period that are greater than or equal to 3. This is an example of a 9-periodic orbit. Then

![Fig. 9.2. A topologically hyperbolic chain of three parallelograms.](image2)
the Product Theorem guarantees that there exists a periodic trajectory with minimal period, in this case, with period 3, the elements of which belong to the corresponding intersections. This assertion is also true for a chain of parallelograms of arbitrary length \( q > 1 \).

Note that in Fig. 9.2, the parallelogram \( P_1 \) is also in a cross-type position with respect to its image \( F(P_1) \). Applying the Product Theorem once more, we can guarantee that there exist periodic orbits with any given period that are greater than, or equal to 3. In Fig. 9.2, there exists a 9-periodic orbit, the elements of which belong to the intersections of the parallelograms and their images. This theorem holds for a chain of parallelograms of arbitrary length \( q > 1 \).

More rigorously, topological hyperbolicity may be defined as follows. Fix two positive integers \( d_u, d_s \) such that \( d_u + d_s = d \). The indices ‘\( u \)’ and ‘\( s \)’ signify ‘unstable’ and ‘stable’, respectively. Let \( V \) and \( W \) be bounded, open and convex product-sets

\[
V = V^{(u)} \times V^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_u}, \quad W = W^{(u)} \times W^{(s)} \subset \mathbb{R}^{d_u} \times \mathbb{R}^{d_s},
\]

satisfying the inclusions \( 0 \in V \), \( 0 \in W \), and let

\[
f : V \to \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}
\]

be a continuous mapping. It is convenient to treat \( f \) as the pair \( (f^{(u)}, f^{(s)}) \), where

\[
f^{(u)} : V \to \mathbb{R}^{d_u} \quad \text{and} \quad f^{(s)} : V \to \mathbb{R}^{d_s}.
\]

**Definition 9.2.1.** The mapping \( f \) is \((V,W)\)-hyperbolic, if

\[
f^{(u)} \left( \partial V^{(u)} \times V^{(s)} \right) \cap \overline{W^{(u)}} = \emptyset, \quad (9.2)
\]

\[
f(V) \cap \left( \overline{W^{(u)}} \times (\mathbb{R}^{d_s} \setminus W^{(s)}) \right) = \emptyset \quad (9.3)
\]

and

\[
\deg(f^{(u)}, V^{(u)}) \neq 0. \quad (9.4)
\]

Condition (9.2) means geometrically that the image of the ‘unstable’ boundary, \( \partial V^{(u)} \times V^{(s)} \) of \( V \), does not intersect the infinite cylinder \( C = \overline{W^{(u)}} \times \mathbb{R}^{d_s} \). Relationship (9.3) means that the image of the entire set \( f(V) \) can intersect the cylinder \( C \) only through its central fragment \( \overline{W^{(u)}} \times W^{(s)} \).

Thus, condition (9.2) means that the mapping ‘expands in a rather weak sense along the first, unstable ‘\( u \)’-coordinate in the Cartesian product \( \mathbb{R}^{d_u} \times \mathbb{R}^{d_s} \), whereas condition (9.3) describes a type of ‘contraction’ along the second, stable ‘\( s \)’-coordinate. Condition (9.4) means that the rotation of the vector field \( f^{(u)} \) at \( V^{(u)} \) is non-zero, that is there exists a fixed point in the set \( V^{(u)} \).
Establishing the \((V, W)\)-hyperbolicity of a particular group of sets provides information on the existence of long periodic trajectories of a particular mapping. In addition, \((V, W)\)-hyperbolicity may be used to establish the existence of chaotic trajectories of such a mapping.

Let \(X = (X_0, \ldots, X_{m-1})\) be a finite family of compact connected subsets of \(\mathbb{R}^d\) and \(A = (a_{i,j}), \ i, j = 1, \ldots, m\).

**Definition 9.2.2.** A continuous mapping \(f\) is called \((\mathcal{X}, \sigma_A)\)-compatible if there exists a mapping \(\varphi: \Sigma_A \to \text{Tr}(f)\) which satisfies the following requirements:

- **R1:** the trajectory \(x = \varphi(\omega)\) satisfies \(x_i \in X_{\omega_i}\) for each \(\omega \in \Sigma_A\) and all integers \(i\);
- **R2:** \(\varphi \sigma_A = \sigma f \varphi\): a shift of \(\omega \in \Sigma_A\) induces a shift of the trajectory \(\varphi(\omega)\);
- **R3:** if \(\omega \in \Sigma_A\) is \(p\)-periodic, then the trajectory \(x = \varphi(\omega)\) is also \(p\)-periodic.

This definition is similar, but significantly weaker than the definition of strong compatibility 9.1.7: neither the uniqueness of the mapping \(\varphi\) nor its continuity are assumed, so \(\varphi\) need not be a semi-conjugacy [65, p. 68]. On the other hand, the subshift \(\sigma_A\) is a factor [65] of a restriction of the system \(f\) to some set \(S \subset \bigcup X_i\), providing that \(f\) is \((\mathcal{X}, \sigma_A)\)-compatible. The \((\mathcal{X}, \sigma_A)\)-compatible mappings have some features of chaotic behavior if \(A\) has some and if sufficiently many subfamilies of \(\mathcal{X}\) have empty intersections.

**Theorem 9.2.3.** Let \(A\) be a square \(m \times m\)-matrix whose entries are either zeros or ones, \(h_i: \mathbb{R}^{d_u} \times \mathbb{R}^{d_u} \to \mathbb{R}^d\) be homeomorphisms, and \(V_i\) be bounded, open and convex product sets. Suppose that \(g_{i,j} = h_j^{-1} f h_i\) is \((V_i, V_j)\)-hyperbolic whenever \(a_{i,j} = 1\). Then \(f\) is \((\mathcal{X}, \sigma_A)\)-compatible for \(\mathcal{X} = \{h_i(V_i), i = 1, \ldots, m\}\).

### 9.3 Semiconductor Lasers Dynamics

External cavity semiconductor lasers present many interesting features for both technological applications and fundamental non-linear science. Their dynamics has been the subject of numerous studies over the last twenty years. Motivations for these studies vary from the need for stable tunable laser sources, for laser cooling or multiplexing, to the general understanding of their complex stability and chaotic behavior. In [118] the split-hyperbolicity concept was applied to study the dynamics of the truncated Lang–Kobayashi equations that describe the behavior of semiconductor lasers with feedback. For particular values of the parameters it was rigorously proved that the system exhibits chaotic behavior in the Smale sense: some power of a suitable first return map is topologically conjugate to the left shift operator in the set of symbolic sequences. The proof is computer assisted, with all errors estimated and taken into account. The main result from [118] is briefly described below.
9.3.1 Truncated Lang–Kobayashi equations

A typical experiment is usually described by a set of delay differential equations introduced by Lang & Kobayashi [81]:

\[
\begin{align*}
\dot{E} &= \kappa (1 + i\alpha) (N - 1) E + \gamma e^{-i\varphi_0} E (t - \tau), \\
\dot{N} &= -\gamma_n (N - J + |E|^2 N).
\end{align*}
\] (9.5)

Here \( E \) is the complex amplitude of the electric field, \( N \) is the carrier density, \( J \) is the pumping current, \( \kappa \) is the field decay rate, \( 1/\gamma_n \) is the spontaneous time scale, \( \alpha \) is the linewidth enhancement factor, \( \gamma \) represents the feedback level, \( \varphi_0 \) is the phase of the feedback if the laser emits at the solitary laser frequency and \( \tau \) is the external cavity round trip time. A typical setup is shown on Fig. 9.3. Numerical simulations of these equations have successfully reproduced many experimental observations, such as mode hopping between external cavity modes [101] and a period doubling route to chaos [83]. However there are few analytical results since delay equations are nonlocal.

This model was recently reduced to a 3-D dynamical system describing the temporal evolution of the laser power \( P = |E|^2 \), carrier density \( N \) and phase difference \( \eta(t) = \varphi(t) - \varphi(t - \tau) \). This was achieved by assuming

\[ P(t - \tau) = P(t) \]

together with the approximation

\[ \varphi = \eta/\tau + \dot{\eta}/2. \]

This expression remains valid when the phase fluctuates on a time scale much shorter than the re-injection time \( \tau \). Under these approximations the Lang–Kobayashi equations (9.5) reduce to:

\[
\begin{align*}
\dot{P} &= 2 (\kappa (N - 1) + \gamma \cos (\eta + \varphi_0)) P, \\
\dot{N} &= -\gamma_n (N - J + PN), \\
\dot{\eta} &= -\frac{1}{\tau_s} \eta + 2\kappa \alpha (N - 1) - 2\gamma \sin (\eta + \varphi_0).
\end{align*}
\]

This model was successfully used to describe low frequency fluctuations commonly observed in semiconductor lasers with optical feedback, but its behavior for a low feedback level (see Fig. 9.4) has not yet been investigated.
9.3.2 Poincaré Map

To improve some estimates crucial for the numerical computations and to somehow center the graph of trajectories while keeping all parameters as rational numbers it is convenient to introduce new scaled variables:

\[ x(1) = \log P, \quad x(2) = 5N - \frac{97}{20}, \quad x(3) = \frac{5(\eta + 2)}{16}. \]

Choosing numerical values for the parameters in line with [127] the system under investigation can be rewritten as

\[
\begin{align*}
\frac{dx(1)}{dt} &= -\frac{3}{50} + \frac{2}{5} x(2) + \frac{3}{25} \cos \left( 1 - \frac{16}{5} x(3) \right), \\
\frac{dx(2)}{dt} &= \frac{609}{2000} - \frac{3}{100} x(2) - \frac{3}{100} e^{x(1)} \left( \frac{97}{20} + x(2) \right), \\
\frac{dx(3)}{dt} &= -\frac{7}{160} + \frac{3}{8} x(2) - \frac{1}{50} x(3) + \frac{3}{80} \sin \left( 1 - \frac{16}{5} x(3) \right)
\end{align*}
\]

and in vector notation as

\[ \dot{x} = \mathcal{F}(x) \quad \text{where} \quad x = \left( x(1), x(2), x(3) \right) \]

for the system (9.6).

For the elements of the plane \( x(3) = 0 \) the notation \( x = (x(1), x(2)) \) will be used as a synonym for \( x = (x(1), x(2), 0) \). Note that the condition of transverse intersection with \( W \), \( x(3) \neq 0 \), is broken only for the set of \( W \) defined by

\[ x(2) = \frac{7}{60} - \frac{3}{10} \sin(1) = x_T(2). \]
Denote by \( S \) the half-plane \( x^{(3)} = 0, x^{(2)} > x_T^{(2)} \). Since \( S \) is, by definition, transversal to trajectories of the system, the Poincaré map \( \Phi \), which is defined as a (partial) map from \( S \) to itself, can be obtained by following trajectories from one intersection with \( S \) to the next.

Let \( \mathcal{R} \) denote the union set of two rectangles \( R_1, R_2 \subset S \) given by

\[
R_1 = \{ (0.32, 0.43) + (-0.3675, 0.3255)\alpha + (0.039, 0.039)\beta : \alpha, \beta \leq 1 \}; \\
R_2 = \{ (-0.04, 0.83) + (0.075, 0)\alpha + (0, 0.1)\beta : \alpha, \beta \leq 1 \}.
\]

**Theorem 9.3.1.** The Poincaré map \( \Phi \) is defined for all \( y \in \mathcal{R} \) and \( \Phi(\mathcal{R}) \subset \mathcal{R} \).

The geometry of the map \( \Phi \) on \( \mathcal{R} \) is illustrated by the numerically constructed Figs. 9.5(a), 9.5(b) and 9.5(c). On the naive level, Fig. 9.5(c) provides a ‘proof’ for the proposition above. However, the detailed justification of this figure is cumbersome. It requires a rigorous proof of continuity of the Poincaré map \( \Phi \) as well as estimates of the accuracy of discrete computer arithmetic and estimates of the precision of the numerical method. A rigorous computer assisted proof of Theorem 9.3.1 was given in [127].

**9.3.3 Main Theorem**

Consider the set \( \mathcal{P} \) consisting of twenty parallelograms \( P_i, i = 1, \ldots, 20 \) given by

\[
P_i = \{ x_i + \alpha p_i + \beta q_i : |\alpha|, |\beta| \leq 0.00002 \}.
\]

Here the coordinates of the center points \( x_i = (x_i^{(1)}, x_i^{(2)}) \) and of the vectors \( p_i = (p_i^{(1)}, p_i^{(2)}) \), \( q_i = (q_i^{(1)}, q_i^{(2)}) \), are given in Table 9.1. Figure 9.6 graphs these parallelograms together with \( R_1, R_2 \).
Theorem 9.3.2. The Poincaré map

By definition the union set $\bigcup P_i$ has 15 connected components $U_1^*, U_2^*, \ldots, U_{15}^*$.
(see Fig. 9.6). Since $\Phi$ is an homeomorphism, Theorem 9.3.2 implies the following corollary.

**Corollary 9.3.3.** The restriction of $\Phi^{38}$ to a suitable $\Phi$-invariant Cantor set $K \subset \bigcup P_i$ is topologically conjugate to the left shift $\sigma_{15}$. Moreover, the inclusion

$$\Phi^{38i}(\varphi(\omega)_0) \in U_{\omega_i}, \quad \omega \in \Sigma(15)$$

is satisfied and the topological entropy is bounded below by

$$\mathcal{E}_{\text{top}}(\Phi) \geq \frac{(\log 15)}{38} > 0.07.$$ 

**Proof.** This follows from the fact that the matrix $A$ is 38-transitive. \qed

### 9.4 Avian Influenza in a Seabird Colony

Novel pathogens pose a significant threat to the health status of man, of domestic animals and of wildlife. Therefore, it is important to know how such pathogens spread in immunologically naive host populations. A model of the long-term dynamics of a novel bird flu pathogen H5N1 in a mixed population of marine birds, which takes into account seasonality effects that affect multiple processes in a seabird population is discussed here. Such seasonal variations may considerably affect the transmission of a pathogen in such a population. In particular, a seasonally perturbed Susceptible-Infected-Recovered (SIR) model is considered. Such systems are known to exhibit complex dynamics as the amplitude of the seasonal perturbation term is increased. Furthermore, it has been long observed that chaotic solutions may appear in such a model: computer simulations of a seasonally perturbed SIR system suggest the existence of chaos in the model. A rigorous proof of the existence of chaos was given in [106] based on the concept of topological hyperbolicity. The main result from [106] is briefly described here.

#### 9.4.1 Basic Model

An SIR (Susceptible-Infected-Recovered) model is used to describe the spread of H5N1 in a seabird colony. According to the classic assumption the population is divided into three classes of epidemiological significance, namely the susceptible birds $S(t)$, the infected bird $I(t)$ and recovered (and immune) birds $R(t)$. The infection is transmitted via direct contacts between the susceptible and the infected birds. frequency-dependent bi-linear (or mass-action) transmission is assumed in which the number of contacts between the susceptibles ($S$) and the infectives ($I$) is independent of the population size. After an instance of infection, the infected bird immediately becomes infectious; that is this model disregards a latent period. This is a reasonable assumption because the latent period of influenza is comparatively short. The infected birds
9.4 Avian Influenza in a Seabird Colony

recover and become immune to the disease at a rate $\gamma$. The size of a colony $N(t)$ is limited only by the availability of the breeding spots, and the number of these are constant. Space is scarce, and a vacant nesting spots are immediately occupied by a candidate. Thus, for a colony the recruitment exactly balances the mortality disregarding whether it is inflicted by the disease (at a rate $\omega_I$) or by natural causes (at a rate $\omega$). Under these assumptions, the SIR model is given by the following system of differential equations (cf. [105]):

\[
\begin{align*}
\dot{S} &= \omega(S + I + R) + \omega_I I - \beta SI - \omega S \\
\dot{I} &= \beta SI - (\gamma + \omega + \omega_I)I \\
\dot{R} &= \gamma I - \omega R.
\end{align*}
\]

(9.7)

The number of adults in a colony is limited by the available nesting spots, and hence is constant. However, the total colony population varies in time according to the birds seasonal breeding pattern: chicks appear increasing the colony population, and then, after some time, the chicks leave the colony. For example, chicks of the common guillemot ($Uria aalge$) leave a colony after about 21 days after the birth, to return to the colony after 5 years to breed. The breeding seasonal pattern of the birds is known, as well as is known an average number of the chicks per an adult bird. Therefore it can be safely assumed that the colony size is a given function of time, $N(t)$. The population is composed of three classes, that is $S(t) + I(t) + R(t) = N(t)$, and hence one of the three equations of system (9.7) may be omitted. Usually, it is the equation for $R(t)$ that is omitted.

However, any of the other two equations may be as well omitted instead, and omitting either of these two, rather than one for $R(t)$, has some advantages. Here, the equation for $S(t)$ is omitted. This choice has an advantages that, for instance, the resulting two-dimensional system

\[
\begin{align*}
\dot{I} &= (\beta(N - I - R) - \gamma - \omega_I - \omega)I, \\
\dot{R} &= \gamma I - \omega R
\end{align*}
\]

is equivalent for SIR and SIRS models, including the models with the vertical transmission. This system always has an infection-free equilibrium state that is conveniently located in the origin. Furthermore, it is readily seen that the positive quadrant is a positive invariant set of this system, and hence any phase trajectory initiated in this quadrant indefinitely remains there. Finally, when the system parameters are constant, this system is globally asymptotically stable [107]. That is, this system always has a globally attractive equilibrium state. Depending on the basic reproduction number $R_0 = \beta/(\gamma + \omega + \omega_I)$, this equilibrium state being either positive (endemic) or infection-free.

Seasonality may be introduced into a SIR model by incorporating a periodical forcing term into the system. For example, the most common approach of introducing seasonality is periodically perturbing the transmission rate $\beta(t)$. However, as mentioned above, the seasonality in a seabird colony is mostly
associated with the seasonal breading pattern of the birds, and respectively with seasonally-varying population size. It is reasonable to assume, therefore, that in this case the seasonally perturbed quantity is the total population \( N(t) \) of the colony. That is, it is assumed that 

\[
N(t) = S(t) + I(t) + R(t)
\]

is a given 1-periodic function, such that \( N(t+1) = N(t) \) holds. This leads to the following system of non-autonomous differential equations:

\[
\begin{align*}
\dot{I} &= (Q(t) - \beta I - \beta R)I, \\
\dot{R} &= \gamma I - \omega R,
\end{align*}
\]

where \( Q(t) \) is a given 1-periodic function defined by the equation 

\[
Q(t) = Q_0(1 + \varepsilon(t)).
\]

Here the constant 

\[
Q_0 = \beta N - (\gamma + \omega + \omega_I)
\]

is the mean value of the function \( Q(t) \), and \( |\varepsilon(t)| < Q_0 \). The seasonal perturbation \( \varepsilon(t) \) may be assumed to be a sinusoidal or step function. The parameter \( Q_0 \) incorporates the effects of the transmission rate \( \beta \), the recovery rate \( \gamma \), the death rates \( \omega, \omega_I \) and, in particular, the population size \( N \). Below it is assumed that 

\[
Q(t) = Q_0(1 - Q_1 \sin 2\pi t).
\]

**9.4.2 Numerical Evidence of Chaotic Trajectories**

The effect of varying the amplitude \( Q_1 \) of 

\[
Q(t) = Q_0(1 - Q_1 \sin 2\pi t)
\]

on the long-term dynamics of the virus in a population has been investigated via numerical experiments. The numerical values used for these experiments were chosen to realistically simulate the dynamics of H5N1 virus in a seabird colony (see [33] for relevant details).

The output of the translation operator for the time of order 1 along a particular trajectory of system (9.8) provides further evidence of chaotic behavior. The output of the translation operator is the solution of system (9.8) shifted by time \( t = 1 \) years. This is achieved by iterating the map from a given initial condition, recording the solution at the end of 1 year and then repeating the iteration from this point. Therefore, the translation operator for system (9.8) is a discrete mapping, which outputs the values of the infected and recovered populations at the end of each year. (Note that it is not a Poincaré mapping.) Let \( \Phi \) denote the time-1 translation operator along trajectories of the seasonally perturbed system with the parameters given in Table 9.2.

The output of \( \Phi \) is plotted in Fig. 9.7(a), after discarding the initial transient values of this. It is noteworthy that this set closely resembles the famous
Table 9.2. Parameter values for the mapping $\Phi$ along numerical trajectories of system (9.8).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Parameter value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total population</td>
<td>$N$</td>
<td>10000</td>
</tr>
<tr>
<td>Basic reproductive number</td>
<td>$R_0$</td>
<td>10</td>
</tr>
<tr>
<td>Transmission rate</td>
<td>$\beta$</td>
<td>0.0924</td>
</tr>
<tr>
<td>Recovery rate</td>
<td>$\gamma$</td>
<td>365/4</td>
</tr>
<tr>
<td>Natural death rate</td>
<td>$\omega$</td>
<td>1/10</td>
</tr>
<tr>
<td>Disease-associated death rate</td>
<td>$\omega_I$</td>
<td>3/4</td>
</tr>
<tr>
<td>Mean value of $Q(t)$</td>
<td>$Q_0$</td>
<td>831.928</td>
</tr>
<tr>
<td>Amplitude</td>
<td>$Q_1$</td>
<td>0.025</td>
</tr>
</tbody>
</table>

strange attractor of the Hénon mapping [59]

$$f(x, y) = (1 - 1.4x^2 + y, 0.3x),$$

shown in Fig. 9.7(b) for comparison. The existence of the set in Fig. 9.7(a) will provide the foundation for the proof of the existence of chaotic trajectories of system (9.8). From this figure, it is also clear that the infected population persists, albeit at very low levels.

![Fig. 9.7.](image)

To prove that the translation operator $\Phi$ is a chaotic mapping, and hence, prove the existence of chaotic trajectories of system (9.8), the topological hyperbolicity method was employed.

9.4.3 Main Assertion

The topological hyperbolicity method will be used to prove that the translation operator $\Phi$ is a chaotic mapping and, hence, to prove the existence of chaotic trajectories of system (9.8).
Theorem 9.4.1. The system \((9.8)\) has periodic orbits with all minimal periods greater than 15.

Proof. The formal definition of topological hyperbolicity will be used here to prove the assertion rigorously. Firstly, define the parallelograms \(P_i\) oriented around the points \(x_i = (I_i, R_i)\) of the quasi-periodic orbit of period 15,

\[
P_i = \left\{ x_i + \kappa_1 a_i^{(u)} p_i + \kappa_2 a_i^{(s)} q_i \right\},
\]

where \(|\kappa_1|, |\kappa_2| \leq 1\). Numerical values for the points \(x_i = (I_i, R_i)\) of the quasi-periodic orbit, sizes of parallelograms \(a_i^{(u)}\) and \(a_i^{(s)}\) and vectors \(p_i\) and \(q_i\), which define the orientations of parallelograms \(P_i\) are given in Table 9.3.

The procedure for locating parallelograms was the following. The first step in identifying suitable topologically hyperbolic parallelograms is to find a quasi-periodic orbit of length \(q\) of the mapping \(\Phi\), the translation operator of the time of order 1 along trajectories of system \((9.8)\). The parallelograms are defined so that they are oriented around the points of the quasi-periodic orbit. The orbit should have elements \(x_i, i = 1, \ldots, q\), in a small vicinity of a fixed point of \(\Phi\) and have elements that are far away from this fixed point. The fixed point may be found using the broken orbits method \([123]\). However, due to the stiffness of system \((9.8)\), this method may not be precise enough for identifying a suitable quasi-periodic point close to the fixed point of \(\Phi\). Instead, an alternative procedure may be used: a small rectangle around the fixed point may be identified, the iterated mapping \(\Phi^q\) may be defined and a loop may be used to search for a suitable quasi-periodic point \(x_q\) such that \(|x_q - \Phi^q(x_q)| \ll 1\). By using smaller rectangles, this point was refined by choosing the point with the smallest norm.

Once a suitable quasi-periodic point is found, a linearization of the iterated mapping \(\Phi^q\) about each point \(x_i, i = 1, \ldots, q\) of the quasi-periodic orbit may be obtained. This gives information about the local dynamics of the system in the neighborhood of each point \(x_i\), and therefore allows the orientations of the parallelograms to be determined. The solution of the linearized system at \(x_i\) is a \(2 \times 2\) matrix. The eigenvectors of this matrix were used as first approximations of the orientations of the parallelograms. However, the sizes \((a_i^{(u)}, a_i^{(s)})\) of the parallelograms had to be adjusted until the desired results were obtained. This is a delicate operation involving much trial and error and experiment. If this procedure is successfully applied, then the mapping \(\Phi\) is topologically hyperbolic.

Denote the product sets in \(\mathbb{R}^2\) by \(V_i = \left\{ (a_i^{(u)}, a_i^{(s)}) \in \mathbb{R}^2 \right\}\), where \(i = 1, 2, \ldots, 15\). These are mapped into the IR plane by the functions

\[
h_i : \mathbb{R}^2 \to \mathbb{R}^2, \quad i = 1, 2, \ldots, 15,
\]

as follows:

\[
h_i : \mathbb{R}^2 \to \mathbb{R}^2, \quad (a_i^{(u)}, a_i^{(s)}) \mapsto x_i + a_i^{(u)} p_i + a_i^{(s)} q_i.
\]
Table 9.3. The points \( x_i = (I_i, R_i) \), sizes of parallelograms \((a_i^{(u)}, a_i^{(s)})\) and vectors \(p_i, q_i\).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(x_i)</th>
<th>((a_i^{(u)}, a_i^{(s)}))</th>
<th>(p_i)</th>
<th>(q_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(37.1130, 9307.10)</td>
<td>(3.8, 6)</td>
<td>(-0.19358, 0.98108)</td>
<td>(0.38154, -0.92435)</td>
</tr>
<tr>
<td>2</td>
<td>(36.6294, 9309.91)</td>
<td>(8, 8)</td>
<td>(-0.26387, 0.96456)</td>
<td>(0.38422, -0.92324)</td>
</tr>
<tr>
<td>3</td>
<td>(39.0891, 9300.52)</td>
<td>(12, 5)</td>
<td>(-0.25980, 0.96565)</td>
<td>(0.37556, -0.92680)</td>
</tr>
<tr>
<td>4</td>
<td>(33.9295, 9319.77)</td>
<td>(4, 15)</td>
<td>(-0.04955, -0.99877)</td>
<td>(-0.39408, 0.91908)</td>
</tr>
<tr>
<td>5</td>
<td>(45.0663, 9277.67)</td>
<td>(25, 12)</td>
<td>(-0.25791, 0.96620)</td>
<td>(0.35547, -0.93469)</td>
</tr>
<tr>
<td>6</td>
<td>(24.5481, 9354.05)</td>
<td>(5, 10)</td>
<td>(0.28701, -0.95792)</td>
<td>(-0.43556, 0.90016)</td>
</tr>
<tr>
<td>7</td>
<td>(73.2449, 9150.18)</td>
<td>(4, 20)</td>
<td>(-0.00044, 1.00000)</td>
<td>(0.25431, -0.96712)</td>
</tr>
<tr>
<td>8</td>
<td>(5.4876, 9429.40)</td>
<td>(2.5, 0.3)</td>
<td>(0.23409, -0.97222)</td>
<td>(-0.98782, 0.15558)</td>
</tr>
<tr>
<td>9</td>
<td>(25.3733, 8601.20)</td>
<td>(2, 10)</td>
<td>(0.52190, -0.85301)</td>
<td>(0.08882, 0.99605)</td>
</tr>
<tr>
<td>10</td>
<td>(0.0510, 9431.51)</td>
<td>(0.9, 0.8)</td>
<td>(-0.00032, -1.00000)</td>
<td>(-0.00610, -0.99998)</td>
</tr>
<tr>
<td>11</td>
<td>(0.6692, 8535.46)</td>
<td>(1, 2.5)</td>
<td>(-0.00023, -1.00000)</td>
<td>(-0.02036, -0.99979)</td>
</tr>
<tr>
<td>12</td>
<td>(0.0524, 9339.88)</td>
<td>(0.1, 0.7)</td>
<td>(0.04666, 0.99891)</td>
<td>(0.00586, 0.99988)</td>
</tr>
<tr>
<td>13</td>
<td>(90.2127, 9153.63)</td>
<td>(1, 6)</td>
<td>(-0.05102, 0.99870)</td>
<td>(0.27353, -0.96186)</td>
</tr>
<tr>
<td>14</td>
<td>(31.1815, 9321.67)</td>
<td>(5.5, 5)</td>
<td>(-0.34241, 0.93955)</td>
<td>(0.39658, -0.91800)</td>
</tr>
<tr>
<td>15</td>
<td>(38.5150, 9304.28)</td>
<td>(5, 4)</td>
<td>(0.29173, -0.95650)</td>
<td>(-0.37898, 0.92541)</td>
</tr>
</tbody>
</table>

It is easy to see that each \( h_i \) is a homeomorphism and thus, the mappings \( h_j^{-1} \Phi h_i \) are meaningful. Note also that \( P_i = h_i(V_i) \).

**Lemma 9.4.2.** The mapping \( h_j^{-1} \Phi h_i \) is \((V_i, V_j)\)-hyperbolic for all \( j = i + 1 \) mod 15. Furthermore, the mapping \( h_1^{-1} \Phi h_1 \) is \((V_1, V_1)\)-hyperbolic, \((V_2, V_1)\)-hyperbolic, \((V_3, V_1)\)-hyperbolic, and \((V_3, V_2)\)-hyperbolic.

Introduce a 15 \times 15 matrix \( A = (a_{i,j}) \) \( i, j = 1, \ldots, 20 \) by

\[
a_{1,1} = a_{2,1} = a_{3,1} = a_{15,1} = 1,
\]

\[
a_{i,i+1} = 1 \text{ for all } 1 \leq i \leq 14,
\]

\[
a_{i,j} = 0 \text{ if pair } (i, j) \text{ is none of above.}
\]

**Theorem 9.4.3.** The Poincaré map \( \Phi \) is \((\mathcal{P}, \sigma_A^R)\) compatible.

Clearly, some pairs of parallelograms \( P_i, P_j \) do not intersect. Therefore, Theorem 9.4.3 implies Theorem 9.4.1 \( \Box \)

A lower bound on the topological entropy \( E_{\text{top}}(\Phi) \) of the mapping \( \Phi \), can also be calculated. The topological entropy is greater than or equal to the natural logarithm of the maximal eigenvalue \( \lambda_{\text{max}} \) of the \( k \)-transitive matrix \( A \), namely

\[
E_{\text{top}}(\Phi) \geq \log \lambda_{\text{max}} = 0.693168.
\]
9.4.4 Existence of Chaotic Behavior

Let \( \mathcal{U} = \{U_1, \ldots, U_m\}, \ m > 1, \) be a family of disjoint subsets of \( \mathbb{R}^d \) and let us denote the set of one-sided sequences \( \omega = \omega_0, \omega_1, \ldots \) by \( \Sigma^R(m) \). Sequences in \( \Omega^R(m) \) will be used to prescribe the order in which sets \( U_i \) are to be visited. For \( x \in \bigcup_{i=1}^m U_i \) define \( I(x) \) to be the number \( i \) satisfying \( x \in U_i \).

A small modification of Definition 7.3.1 will be used below.

**Definition 9.4.4.** A mapping \( f \) is called \((\mathcal{U}, k)\)-chaotic, if there exists a compact \( f \)-invariant set \( S \subset \bigcup_i U_i \) with the following properties:

P1: for any \( \omega \in \Sigma^R_m \) there exists \( x \in S \) such that \( f^{ik}(x) \in U_{\omega_i} \) for \( i \geq 1 \);

P2: for any \( q \)-periodic sequence \( \omega \in \Sigma^R_m \) there exists a \( q \)-periodic point \( x \in S \) with \( f^{ik}(x) \in U_{\omega_i} \);

P3: for each \( \eta > 0 \) there exists an uncountable subset \( S(\eta) \) of \( S \), such that the simultaneous relationships

\[
\limsup_{i \to \infty} |I(f^{ik}(x)) - I(f^{ik}(y))| \geq 1, \quad \liminf_{i \to \infty} |f^{ik}(x) - f^{ik}(y)| < \eta
\]

hold for all \( x, y \in S(\eta), x \neq y \).

Similarly to Definition 7.3.1, this definition describes rigorously the important attributes of chaotic behavior of a mapping \( f : \mathbb{R}^d \to \mathbb{R}^d \). These are the sensitive dependence on initial conditions, an abundance of periodic trajectories and an irregular mixing effect, that is the existence of a finite number of disjoint sets which can be visited by trajectories of \( f \) in any prescribed order.

This definition of chaos can be applied to the family \( \mathcal{U} \) of connected components of the set of 15 parallelograms that are defined above. By inspection, the family \( \mathcal{U} \) has 9 elements. Therefore, the following theorem was established:

**Theorem 9.4.5.** The mapping \( \Phi \) is \((\mathcal{U}, 26)\)-chaotic.

9.5 Chaotic Canard-Type Trajectories

In this section topological methods are used to analyze canard-type periodic and chaotic trajectories following [122].

Consider the slow-fast system

\[
\dot{x} = X(x, y, \varepsilon), \quad \varepsilon \dot{y} = Y(x, y, \varepsilon),
\]

where \( x \in \mathbb{R}^2, \ y \in \mathbb{R}^1, \) and \( \varepsilon > 0 \) is a small parameter.

The subset
of the phase space is called a slow surface of the system (9.9): on this surface the derivative $\dot{y}$ of the fast variable is zero, the small parameter $\varepsilon$ vanishes. The part of $S$ where

$$Y'_y(x,y,0) < 0 \quad (> 0)$$

is called attractive (repulsive, respectively). The subsurface $L \subset S$ which separates attractive and repulsive parts of $S$ is called a turning subsurface.

Trajectories which at first pass along, and close to, an attractive part of $S$ and then continue for a while along the repulsive part of $S$ are called canards or duck-trajectories [15,100].

Topological hyperbolicity will be used to prove the existence of and to locate with a given accuracy chaotic canards of system (9.9). The canards which are found in this way are topologically robust: they vary only slightly if the right hand side of the system is disturbed.

**Assumption 9.5.1** Suppose that the function $X$ is continuously differentiable and that the function $Y$ is twice continuously differentiable.

A point $(x_c,y_c)$ is called a critical point of the system (9.9), if it satisfies the equations

$$\langle X(x_c,y_c,0), Y'_x(x_c,y_c,0) \rangle = 0,$$

$$Y(x_c,y_c,0) = 0,$$

$$Y'_y(x_c,y_c,0) = 0.$$

This is a system of three equations with three variables, so in general case it is expected to have solutions. The existence of critical points is important for the phenomenon of canard type solutions, because every canard, which first goes along the stable slow integral manifold for $t < 0$ and then along the unstable slow integral manifold for $t > 0$, must pass through a small vicinity of a critical point. Thus, attention will focus on critical points and the behavior of (9.9) in a vicinity of such a point.

A critical point is called non-degenerate, if the following inequalities hold:

$$X(x_c,y_c,0) \neq 0, \quad Y'_x(x_c,y_c,0) \neq 0, \quad Y''_{yy}(x_c,y_c,0) \neq 0.$$ 

Note that non-degeneracy is stable with respect to small perturbations of the right-hand side of (9.9). Only non-degenerate critical points are considered here. Without loss of generality it can be assumed that the critical point is situated at the origin:

$$x_c = y_c = 0.$$ 

(9.14)

Consider the auxiliary system

$$\dot{x} = X(x,y,0), \quad \langle (\dot{x}, \dot{y}), Y'_x(x,y) \rangle = 0.$$ 

(9.15)
If the initial point \((x_0, y_0)\) lies on the slow surface
\[ S_0 = \{(x, y) \in \mathbb{R}^3 : Y(x, y, 0) = 0\} \quad (9.16) \]
of the system (9.9), then (9.15) is equivalent to
\[ \dot{x} = X(x, y, 0), \quad (x, y) \in S_0. \quad (9.17) \]
The system (9.17) is important because it describes the singular limits of the solutions of (9.9) which lie on the slow surface. Equations (9.15) can also be rewritten in the form:
\[ \dot{x} = X(x, y, 0), \quad \dot{y}Y_y(x, y, 0) = -\langle X(x, y, 0), Y_x(x, y, 0) \rangle, \quad (9.18) \]
which, by (9.13), has a singularity at the origin. Therefore, the existence and uniqueness of a solution of (9.18) which starts at the origin requires an additional assumption and will be discussed in detail later.

To describe the dynamics of the system (9.9) near the origin, a special coordinate system \((x^{(1)}, x^{(2)}, y)\) is introduced in the three-dimensional space of pairs \((x, y)\). Choose \(x^{(1)}\) to be co-directed with the gradient \(Y'(0, 0, 0, 0)\), and \(x^{(2)}\) to be orthogonal to \(x^{(1)}\) and \(y\). In the coordinate system \((x^{(1)}, x^{(2)}, y)\) the gradient of \(Y(x^{(1)}, x^{(2)}, y, 0)\) at the origin takes the form
\[ Y'(0, 0, 0, 0) = (\xi, 0, 0), \quad \xi > 0, \quad (9.19) \]
and the equation (9.9) takes the form
\[ \dot{x}^{(1)} = X^{(1)}(x^{(1)}, x^{(2)}, y, 0), \]
\[ \dot{x}^{(2)} = X^{(2)}(x^{(1)}, x^{(2)}, y, 0), \]
\[ \dot{y} = Y(x^{(1)}, x^{(2)}, y, 0). \]

Equations (9.19) and (9.11) imply
\[ X^{(1)}(0, 0, 0, 0) = 0. \]
The non-degeneracy of the origin guarantees the inequalities
\[ X^{(2)}(0, 0, 0, 0) > 0, \quad Y''_{yy}(0, 0, 0, 0) = \zeta > 0, \]
by changing, if necessary, the directions of the \(x^{(2)}\) and \(y\) axes.

The existence of canards and uniqueness of solutions of (9.18) is guaranteed by the following assumption.

**Assumption 9.5.2**
\[ 2X^{(1)}_{x^{(2)}}(0, 0, 0, 0)Y''_{yy}(0, 0, 0, 0) - X^{(1)}(0, 0, 0, 0)Y''_{x^{(2)y}}(0, 0, 0, 0) < 0, \]
\[ X^{(1)}_{y}(0, 0, 0, 0) > 0. \]
Lemma 9.5.3. There exist $T_a < 0 < T_r$ such that system \((9.15)\) with the initial condition

\[ x(0) = y(0) = 0, \]

has the unique solution

\[ w^*(t) = (x^*(t), y^*(t)), \quad T_a < t < T_r, \]

and the inequalities

\[ Y_y(x^*(t), y^*(t)) > 0, \quad 0 < t < T_r, \]
\[ Y_y(x^*(t), y^*(t)) < 0, \quad T_a < t < 0 \]

hold. In other words, the half-trajectory \((x^*(t), y^*(t)), T_a < t < 0\), lives on the attractive part of the slow surface \((9.16)\), while the half-trajectory \((x^*(t), y^*(t)), 0 < t < T_r\), lives on the repulsive part.

Lemma 9.5.3 implies a strict restriction on the possible location of canards of system \((9.9)\) that passing near the origin: such canards should follow closely the solution \(w^*(t)\) for a certain interval \(t_a < 0 < t_r\). The above argument shows that a canard should have segments of fast motion from a small neighborhood of some point of the repulsive part of \(w^*(t)\) to a small neighborhood of the attractive part of \(w^*(t)\); this fast motion is, consequently, almost vertical (that is almost parallel to the \(y\) axis). More precisely, if there is a limit of a canard as \(\varepsilon \to 0\), then the limiting curve has necessarily vertical segments connecting the repulsive and attractive parts of \(w^*(t)\).

Assumption 9.5.4 Let the trajectories \(\Gamma_a\) and \(\Gamma_r\) intersect \(K \geq 2\) times, that is, there exist \(\tau_i\) and \(\sigma_i, i = 1, \ldots, K\), such that

\[ x^*(\tau_i) = x^*(\sigma_i) = x_i^*, \]

with

\[ T_a < \tau_i < 0 < \sigma_i < T_r, \]

and \(\tau_i \neq \tau_j\) for \(i \neq j\). It is also required that the curves \(\Gamma_a\) and \(\Gamma_r\) do not self-intersect and that

\[ Y(x_i^*, y) < 0, \quad y \in [y^*(\tau_i), y^*(\sigma_i)], \quad i = 1, \ldots, K. \]

Assumption 9.5.5 All the intersections are transversal, \(i = 1, \ldots, K\).

Theorem 9.5.6. There exist disjoint sets \(P_i \subset \Pi\), the plan \(y = 0\), with \(P_i \ni x^*(\tau_i)\), and \(\varepsilon_0 > 0\) such that for any \(\varepsilon < \varepsilon_0\) the Poincaré map \(\Phi : \bigcup_i P_i \to \Pi\) of system \((9.9)\) is \((U, 1)\)-chaotic, where \(U = \{P_1, \ldots, P_K\}\).

Let \(S \subset \bigcup_i P_i\), \(i = 1, \ldots, K\), be the compact \(\Phi\)-invariant set from Definition 9.4.4; its existence is guaranteed by Theorem 9.5.6. Denote by \(\mathcal{S}_S\) the topological entropy of the Poincaré map \(\Phi\) with respect to the compact set \(S\), see \([65]\) p. 109\).
Corollary 9.5.7. Under conditions of Theorem 9.5.6, for sufficiently small $\varepsilon$ the following inequality holds:

$$\varepsilon S \geq \log K.$$ 

This corollary follows from the $\{P_i\}$-chaoticity of $\Phi$ and the definition of topological entropy. See [39] and references therein for another approach to understanding chaotic canards.

As an example consider the system

\begin{align*}
\dot{x}^{(1)} &= -ax^{(2)} + y/3, \\
\dot{x}^{(2)} &= x^{(1)} + 1, \\
\varepsilon y &= x^{(1)} + y^2 + x^{(2)} y.
\end{align*}

The curves $\Gamma_a$ and $\Gamma_r$ intersect transversally on the plane $(x^{(1)}, x^{(2)})$, see Fig. 9.8.

According to the theorem above this system has a chaotic canard for a small $\varepsilon$.

9.6 Other Applications

9.6.1 Kaldor Model of the Business Cycle

The Kaldor model of the trade cycle [63] is one of the earliest nonlinear models in macroeconomics. Kaldor was among the foremost economists in the pre-war period and based his work on Keynes’ General Theory [67]. According to Keynes, an economy should reach an equilibrium when the levels of savings and investments are equal, however such equilibria are not observed in the real world, and Kaldor’s model attempted to explain why. In Kaldor’s model
investment and savings depend on the economic activity through a nonlinear relationship. He explained this by noting that investment opportunities saturate at high activity (i.e., all of the easy investments are taken, leaving higher-cost, higher-risk investments), and at low activity the level of investment will also level out due to the low levels of expected income. This leads to a sigmoidal dependance on activity.

From the mathematical perspective Kaldor model of the business cycle is a discrete time nonlinear business cycle model comprised of the following difference equations for income \(Y\) and capital stock \(K\):

\[
Y_{t+1} = Y_t + \alpha (I_t - S_t) \tag{9.20}
\]

\[
K_{t+1} = (1 - \delta)K_t + I_t \tag{9.21}
\]

with the following parameters:

- \(\alpha\) is the speed of adjustment (\(\alpha > 0\)) i.e. it measures the entrepreneurs reaction to the demand excess. A small value of \(\alpha\) means a prudent or careful reaction from the entrepreneur. A high value of \(\alpha\), (\(\alpha > 1\)) means a rash reaction from the entrepreneur.
- \(\delta\) is the depreciation rate of the capital stock (the rate at which the value of stock is reduced).
- The dynamic variables \(Y_t\) and \(K_t\) represent the income level in period \(t\) and the capital stock in period \(t\), respectively. \(I_t\) represents the investment demand in period \(t\) and \(S_t\) represents savings in period \(t\).
- Savings are assumed to be proportional to the current level of income: \(S_t = sY_t\), where \(s\), \(0 < s < 1\), represents the propensity to save.

The main reason for more recent interest in the Kaldor model is the discovery of the presence of rich and complicated dynamics in its behavior. The existence of multiple attractors, and the presence of global attractors, was proved in \cite{17}. Here they also mention the apparent existence of chaotic behavior in certain regions of the parameter space. This behavior had been noted in the literature before and since, however no formal proof of the existence of chaos in the Kaldor mapping was presented until recently, by Lisa Cronin in her M.Sc. Dissertation.

In \cite{96} the Kaldor model of the business cycle is modified by the incorporation of a Preisach nonlinearity. The paper then presents a rigorous proof of chaotic behavior, including the existence of periodic orbits of all minimal periods \(p > 57\). The corresponding control problems were considered in \cite{10} using the split-hyperbolicity technique.

### 9.6.2 Periodically Forced KdVB Equations

The paper \cite{36}, which was motivated by experimental observations presented in \cite{28}, is concerned with the existence of chaotic behavior of second order
periodically forced ordinary differential equations. These equations are a reduced form of a periodically forced Korteweg-de Vries–Burgers’ (KdVB) and extended KdVB (eKdVB) equations when viscous damping is included, and which are given by

\[
\frac{\partial u}{\partial t} + \Delta \frac{\partial u}{\partial x} + au^2 \frac{\partial u}{\partial x} + bu \frac{\partial u}{\partial x} + d \frac{\partial^3 u}{\partial x^3} + \nu \frac{\partial^2 u}{\partial x^2} = R'(x).
\]  

(9.22)

Here \( \Delta, a, b, d \) and \( \nu \) are real parameters, \( R(x) \) is a periodic function and \( R' \) is the derivative of \( R \). When \( a = 0 \), (9.22) reduces to the forced KdVB equation, which is a well known model for weakly nonlinear dispersive waves with viscous damping. When \( b = 0 \), (9.22) reduces to the forced eKdVB equation.

The ordinary differential equation

\[
\Delta u' + au^2 u' + buu' + du'' + \nu u'' = R'(x),
\]  

(9.23)

where \( u = u(x) \) describes the steady, time independent profiles for the eKdVB equation. In [36] the existence of profiles which are close to any shuffling of two basic profiles is proved, and hence the existence of spatially chaotic and recurrent solutions. The proofs are based on the topological hyperbolicity technique.

### 9.6.3 Piecewise Linear Oscillator

Consider a semi-linear equation of the type

\[
x'' + \alpha^2 x = b(t) + f(x). \tag{9.24}
\]

Here \( f \) is a scalar continuous bounded nonlinearity and \( b \) is \( 2\pi \)-periodic. The behavior of this equation is a classical object of studies in the theory of dynamical systems. Nevertheless a lot of important questions are still open and just some recent developments in this area are mentioned here.

Consider the following equation:

\[
\ddot{x} + x = \sin(\sqrt{2}t) + s(x), \tag{9.25}
\]

where

\[
s(x) = \begin{cases} 
-1 & \text{if } x \leq -\frac{1}{5}, \\
5x & \text{if } -\frac{1}{5} < x < \frac{1}{5}, \\
1 & \text{if } x \geq \frac{1}{5}.
\end{cases}
\]

This equation has been already considered in the engineering literature as a model of the saturation effects. Bifurcation phenomena and boundedness of solutions have been studied respectively in [86] and [108].

Denote by \( F \) the shift operator along the trajectories of equation (9.25) for the time \( \sqrt{2}\pi \), i.e., the period of the forcing term. Then for any pair \( (x_0, x'_0) \in \mathbb{R}^2 \) the value \( F(x_0, x'_0) \) is a two-dimensional vector \( (x(\sqrt{2}\pi), x'(\sqrt{2}\pi)) \), where \( x(t) \) denotes the solution of the initial value problem \( x(0) = x_0, x'(0) = x'_0 \) for equation (9.25).
Fig. 9.9 shows three typical long trajectories of (9.25):

- an outer invariant curve bounding all trajectories inside itself;
- two filled black circles representing a so called ‘chain of islands’, see [85];
- a cloud of circles, which is a long trajectory of $F$ identifying an area of instability; the structure of this area is similar to the classical Aubry–Mather instability area for twist maps, see [65].

In general, an understanding of the dynamics of Aubry–Mather instability area is not complete. It is known that there are signs of homoclinic behavior, see for example [65]. Nevertheless, the area in this case is not a chaotic attractor in a traditional sense: for example, it seems that the box counting dimension of a typical trajectory is fractional, while the correlation dimension is not.

The existence of chaotic behavior in the Smale sense inside the instability area was proved in [119], i.e., that a suitable restriction of some iteration $F^m$ of the mapping $F$ is topologically conjugated to the left shift operator in the set of symbolic sequences. A typical result from [119] is as follows:

**Theorem 9.6.1.** The point $(0, -\frac{\sqrt{2}}{6})$ is a transversal homoclinic fixed point of $F$.

The results follow from the application of topological hyperbolicity technique and the computer-assisted verification of a set of inequalities. The beauty of this work resides in its simplicity: even though the construction of the proof is computer-assisted, the set of inequalities is really small and it could be verified by hand.
9.6.4 Oscillations of a Ferromagnetic Pendulum

Consider an iron pendulum oscillating in an external magnetic field, see Fig. 9.10.

![Diagram of an iron pendulum in a magnetic field](image)

**Fig. 9.10.** An iron pendulum in a magnetic field

Such oscillators are common in nature and in industry: a satellite orbiting the Earth, or swings in a playground (in both cases because the Earth is a huge magnet), numerous cases in microelectronics etc. Without a magnetic field, the pendulum dynamics are described (in the linear approximation) by the equation

\[ x'' + ax' + x = \sin(\omega t), \]

which can be integrated explicitly. The situation when the magnetic field is present is, however, much more complex. The iron substance of the pendulum will be magnetized and demagnetized. This process is called *ferromagnetic hysteresis*. The magnetization will interact with the external magnetic field, and thus the equation will have the form

\[ x'' + ax' + x = \sin(\omega t) + y(t), \quad y(t) = \Gamma x(t), \quad (9.26) \]

where the nonlinearity \( \Gamma x(t) \) describes the interaction between the external magnetic field and the magnetized pendulum itself, and will be described more fully later. This term, \( \Gamma x(t) \), depends on the pendulum position as well as on the pendulum magnetization. The pendulum’s current magnetization depends in turn on the whole previous history of the pendulum motion. Thus the actual equation is of a differential-operator type.

Strange attractors of fractional dimension in the system (9.26) were discovered in [22] and a rigorous computer-aided proof of chaotic behavior, based on topological hyperbolicity, was presented there.
A

Semi-Hyperbolicity: Estimations

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In this appendix we look at the relation between the semi-hyperbolicity and hyperbolicity conditions from the point of view of possible numerical applications. We provide some estimations that can help in strict numerical verification of the hyperbolicity condition via the semi-hyperbolicity one.

Due to its rather numerical character and frequent use of Banach algebra tools, we present a mainly self-sufficient approach, which partially differs from that presented in Chaps. 4 and 5.

A.1 Linear Case

The aim of this section is to recall and generalize some results of Sect. 4.3.1 and to present comparison between hyperbolic and semi-hyperbolic linear operators. In the following sections we will generalize this approach to obtain some numerical estimations of hyperbolicity constants (in the nonlinear case) with the use of semi-hyperbolicity.

Let $E$ be a Banach space. Let us begin with the following

**Lemma A.1.1.** Let $S, T : E \to E$ be commuting bounded linear operators. If $ST$ is invertible, then $T$ is invertible.

**Proof.** Let $U$ be the inverse operator to $ST$. Then $(US)T = U(ST) = I$ and $T(SU) = (ST)U = I$, which means that $T$ has a left and right inverse, so $T$ is invertible. \( \square \)

**Proposition A.1.2.** Let $E = E_1 \oplus E_2$ and $T_{21} : E_1 \to E_2$, $T_{12} : E_2 \to E_1$ be given bounded linear operators. Define the operator $T : E \to E$ by the following matrix formula

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\[ T := \begin{bmatrix} I_E & T_{12} \\ T_{21} & I_E \end{bmatrix}. \]

If \( \|T_{12}T_{21}\| < 1 \) and \( \|T_{21}T_{12}\| < 1 \), then \( T \) is invertible.

**Proof.** Let

\[ S := \begin{bmatrix} I_E & -T_{12} \\ -T_{21} & I_E \end{bmatrix}. \]

One can easily check that

\[ ST = TS = \begin{bmatrix} I_E & -T_{12}T_{21} & 0 \\ 0 & I_E - T_{21}T_{12} \end{bmatrix}. \]

Then, using Lemma [A.1.1] we conclude that \( T \) is invertible (note, that \( I_E - T_{12}T_{21} \) and \( I_E - T_{21}T_{12} \) are invertible).

\[ \square \]

From now on, let \( A : E \to E \) be a bounded linear operator.

The following theorem was used in Sect. 4. We now present its proof that comes from [93].

**Theorem 4.3.5.** If the operator \( A \) is semi-hyperbolic with respect to a split \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) and a norm \( \| \cdot \| \), then \( A \) is spectrally hyperbolic.

**Proof.** Take a semi-hyperbolic splitting \( E = E^s \oplus E^u \) and corresponding projections \( P^s, P^u \). Put

\[ A_{11} := P^s A|_{E^s}, \quad A_{12} := P^s A|_{E^u}, \quad A_{21} := P^u A|_{E^s}, \quad A_{22} := P^u A|_{E^u}. \]

To show that \( A \) is hyperbolic is enough to check that the operator

\[ \lambda I_E - A := \begin{bmatrix} \lambda I_{E^s} - A_{11} & -A_{12} \\ -A_{21} & I_{E^u} - A_{22} \end{bmatrix} \]

is invertible for any \( \lambda \in \mathbb{C} \), \( |\lambda| = 1 \). Clearly, \( \lambda I_{E^s} - A_{11} \) and \( \lambda I_{E^u} - A_{22} \) are invertible and

\[ (\lambda I_{E^s} - A_{11})^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1} A_{11})^n, \]

\[ (\lambda I_{E^u} - A_{22})^{-1} = -A_{22}^{-1} \sum_{n=0}^{\infty} (\lambda A_{22}^{-1})^n. \]

Thus the following estimates hold

\[ \|\lambda I_{E^s} - A_{11}\|^{-1} \leq \sum_{n=0}^{\infty} \|A_{11}\|^n \leq \frac{1}{1 - \lambda_s}, \]

\[ \|(\lambda I_{E^u} - A_{22})^{-1}\| \leq \|A_{22}^{-1}\| \sum_{n=0}^{\infty} \|A_{22}^{-1}\|^n \leq \frac{1}{\lambda_u - 1}. \]
Since $\lambda I_E - A = ST$, where

$$S := \begin{bmatrix} \lambda I_E - A_{11} & 0 \\ 0 & I_E - A_{22} \end{bmatrix}$$

is an invertible operator and

$$T := \begin{bmatrix} I_E & -(\lambda I_E - A_{11})^{-1}A_{12} \\ -(\lambda I_E - A_{22})^{-1}A_{21} & I_E \end{bmatrix},$$

to obtain the conclusion we need to check the invertibility of $T$. But due to the following estimates

$$\| (\lambda I_E - A_{11})^{-1}A_{12} (\lambda I_E - A_{22})^{-1}A_{21} \| \leq \frac{\mu_s \mu_u}{(1 - \lambda_s)(\lambda_u - 1)} < 1,$$

$$\| (\lambda I_E - A_{22})^{-1}A_{21} (\lambda I_E - A_{11})^{-1}A_{12} \| \leq \frac{\mu_s \mu_u}{(1 - \lambda_s)(\lambda_u - 1)} < 1,$$

this is an immediate consequence of Proposition A.1.2. ⊓ ⊔

Now we show how to easily obtain a strengthened version of Theorem 4.3.5.

**Theorem A.1.3.** If the operator $A$ is semi-hyperbolic with respect to a split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and a norm $\| \cdot \|$, then the spectrum of $A$ does not intersect the ring

$$\mathcal{R}(\lambda_s^*, \lambda_u^*) := \{ \lambda \in \mathbb{C} : \lambda_s^* < |\lambda| < \lambda_u^* \},$$

where

$$\lambda_s^* = \frac{\lambda_s + \lambda_u}{2} - \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s \mu_u}}{2} < 1,$$

$$\lambda_u^* = \frac{\lambda_s + \lambda_u}{2} + \frac{\sqrt{(\lambda_u - \lambda_s)^2 - 4\mu_s \mu_u}}{2} > 1. \quad (A.1)$$

Before proceeding to the proof let us mention that the estimations obtained in (A.1) are sharp. To observe this, consider the linear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A = \begin{bmatrix} \lambda_s & \mu_s \\ \mu_u & \lambda_u \end{bmatrix}.$$
Obviously, if $\lambda \in \sigma(A)$ then $A_\lambda$ is not hyperbolic. Since

$$\|P^s A_\lambda|_{E^s}\| = \frac{\|P^s A|_{E^s}\|}{|\lambda|} \leq \frac{\lambda_s}{|\lambda|},$$

and, analogously,

$$\|P^s A_\lambda|_{E^u}\| \leq \frac{\mu_s}{|\lambda|}, \quad \|P^u A_\lambda|_{E^s}\| \leq \frac{\mu_u}{|\lambda|} \quad \text{and} \quad \|(P^u A_\lambda|_{E^u})^{-1}\| \leq \frac{|\lambda|}{\lambda_u},$$

we conclude that the operator $A_\lambda$ is semi-hyperbolic if the following inequalities hold

$$\lambda_s < |\lambda| < \lambda_u \quad \text{and} \quad \mu_s \mu_u < (|\lambda| - \lambda_s)(\lambda_u - |\lambda|).$$

Solving the above system we obtain

$$\lambda \in \mathcal{R}(\lambda^*_s, \lambda^*_u).$$

The proof is completed by the following sequence of implications

$$\lambda \in \mathcal{R}(\lambda^*_s, \lambda^*_u) \Rightarrow A_\lambda \text{ is semi-hyperbolic}$$

$$\Rightarrow A_\lambda \text{ is spectrally hyperbolic} \Rightarrow \lambda \notin \sigma(A).$$

At the end of this section we would like to fix our attention on the inverse problem. Thus we would like to ask the following question

**Problem A.1.4.** Suppose that the spectrum of the operator $A$ does not intersect the ring $\mathcal{R}(\lambda^*_s, \lambda^*_u)$. Does there exist an equivalent norm on $E$ such that for arbitrary $\lambda_s \in [0, \lambda^*_s]$ the operator $A$ is semi-hyperbolic with respect to a split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, for some $\lambda_u, \mu_s, \mu_u$ for which (A.1) holds?

Roughly speaking we ask if we can ‘decrease’ the value of $\lambda^*_s$ by ‘putting’ it into the not invariant splitting (according to the semi-hyperbolicity condition). Of course, one can also ask the dual question for $\lambda_u \in [\lambda^*_u, \infty)$ but since this is an analogue, we consider only the case of $\lambda_s$.

Let us first show that under no additional assumptions the answer for the above question is negative.

**Example A.1.5.** Consider the mapping

$$A : x \mapsto x/2, \quad x \in \mathbb{R}.$$ 

Then the only spectral value of $A$ is $1/2$, hence

$$\sigma(A) \cap \mathcal{R}(1/2, \infty) = \emptyset.$$ 

However, one can easily notice that for every $\lambda_s < 1/2$ and $\lambda_u, \mu_s, \mu_u$ such that (A.1) holds, the operator $A$ does not satisfy semi-hyperbolicity conditions with respect to the split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ and any norm on $\mathbb{R}$.  

Nonetheless, we show that in some cases, the inverse holds.

**Example A.1.6.** Let $\lambda_s^* \in [0, 1)$, $\lambda_u^* \in (1, \infty)$. Consider the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ given in the matrix form by

$$A := \begin{bmatrix} \lambda_s^* & 0 \\ 0 & \lambda_u^* \end{bmatrix}.$$ 

Then

$$\sigma(A) = \{\lambda_s^*, \lambda_u^*\} \subset \mathbb{C} \setminus \mathcal{R}(\lambda_s^*, \lambda_u^*).$$

To show that in this case the answer to Problem A.1.4 is positive, let us fix $b_1, b_2 \in \mathbb{R}$ such that

$$b_1 \cdot b_2 \in [0, \lambda_s^*/\lambda_u^*].$$

Consider the operator $B : \mathbb{R}^2 \to \mathbb{R}^2$, defined, in the matrix form, by

$$B = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}.$$ 

Put

$$E^s = (I - B)(\mathbb{R} \times \{0\}) \quad \text{and} \quad E^u = (I - B)(\{0\} \times \mathbb{R}),$$

where $I$ denotes the identity operator. It is easy to see that the projections corresponding to the splitting $E^s \oplus E^u$ are given by

$$P^s = \frac{1}{1 - b_1 b_2} \begin{bmatrix} 1 & b_1 \\ -b_2 & -b_1 b_2 \end{bmatrix},$$

$$P^u = \frac{1}{1 - b_1 b_2} \begin{bmatrix} -b_1 b_2 & -b_1 \\ b_2 & 1 \end{bmatrix}.$$ 

Then, choosing a norm $\| \cdot \|$ on $\mathbb{R}^2$ such that $\|e^s\| = \|e^u\| = 1$, where $e^s = [1, -b_2]^T \in E^s$ and $e^u = [-b_1, 1]^T \in E^u$, we obtain

$$\|P^s A e^s\| = \frac{\lambda_s^* - \lambda_u^* b_1 b_2}{1 - b_1 b_2} \|e^s\| = \frac{\lambda_s^* - \lambda_u^* b_1 b_2}{1 - b_1 b_2} = \frac{1 - \frac{\lambda_u^*}{\lambda_s^*} b_1 b_2}{1 - b_1 b_2} \lambda_s^* \leq \lambda_u^* < 1,$$

$$\|P^u A e^s\| = \frac{\lambda_u^* b_2 - \lambda_u^* b_2}{1 - b_1 b_2} \|e^s\| = \frac{\lambda_u^* - \lambda_s^*}{1 - b_1 b_2} |b_2|,$$

$$\|P^u A e^u\| = \frac{\lambda_u^* b_1 - \lambda_u^* b_1}{1 - b_1 b_2} \|e^u\| = \frac{\lambda_u^* - \lambda_s^*}{1 - b_1 b_2} |b_1|,$$

$$\|P^s A e^u\| = \frac{\lambda_u^* - \lambda_s^* b_1 b_2}{1 - b_1 b_2} \|e^u\| = \frac{\lambda_u^* - \lambda_s^* b_1 b_2}{1 - b_1 b_2} = \frac{1 - \frac{\lambda_s^*}{\lambda_u^*} b_1 b_2}{1 - b_1 b_2} \lambda_u^* \geq \lambda_u^* > 1.$$ 

Thus, putting

$$\lambda_s = \frac{\lambda_u^* - \lambda_u^* b_1 b_2}{1 - b_1 b_2}, \quad \lambda_u = \frac{\lambda_u^* - \lambda_s^* b_1 b_2}{1 - b_1 b_2},$$
and
\[
\mu_s = \frac{\lambda_u^* - \lambda_s^*}{1 - b_1 b_2}|b_1|, \quad \mu_u = \frac{\lambda_u^* - \lambda_s^*}{1 - b_1 b_2}|b_2|,
\]
one can easily check, by direct computations, that (A.1) holds and that
\[
\lambda_s \in [0, \lambda_s^*], \quad \mu_s \mu_u < (1 - \lambda_s)(\lambda_u - 1).
\]
We have obtained even more then we wanted — by modifying the values of \(b_1\) and \(b_2\) we can not only realize any value \(\lambda_s \in [0, \lambda_s^*]\), but we control also one of the constants \(\mu_s, \mu_u\).

The above example leads us to the following

**Conjecture A.1.7.** Let \(A\) be a hyperbolic operator such that the spaces \(E^s\) and \(E^u\) are isomorphic. Then the answer to Problem A.1.4 is positive.

### A.2 Functional Calculus and its Consequences

Since we will need some basic results from the operator functional calculus, for the convenience of the reader and to establish notation we present them in this section. For more information on this subject we refer the reader to [64].

By \(S_r = \{\lambda \in \mathbb{C} : |\lambda| = r\}\) we denote the positively oriented circle with the radius \(r\). Assume that
\[
\sigma(A) \cap \mathbb{R}(\lambda_s^*, \lambda_u^*) = \emptyset
\]
for some constants \(\lambda_s^* < 1\) and \(\lambda_u^* > 1\). Hence, by Theorem 4.3.5 \(A\) is metrically hyperbolic in some equivalent norm.

We put
\[
P_s^* = \frac{1}{2\pi i} \int_{S_r} (\lambda I - A)^{-1} d\lambda, \quad P_u^* = I - P_s^* \quad \text{for} \quad r \in (\lambda_s^*, \lambda_u^*). \quad (A.2)
\]
Then \(P_s^*\) and \(P_u^*\) does not depend on the choice of \(r \in (\lambda_s^*, \lambda_u^*)\). Moreover, \(P_s^*\) and \(P_u^*\) are the projections corresponding to the unique splitting from the metric definition of hyperbolicity (we have \(E = E^s_s \oplus E^u_u\) for \(E^{s,u} = P_s^{s,u}(E)\)). Moreover, we have the following formulas for the iterations of \(A\) on spaces \(E^s_s\) and \(E^u_u\)
\[
A^k x_s = \int_{S_r} \lambda^k (\lambda I - A)^{-1} x_s d\lambda \quad \text{for} \quad x_s \in E^s_s, k \geq 0, \quad (A.3)
\]
\[
(A|_{E^u_u})^{-k} x_u = -\int_{S_r} \lambda^{-k} (\lambda I - A)^{-1} x_u d\lambda \quad \text{for} \quad x_u \in E^u_u, k \geq 1. \quad (A.4)
\]

Our aim in this section it to give exact estimations of respective iterates of \(A\) on subspaces \(E^s_s\) and \(E^u_u\).
Theorem A.2.1. Let $A$ be a semi-hyperbolic operator with respect to a split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, a constant $h$ and a norm $\| \cdot \|$, $\lambda^*_s, \lambda^*_u$ be given by (A.1) and $r \in (\lambda^*_s, \lambda^*_u)$ be arbitrary.

We put

$$C_r := \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(r - \lambda_s)(\lambda_u - r) - \mu_s \mu_u} - h.$$

Then

$$\| (A|_{E^u})^k \| \leq C_r \cdot r^k, \quad \| (A|_{E^u})^{-k} \| \leq C_r \cdot r^{-k} \quad \text{for} \quad k \in \mathbb{N}, k \geq 1.$$

Clearly, if we are interested in the estimation of $\| (A|_{E^u})^k \|$ we should take $r \in (\lambda^*_s, 1)$, and $r \in (1, \lambda^*_u)$ if we need to estimate $\| (A|_{E^u})^{-k} \|$.

Theorem A.2.1 is a direct consequence of (A.3), (A.4) and the following proposition:

Proposition A.2.2. Let $A$ be a semi-hyperbolic operator with respect to a split $s = (\lambda_s, \lambda_u, \mu_s, \mu_u)$, a constant $h$ and a norm $\| \cdot \|$, (Hence, by Theorem A.1.3, $\sigma(A) \cap \mathcal{R}(\lambda^*_s, \lambda^*_u) = \emptyset$ where $\lambda^*_s$ and $\lambda^*_u$ are given by (A.1).) Then

$$\| (\lambda I - A)^{-1} \| \leq \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{|\lambda| - \lambda_s}(\lambda_u - |\lambda|) - \mu_s \mu_u - h \quad \text{for} \quad \lambda \in \mathcal{R}(\lambda^*_s, \lambda^*_u).$$

Proof. Take a semi-hyperbolic splitting $E = E^s \oplus E^u$, corresponding projections $P^s$ and $P^u$. Choose $x, y \in E$, $\lambda \in \mathcal{R}(\lambda^*_s, \lambda^*_u)$ such that

$$(\lambda I - A)x = y.$$  

Then $\lambda x = y + Ax$ and hence

$$\lambda P^s x = P^s y + P^s AP^s x + P^s AP^u x,$$

$$P^u x = P^u y + P^u AP^s x + P^u AP^u x.$$  

Since

$$U = P^u A|_{E^s} : E^u \rightarrow E^u$$

is an invertible operator with $\| U^{-1} \| \leq (\lambda_u)^{-1}$, we obtain

$$P^u x = U^{-1}(\lambda P^u x - P^u y - P^u AP^s x).$$

It follows that

$$|\lambda| \| P^s x \| \leq h \| y \| + \lambda_s \| P^s x \| + \mu_s \| P^u x \|,$$

$$\lambda_u \| P^u x \| \leq h \| y \| + |\lambda| \| P^u x \| + \mu_u \| P^s x \|$$

and, in consequence,

$$|\lambda - \lambda_s| \| P^s x \| - \mu_s \| P^u x \| \leq h \| y \|,$$

$$(\lambda_u - |\lambda|) \| P^u x \| - \mu_u \| P^s x \| \leq h \| y \|.$$
Then
\[ \mu_u(|\lambda| - \lambda_s)\|P^u x\| - \mu_s \mu_u \|P^u x\| \leq \mu_u h\|y\|, \]
\[ (|\lambda| - \lambda_s)(\lambda_u - |\lambda|)\|P^u x\| - \mu_u(|\lambda| - \lambda_s)\|P^u x\| \leq (|\lambda| - \lambda_s)h\|y\|, \]
and hence
\[ \|P^u x\| \leq \frac{\mu_u + |\lambda| - \lambda_s}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s \mu_u} h\|y\|. \]

Analogously,
\[ \|P^s x\| \leq \frac{\mu_s + \lambda_u - |\lambda|}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s \mu_u} h\|y\|. \]

Let us note that the above calculations are based on the following estimates
\[ \lambda_s < |\lambda| < \lambda_u \quad \text{and} \quad \mu_s \mu_u < (|\lambda| - \lambda_s)(\lambda_u - |\lambda|), \]
which can be easily verified as in the proof of Proposition A.1.3. Thus we have
\[ \|(\lambda I - A)^{-1} y\| = \|x\| \leq \|P^s x\| + \|P^u x\| \]
\[ \leq \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(|\lambda| - \lambda_s)(\lambda_u - |\lambda|) - \mu_s \mu_u} h\|y\|. \]

\[ \square \]

As a direct consequence of Proposition A.2.2 and (A.2) we obtain the following

**Corollary A.2.3.** Let A be a semi-hyperbolic operator with respect to a split \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \), a constant \( h \) and a norm \( \| \cdot \| \). Then
\[ \|P^s_*\| \leq L, \quad \|P^u_*\| \leq L + 1, \]
where \( P^s_* \), \( P^u_* \) are given by (A.2) and
\[ L = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u} h. \]

**A.3 General Case**

The results of this section are strictly related to ones obtained in Chap. 5. Generally, we show that any diffeomorphism of a compact Riemannian manifold, which is semi-hyperbolic on some compact invariant set is, in fact, hyperbolic on this set. We consider here a variable norm version of semi-hyperbolicity condition (see Definition 3.1.4). Nevertheless, one can easily adapt all the proofs to the other concepts of semi-hyperbolicity that were presented in Sect. 3.1.

Let \( f : M \to M \) be a diffeomorphism of a compact Riemannian manifold \( M \). For \( x \in M \) consider the Banach space \( \mathcal{C}_x \subset l^\infty(\mathbb{Z}, TM) \) consisting of all
bounded sequences $\mathbf{v} = \{v_n\}$ of vectors $v_n \in T_{f^n(x)}M \subset TM$, endowed with the supremum norm

$$\|\mathbf{v}\|_{x}^{\infty} = \sup_{n \in \mathbb{Z}} \|v_n\|_{x,n}.$$ 

Here and later on $\| \cdot \|_{x,n}$ denotes, for each $n \in \mathbb{Z}$, an underlying Riemannian norm on $T_{f^n(x)}M$.

We define the bounded operator $\mathcal{A}_x$ on $\mathcal{E}_x$ by the following formula

$$(\mathcal{A}_x \mathbf{v})_{n+1} = T_{f^n(x)}v_n \quad \text{for} \quad \mathbf{v} \in \mathcal{E}_x.$$ 

From the above definition it directly follows that we can identify the operator $\mathcal{A}_x$ with $\mathcal{A}_y$ in the case of $x$ and $y$ lying on the same trajectory of $f$. (Compare the above notation with that used in Chap. 4.)

Now we are ready to present the main result of this appendix (cf. [95]).

**Theorem A.3.1.** Assume $K$ is a compact $f$-invariant subset of $M$, $\mathbf{s} = (\lambda_s, \lambda_u, \mu_s, \mu_u)$ is a split and $h$ is a real positive number. Let $\| \cdot \|_x$ be a Riemannian norm on $T_xM$ for $x \in K$.

(i) If $f$ is $(\mathbf{s}, h)$-semi-hyperbolic for $f$ according to the norm $\| \cdot \|_x$, then for every $x \in K$ the operator $\mathcal{A}_x$ is semi-hyperbolic with respect the split $\mathbf{s}$, the constant $h$ and the norm $\| \cdot \|_x^{\infty}$.

(ii) Assume that for every $x \in K$ the operator $\mathcal{A}_x$ is semi-hyperbolic with respect the split $\mathbf{s}$, the constant $h$ and the norm $\| \cdot \|_x^{\infty}$. Let $\lambda^*_s, \lambda^*_u$ be given by (A.1) and $\gamma^*_s, \gamma^*_u$ be arbitrary reals such that

$$\lambda^*_s < \gamma^*_s < 1 < \gamma^*_u < \lambda^*_u.$$ 

Put

$$C^* = \max \left\{ \frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma^*_s - \lambda_s)(\lambda_u - \gamma^*_u) - \mu_s \mu_u}, \frac{(\lambda_u - \lambda_s + \mu_s + \mu_u)h}{(\gamma^*_u - \lambda_s)(\lambda_u - \gamma^*_u) - \mu_s \mu_u} \right\}.$$ 

Then there exists a splitting $T_xM = E^s_x \oplus E^u_x$ varying continuously in $x \in K$, such that

$$T_{f^n(x)}(E^s_{f^n(x)}) = E^s_{f^{n+1}(x)}; \quad T_{f^n(x)}(E^u_{f^n(x)}) = E^u_{f^{n+1}(x)}, \quad \text{(A.5)}$$

$$\|T_{f^n(x)}v^s\|_{x,n+k} \leq C^* (\gamma^*_s)^k \|v^s\|_{x,n}, \quad \|T_{f^n(x)}v^u\|_{x,n-k} \leq C^* (\gamma^*_u)^{-k} \|v^u\|_{x,n} \quad \text{(A.6)}$$

for all $x \in K$, $n, k \in \mathbb{Z}$, $k > 0$, $v^s \in E^s_{f^n(x)}$ and $v^u \in E^u_{f^n(x)}$.

Before we proceed with the proof let us quote an important direct consequence of Theorem A.3.1

**Theorem A.3.2.** Assume that the diffeomorphism $f$ is semi-hyperbolic on a compact $f$-invariant subset $K$ of the manifold $M$. Then the set $K$ is hyperbolic for $f$. 


Proof of Theorem A.3.1. (i) Take \( x \in K \) and, for each \( n \in \mathbb{Z} \), projections \( P_{x,n}^s \) and \( P_{x,n}^u \) corresponding to a given semi-hyperbolic splitting \( T_{f^n(x)}M = E_{x,n}^s \oplus E_{x,n}^u \). Then it is easy to see that the operators \( \mathcal{P}_x^s \) and \( \mathcal{P}_x^u \), defined by

\[
\mathcal{P}_x^s = \prod_{n \in \mathbb{Z}} P_{x,n}^s \quad \text{and} \quad \mathcal{P}_x^u = \prod_{n \in \mathbb{Z}} P_{x,n}^u,
\]

are bounded projections inducing a splitting of the space \( E_x \) that makes \( A_x \) a semi-hyperbolic operator with respect to \( s, h \) and \( \| \cdot \|_x^\infty \).

(ii) Fix \( \gamma_s^* \in (\lambda_s^*, 1) \), \( \gamma_u^* \in (1, \lambda_u^*) \) and take \( x \in K \). For the operator \( A_x \) consider a hyperbolic splitting \( E_x = E_x^s \oplus E_x^u \), corresponding to projections \( \mathcal{P}_x^s \) and \( \mathcal{P}_x^u \). Then by Corollary A.2.3 and Theorem A.2.1 we obtain that

\[
\| (A_x | E_x^s)^k \|_x^\infty \leq C^*(\gamma_s^*)^k, \quad \| (A_x | E_x^u)^{-k} \|_x^\infty \leq C^*(\gamma_u^*)^{-k}
\]

for all \( k > 0 \),

\[
\max \{ \| \mathcal{P}_x^s \|_x^\infty, \| \mathcal{P}_x^u \|_x^\infty \} \leq L + 1.
\]

where

\[
L = \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{(1 - \lambda_s)(\lambda_u - 1) - \mu_s\mu_u} h,
\]

\[
C^* = \max \left\{ \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{\lambda_s - \lambda_u} h, \frac{\lambda_u - \lambda_s + \mu_s + \mu_u}{\lambda_u - \gamma_u^*} h, \frac{\gamma_u^* - \lambda_u}{\lambda_u - \gamma_u^*} h \right\}.
\]

For each \( n \in \mathbb{Z} \) put

\[
E_{x,n} = T_{f^n(x)}M \quad \text{and} \quad A_{x,n} = T_{f^n(x)}
\]

and note that

\[
T_{f^n(x)}^{k} = T_{f^{n+k-1}} \cdots T_{f^n(x)} = A_{x,n+k-1} \cdots A_{x,n},
\]

\[
T_{f^n(x)}^{-k} = T_{f^{-1}}^{-k+1} \cdots T_{f^{-1}} = A_{x,n-k}^{-1} \cdots A_{x,n-1}^{-1}.
\]

Consider the spaces \( E_{x,n}^s, E_{x,n}^u \subset E_{x,n} \) defined as follows

\[
E_{x,n}^s = \left\{ v \in E_{x,n} : \sup_{k \geq 0} \| A_{x,n+k-1} \cdots A_{x,n} v \|_{x,n+k} < \infty \right\},
\]

\[
E_{x,n}^u = \left\{ v \in E_{x,n} : \sup_{k \geq 0} \| A_{x,n-k}^{-1} \cdots A_{x,n-1}^{-1} v \|_{x,n-k} < \infty \right\}.
\]

Obviously, since \( A_{x,n} \) is invertible,

\[
A_{x,n}(E_{x,n}^s) = E_{x,n+1}^s \quad \text{and} \quad A_{x,n}(E_{x,n}^u) = E_{x,n+1}^u.
\]

Firstly note that \( P_{x}^n(E_x^s) \subset E_{x,n}^s \) and \( P_{x}^n(E_x^u) \subset E_{x,n}^u \), where \( P_{x}^n \) denotes the canonical projection on the \( n \)-th coordinate in \( \prod_{n \in \mathbb{Z}} E_{x,n} \). Indeed, taking sequences \( v^s \in E_x^s \) and \( v^u \in E_x^u \) it is easily seen that for \( k \in \mathbb{N}, k \geq 1 \).
and note that the following estimates hold for all $k > 0$

$$
\|A_{x,n+k-1} \cdots A_{x,n} v^s_{x,n+k} \|_{x,n+k} \leq \| A^k_x v^s \|_x \leq C^* (\gamma^*_s)^k \| v^s \|_x \leq C^* \| v^s \|_x, \\
\|A_{x,n-k}^{-1} \cdots A_{x,n-1} v^u_{x,n-k} \|_{x,n-k} \leq \| A^{-k}_x v^u \|_x \leq C^* (\gamma^*_u)^{-k} \| v^u \|_x \leq C^* \| v^u \|_x.
$$

Now we show that, for each $n \in \mathbb{Z}$, the spaces $E^s_{x,n}$ and $E^u_{x,n}$ form splitting of the space $E_{x,n}$.

Indeed, if $v \in E^s_{x,n} \cap E^u_{x,n}$ then

$$
v = (\ldots, A_{x,n-2} A^{-1}_{x,n-1} v, A_{x,n-1}^{-1} v, v_n = v, A_{x,n} v, A_{x,n+1} A_{x,n} v, \ldots) \in \mathcal{E}_x
$$

and since $\mathcal{A}_x$ is hyperbolic and $\mathcal{A}_x v = v$ we obtain $v = 0$. The equality $E_{x,n} = E^s_{x,n} + E^u_{x,n}$ follows immediately from the following observation

$$
v = v_n = (\mathcal{P}^s_x v + \mathcal{P}^u_x v)_n = (\mathcal{P}^s_x v)_n + (\mathcal{P}^u_x v)_n \\
\in P^u_x (\mathcal{E}^s_x) + P^n_x (\mathcal{E}^u_x) \subset E^s_{x,n} + E^u_{x,n}
$$

for any $v \in \mathcal{E}_x$ with $v_n = v \in E_{x,n}$, e.g., $v = (\ldots, 0, v_n = v, 0, \ldots)$.

This finishes the proof of the condition $E_{x,n} = E^s_{x,n} \oplus E^u_{x,n}$. Note that (A.5) is a trivial consequence of (A.7).

Finally, we also obtain that

$$
\mathcal{P}^s_x = \prod_{n \in \mathbb{Z}} P^s_{x,n} \quad \text{and} \quad \mathcal{P}^u_x = \prod_{n \in \mathbb{Z}} P^u_{x,n},
$$

where $P^s_{x,n}$ and $P^u_{x,n}$ denote the projections corresponding to the splitting $E_{x,n} = E^s_{x,n} \oplus E^u_{x,n}$, as the result of the earlier remarks and the following sequence of implications that hold for each $n \in \mathbb{Z}$

$$
v \in \mathcal{E}_x \Rightarrow v_n = (\mathcal{P}^s_x v)_n + (\mathcal{P}^u_x v)_n = E^s_{x,n} + E^u_{x,n} \\
\Rightarrow P^u_{n,x} v_n = (\mathcal{P}^u_x v)_n \quad \text{and} \quad P^u_{n,x} v_n = (\mathcal{P}^u_x v)_n.
$$

In particular, if $v = (\ldots, 0, v_n = v, 0, \ldots)$ then

$$
\mathcal{P}^s_x v = (\ldots, 0, P^s_{x,n} v, 0, \ldots) \quad \text{and} \quad \mathcal{P}^u_x v = (\ldots, 0, P^u_{x,n} v, 0, \ldots).
$$

Now take $v^s \in E^s_{x,n}$, $v^u \in E^u_{x,n}$, corresponding sequences

$$
v^s = (\ldots, 0, v^s_n = v^s, 0, \ldots) = (\ldots, 0, P^s_{x,n} v^s, 0, \ldots) = \mathcal{P}^s_x v^s \in \mathcal{E}^s_x, \\
v^u = (\ldots, 0, v^u_n = v^u, 0, \ldots) = (\ldots, 0, P^u_{x,n} v^u, 0, \ldots) = \mathcal{P}^u_x v^u \in \mathcal{E}^u_x,
$$

and note that the following estimates hold for all $k > 0$

$$
\|A_{x,n+k-1} \cdots A_{x,n} v^s\|_{x,n+k} \leq \| A^k_x v^s \|_x \leq C^* (\gamma^*_s)^k \| v^s \|_x \leq C^* \| v^s \|_x.
$$
\[
\left\| A_{x,n-k}^{-1} \cdots A_{x,n-1}^{-1} v^u \right\|_{x,n-k} \leq \left\| \mathcal{A}_x^{-k} v^u \right\|_x^{\infty} \leq C^* (\gamma_u^*)^{-k} \| v^u \|_x^{\infty} = C^* (\gamma_u^*)^{-k} \| v^u \|_{x,n},
\]

which completes the proof of the condition (A.6). Since the uniform boundedness of projections guarantees continuity of a hyperbolic splitting (see [110, Sect. 2.2]), it is enough to verify that the projections \( P_{s,x,n} \) and \( P_{u,x,n} \) are uniformly bounded, that is

\[
\sup_{x \in K, n \in \mathbb{Z}} \left\{ \left\| P_{s,x,n} \right\|_{x,n}, \left\| P_{u,x,n} \right\|_{x,n} \right\} < \infty.
\]

Indeed, for \( v \in E_{x,n} \) and \( v = (\ldots, 0, v_n = v, 0, \ldots) \) we have

\[
\mathcal{P}_x^s v = (\ldots, 0, P_{s,x,n}^v, 0, \ldots) \quad \text{and} \quad \mathcal{P}_x^u v = (\ldots, 0, P_{u,x,n}^v, 0, \ldots)
\]

and then we obtain the following estimates

\[
\left\| P_{s,x,n} v \right\|_{x,n} = \left\| \mathcal{P}_x^s v \right\|_x^{\infty} \leq \left\| \mathcal{P}_x^s \right\|_x^{\infty} \left\| v \right\|_x^{\infty} = \left\| \mathcal{P}_x^s \right\|_x^{\infty} \left\| v \right\|_{x,n} \leq (L + 1) \left\| v \right\|_{x,n},
\]
\[
\left\| P_{u,x,n} v \right\|_{x,n} = \left\| \mathcal{P}_x^u v \right\|_x^{\infty} \leq \left\| \mathcal{P}_x^u \right\|_x^{\infty} \left\| v \right\|_x^{\infty} = \left\| \mathcal{P}_x^u \right\|_x^{\infty} \left\| v \right\|_{x,n} \leq (L + 1) \left\| v \right\|_{x,n}.
\]

But \( L \) depends neither on \( x \in K \) nor on \( n \in \mathbb{Z} \), which makes the proof complete.  \( \square \)
The concept of semi-hyperbolicity can be extended to set-valued dynamics generated by a set-valued mapping, see, e.g., [115–117]. As this topic is a subject of ongoing research, we can only provide a very limited presentation of this generalization. We will introduce a strong and a weak set-valued notion of semi-hyperbolicity and discuss the issues of expansivity and shadowing.

**B.1 Definition**

Let $\mathcal{C}(\mathbb{R}^d)$ be the collection of all nonempty compact subsets of $\mathbb{R}^d$. For $A, B \in \mathcal{C}(\mathbb{R}^d)$, we define the Hausdorff semi-distance and the Hausdorff distance by

$$\text{dist}(A, B) := \sup_{a \in A} \text{dist}(a, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|$$

and

$$\text{dist}_H(A, B) := \max \{\text{dist}(A, B), \text{dist}(B, A)\},$$

respectively. When equipped with the Hausdorff distance, $\mathcal{C}(\mathbb{R}^d)$ is a complete metric space.

A set-valued mapping $F : \mathbb{R}^d \to \mathcal{C}(\mathbb{R}^d)$ is a function which assigns a nonempty convex and compact subset to any point in $\mathbb{R}^d$. The Hausdorff metric allows us to specify notions of continuity for $F$. We call such a mapping continuous at $x \in \mathbb{R}^d$ if $\lim_{\tilde{x} \to x} \text{dist}_H(F(x), F(\tilde{x})) = 0$, and we say that $F$ is continuous on an open subset $X \subset \mathbb{R}^d$ if it is continuous at any point $x \in X$. Moreover, a set-valued mapping $F$ is called Lipschitz continuous with Lipschitz constant $L \geq 0$ on $X$ if

$$\text{dist}_H(F(x), F(\tilde{x})) \leq L\|x - \tilde{x}\| \quad \text{for any} \quad x, \tilde{x} \in X.$$
Consider a set-valued mapping \( F : X \to \mathcal{C}(X) \), where \( X \) is an open subset of \( \mathbb{R}^d \) and \( \mathcal{C}(X) \) denotes the subcollection of sets from \( \mathcal{C}(\mathbb{R}^d) \) which are contained in \( X \). The mapping \( F \) generates two types of discrete time set-valued dynamical systems by successive iteration.

a) A sequence \( \{x_n\}_{n \in \mathbb{I}} \) in \( X \) is an **individual trajectory** of the system on a finite or infinite interval \( \mathbb{I} \subset \mathbb{Z} \) if

\[
x_{n+1} \in F(x_n)
\]

holds whenever \( n, n+1 \in \mathbb{I} \).

b) A sequence \( \{X_n\}_{n \in \mathbb{I}} \) in \( \mathcal{C}(X) \) is a **set trajectory** of the system on a finite or infinite interval \( \mathbb{I} \subset \mathbb{Z} \) if

\[
X_{n+1} = F(X_n)
\]

holds whenever \( n, n+1 \in \mathbb{I} \). As usual, \( F(X_n) := \bigcup_{x \in X_n} F(x) \).

Both points of view have their merits, as we will see below.

**B.1.1 Strong semi-hyperbolicity**

The notion of strong semi-hyperbolicity is defined in terms of conditions on the full image of the mapping \( F \). In \[116\], a shadowing and an inverse shadowing theorem are provided which require convexity of the images, but are less restrictive with respect to the regularity of \( F \).

**Definition B.1.1.** Let \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) be a split and \( K \) a compact subset of an open set \( X \subset \mathbb{R}^d \). A set-valued mapping \( F : X \to \mathcal{C}(X) \) is called strongly \( s \)-semi-hyperbolic on \( K \) if it can be represented as

\[
F(x) = f(x) + G(x), \quad (B.1)
\]

where

1) \( f : X \to X \) is a Lipschitz continuous single-valued function;

2) there exists a splitting \( \mathbb{R}^d = E_x^s \oplus E_x^u \) on \( K \) with projectors \( P_x^s \) and \( P_x^u \) for each \( x \in K \);

3) there exists a norm \( \| \cdot \| \) on \( \mathbb{R}^d \) and positive constants \( \delta \) and \( h \) such that

\[
\text{(i)} \quad \dim E_x^s = \dim E_y^u \quad \forall x, y \in K \text{ with } \text{dist}(y, F(x)) \leq \delta;
\]

\[
\text{(ii)} \quad \sup_{x \in K} \{\|P_x^s\|, \|P_x^u\|\} \leq h;
\]

\[
\text{(iii)} \quad \text{the inclusion } x + u + v \in X
\]

and the inequalities

\[
\|P_y^s(f(x + u + v) - f(x + \tilde{u} + \tilde{v}))\| \leq \lambda_s\|u - \tilde{u}\|, \quad (B.2)
\]

\[
\|P_y^s(f(x + u + v) - f(x + u + \tilde{v}))\| \leq \mu_s\|v - \tilde{v}\|, \quad (B.3)
\]

\[
\|P_y^u(f(x + u + v) - f(x + \tilde{u} + v))\| \leq \mu_u\|u - \tilde{u}\|, \quad (B.4)
\]

\[
\|P_y^u(f(x + u + v) - f(x + u + \tilde{v}))\| \geq \lambda_u\|v - \tilde{v}\|, \quad (B.5)
\]
hold for all \( x, y \in K \) with \( \text{dist}(y, F(x)) \leq \delta \) and all \( u, \tilde{u} \in E^s_x \) and \( v, \tilde{v} \in E^u_x \) such that \( \|u\|, \|\tilde{u}\|, \|v\|, \|\tilde{v}\| \leq \delta \);

4) \( G : X \to \mathcal{C}(X) \) is a Lipschitz continuous set-valued mapping with Lipschitz constant \( L > 0 \) satisfying

\[
10 \cdot Ld < \nu(s),
\]

where

\[
\nu(s) = \frac{(1 - \lambda_s)(\lambda_u - 1) - \mu_s \mu_u}{\lambda_u - \lambda_s + \mu_s + \mu_u}
\]

is the robustness constant associated with the split \( s \).

This notion of semi-hyperbolicity requires a certain degree of uniformity of the splitting. If the images of \( G \) are large, conditions (B.2) to (B.5) must be satisfied for \( y \) varying in a large region. At the same time, the restriction (B.6) imposed on the Lipschitz constant of \( G \) ensures that the semi-hyperbolicity of \( f \) dominates the behavior of the set-valued mapping \( F \). The factor \( 10 \cdot d \) originates from a geometric principle, the so-called Parametrization Theorem, see Lemma B.1.4. For set-valued mappings with convex and compact values, it is possible to define strong semi-hyperbolicity and obtain strong shadowing results without this factor (see [116]), because in that case, continuous local parametrizations can be obtained from minimal-norm-type selections.

In [55], spatial discretization effects near compact invariant sets are modeled by inflated right hand sides. If the original mapping is semi-hyperbolic, the resulting set-valued mapping is strongly semi-hyperbolic if either the inflation and the variation of the splitting are small enough, see Example B.1.2. The saddle dynamics analyzed in [34] are generated by a strongly semi-hyperbolic set-valued mapping. The focus of this paper is the characterization of analogs of the stable and unstable manifolds.

Example B.1.2. Let \( X = \mathbb{R}^d \) and let \( F(x) = f(x) + B_{g_1(x)}(g_2(x)) \), where \( f : \mathbb{R}^d \to \mathbb{R}^d \) is \( L_f \)-Lipschitz and \( s \)-semi-hyperbolic on some compact set \( K \) and \( g_1 : \mathbb{R}^d \to \mathbb{R}_+ \) and \( g_2 : \mathbb{R}^d \to \mathbb{R}^d \) are \( L_1 \)- and \( L_2 \)-Lipschitz. Assume that the projectors \( P^s_x \) and \( P^u_x \) satisfy

\[
\|P^s_y - P^s_{\tilde{y}}\|_{\text{op}}, \|P^u_y - P^u_{\tilde{y}}\|_{\text{op}} \leq L_P \|y - \tilde{y}\|,
\]

where \( L_P > 0 \) and \( \|\cdot\|_{\text{op}} \) is the operator norm induced by the vector norm according to which \( f \) is semi-hyperbolic.

Let us check under which conditions on \( P^s, P^u, g_1, \) and \( g_2 \), the mapping \( F \) is strongly semi-hyperbolic on \( K \) with respect to the original splitting and a modified split. For any \( x \in K \) and \( y \in \mathbb{R}^d \) with \( \text{dist}(y, F(x)) \leq \delta \), there exists some \( \tilde{y} \in \mathbb{R}^d \) such that \( \|y - \tilde{y}\| \leq g_1(x) + \|g_2(x)\| \) and \( \|\tilde{y} - f(x)\| \leq \delta \), and hence,
\[ \|P^s_y(f(x + u + v) - f(x + \tilde{u} + v))\| = \|(P^s_y + (P^s_y - P^s_{\tilde{y}}))(f(x + u + v) - f(x + \tilde{u} + v))\| \]
\[ \leq \|P^s_y(f(x + u + v) - f(x + \tilde{u} + v))\| + \|P^s_y - P^s_{\tilde{y}}\|_{\text{op}} \cdot \|f(x + u + v) - f(x + \tilde{u} + v)\| \]
\[ \leq \lambda_s \|u - \tilde{u}\| + L_P \|y - \tilde{y}\| L_f \|u - \tilde{u}\| \]
\[ \leq (\lambda_s + L_P L_f (g_1(x) + \|g_2(x)\|)) \|u - \tilde{u}\|, \]
etc., and thus, \( F \) is strongly \( \tilde{s} \)-semi-hyperbolic with respect to the split
\[ \tilde{s} = (\lambda_s + \kappa, \lambda_u - \kappa, \mu_s + \kappa, \mu_u + \kappa), \]
provided that
\[ \kappa = L_P L_f \sup_{x \in K} \{g_1(x) + \|g_2(x)\|\} < \nu(\tilde{s}) \quad (B.7) \]
and the Lipschitz constant \( L = L_1 + L_2 \) of the mapping \( x \mapsto B_{g_1(x)}(g_2(x)) \) satisfies
\[ 10 \cdot L d h < \nu(\tilde{s}). \quad (B.8) \]

By the very definition of semi-hyperbolicity, the Lipschitz constant \( L_f \) of \( f \) is larger than one. Consequently, there is a tradeoff between the tolerable roughness of the splitting measured in terms of the Lipschitz constant \( L_P \) and the maximal sizes of the radii of the images \( g_1 \) and the perturbation \( g_2 \) expressed by inequality (B.7).

### B.1.2 Weak semi-hyperbolicity

The notion of weak semi-hyperbolicity is based on a representation via selections. A single-valued function \( f : X \rightarrow X \) is called a selection of a set-valued mapping \( F : X \rightarrow C(X) \) if \( f(x) \in F(x) \) for all \( x \in X \). We will say that \( F \) is weakly semi-hyperbolic if it is parametrized by a family of hyperbolic selections subject to the same split and splitting. This hyperbolicity condition is more general than the notion of strong semi-hyperbolicity, but less graphic and more difficult to check. For a more detailed presentation of this issue, we refer to [117].

**Definition B.1.3.** Let \( s = (\lambda_s, \lambda_u, \mu_s, \mu_u) \) be a split and let \( K \) be a compact subset of an open set \( X \subseteq \mathbb{R}^d \). The set-valued mapping \( F : X \rightarrow \mathcal{C}(X) \) is called weakly \( s \)-semi-hyperbolic on \( K \) if there exist

1) a splitting \( \mathbb{R}^d = E^s_x \oplus E^u_x \) on \( K \) with projectors \( P^s_x \) and \( P^u_x \) for each \( x \in K \), a norm \( \| \cdot \| \) on \( \mathbb{R}^d \), and positive constants \( \delta \) and \( h \);
2) a parametrization \( f : X \times U \rightarrow X \) of \( F \) such that
   (i) \( U \) is an arbitrary index set;
   (ii) \( F(x) = f(x, U) := \bigcup_{u \in U} f(x, u) \) for any \( x \in X \);
   (iii) for any \( u \in U \), the function \( f(\cdot, u) \) is Lipschitz and \( s \)-semi-hyperbolic with respect to the split \( s \), the above splitting, the norm \( \| \cdot \| \) and \( \delta \) and \( h \) in the sense of Definition 3.1.6.
A strongly semi-hyperbolic mapping is weakly semi-hyperbolic.

**Lemma B.1.4.** Let $F : X \rightarrow \mathcal{C}(X)$ be a set-valued mapping which is strongly $s$-semi-hyperbolic on a compact set $K$. Then $F$ is weakly $\tilde{s}$-semi-hyperbolic with respect to the same splitting and the split $\tilde{s} = (\lambda_s + \kappa, \lambda_u - \kappa, \mu_s + \kappa, \mu_u + \kappa)$, where $\kappa = 10 \cdot Ldh$.

**Proof.** If $F$ is strongly $s$-semi-hyperbolic, then $F(x) = f(x) + G(x)$, where $f$ is $s$-semi-hyperbolic and $G$ is $L$-Lipschitz. According to the Parametrization Theorem (see [12, Th. 9.7.2]), there exists a parametrization $g : X \times B_1(0) \rightarrow X$ of $G$ such that $G(x) = g(x, B_1(0))$ for all $x \in X$ and $g$ is Lipschitz in the first argument with Lipschitz constant $10 \cdot Ld$. Hence, we can parametrize $F(x) = f(x) + g(x, B_1(0))$ for all $x \in X$, and the parametrizing selections $f(\cdot) + g(\cdot, z)$, $z \in B_1(0)$, satisfy

$$
\|P^s_y([f(x + u + v) + g(x + u + v, z)] - [f(x + \tilde{u} + v) + g(x + \tilde{u} + v, z)])
\leq \|P^s_y(f(x + u + v) - f(x + \tilde{u} + v))
+ \|P^s_y\| \cdot \|g(x + u + v, z) - g(x + \tilde{u} + v, z)\|
\leq (\lambda_s + 10 \cdot Ldh)\|u - \tilde{u}\|
$$

eq \text{etc. for all } x, y \in K \text{ with } \|y - f(x) - g(x, z)\| \leq \delta, \text{ which shows that the selections are } \tilde{s}\text{-semi-hyperbolic, because } \kappa < \nu(s). \quad \square

**Example B.1.5.** As in Example B.1.2, consider the set-valued mapping

$$
F(x) = f(x) + B_{g_1}(x)(g_2(x)),
$$

where $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is $L_f$-Lipschitz and $s$-semi-hyperbolic on some compact set $K$ and $g_1 : \mathbb{R}^d \rightarrow \mathbb{R}^+$ and $g_2 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are $L_1$- and $L_2$-Lipschitz. Defining $g(x, z) := g_2(x) + g_1(x)z$, $z \in B_1(0)$, we obtain a parametrization $F(x) = f(x) + g(x, B_1(0))$ such that $g(\cdot, z)$ is $L$-Lipschitz in the first argument with $L = L_1 + L_2$.

Fix any $z \in B_1(0)$. For any $y$ with $\text{dist}(y, F(x)) \leq \delta$, there exists some $\tilde{y}$ such that $\|y - \tilde{y}\| \leq g_1(x) + \|g_2(x)\|$ and $\|\tilde{y} - f(x)\| \leq \delta$, and consequently,

$$
\|P^s_y([f(x + u + v) + g(x + u + v, z)] - [f(x + \tilde{u} + v) + g(x + \tilde{u} + v, z)])
\leq \|P^s_y(f(x + u + v) - f(x + \tilde{u} + v))
+ \|P^s_y\| \cdot \|g(x + u + v, z) - g(x + \tilde{u} + v, z)\|
\leq ((P^s_y + (P^s_y - P^s_{\tilde{y}}))(f(x + u + v) - f(x + \tilde{u} + v))) + hL\|u - \tilde{u}\|
\leq \lambda_s\|u - \tilde{u}\| + L_P\|y - \tilde{y}\|L_f\|u - \tilde{u}\| + hL\|u - \tilde{u}\|
\leq (\lambda_s + L_P L_f(g_1(x) + \|g_2(x)\|) + hL\|u - \tilde{u}\|
etc., so that \( f(\cdot) + g(\cdot, z) \) and consequently \( F \) is \( \tilde{s} \)-semi-hyperbolic with respect to the split

\[
\tilde{s} = (\lambda_s + \kappa, \lambda_u - \kappa, \mu_s + \kappa, \mu_u + \kappa),
\]

if

\[
\kappa = L_P L_f \sup_{x \in K} \{ g_1(x) + \| g_2(x) \| \} + hL < \nu(s). \tag{B.9}
\]

The counterpart of the heavy restriction (B.8) required for strong semi-hyperbolicity is the additional term \( hL \) in (B.9). The factor \( 10 \cdot d \) originating from the Parametrization Theorem is obsolete, because \( F \) has a simple structure and possesses a particularly mild parametrization.

### B.2 Expansivity

Let us adapt Definition 2.2.1 to set trajectories. Fix any norm \( \| \cdot \| \) on \( \mathbb{R}^d \). For any subset \( K \subseteq X \) and \( k \) with \( 0 \leq k \leq \infty \), let \( \text{STr}_{\pm k}(F, K) \) denote the collection of all set trajectories

\[
X = \{ X_{-k}, \ldots, X_0, \ldots, X_k \}
\]

of the mapping \( F \) that are entirely contained in the set \( K \), and for any \( X, \tilde{X} \in \text{STr}_{\pm k}(F, K) \) define

\[
\rho_k(X, \tilde{X}) = \max_{-k \leq n \leq k} \text{dist}_H(X_n, \tilde{X}_n)
\]

if \( k < \infty \) and

\[
\rho_{\infty}(X, \tilde{X}) = \max_{n \in \mathbb{Z}} \text{dist}_H(X_n, \tilde{X}_n).
\]

**Definition B.2.1.** A set-valued dynamical system generated by a mapping \( F : X \to \mathcal{C}(X) \) is said to be \( \xi \)-expansive on \( X \) if for any two infinite set trajectories \( X \) and \( \tilde{X} \), the inequality

\[
\rho_{\infty}(X, \tilde{X}) \leq \xi
\]

implies that \( X_0 = \tilde{X}_0 \).

**Lemma B.2.2.** Let \( F : K \to \mathcal{C}(K) \) be a continuous \( \xi \)-expansive mapping. Then for every \( 0 < \varepsilon \) and \( \theta < \xi \) there exists a positive integer \( N(\varepsilon, \theta) \) such that \( \rho_k(X, \tilde{X}) > \theta \) holds for all \( X, \tilde{X} \in \text{STr}_{\pm k}(F, K) \) with \( \text{dist}_H(X_0, \tilde{X}_0) \geq \varepsilon \) and \( k \geq N(\varepsilon, \theta) \).

In order to prove this lemma, we need a set-valued analog of the Bolzano-Weierstrass Theorem.

**Lemma B.2.3.** Let \( K \subseteq \mathbb{R}^d \) be compact and let \( \{ K_n \}_{n \in \mathbb{N}} \) be a sequence of compact sets such that \( K_n \subseteq K \). Then there exists a subsequence \( \{ K_{n_j} \}_{j \in \mathbb{N}} \) of \( \{ K_n \}_{n \in \mathbb{N}} \) and a compact set \( K^* \subseteq K \) such that \( \text{dist}_H(K_{n_j}, K^*) \to 0 \) as \( j \to \infty \).
Proof. Define a grid
\[ \Delta_0 := \left\{ x_{1}^{(0)}, \ldots, x_{l(0)}^{(0)} \right\} := K \cap 2^{0}Z \]
and a system of sets
\[ \mathcal{S}_0 := \left\{ \bigcup_{j=1}^{m} Q(x_{i_j}^{(0)}, 2^{-1}) : \right\} \]
\[ i_j \in \{1, \ldots, l(0)\}, i_j \neq i_j' \quad \text{for} \quad j \neq j', \ 1 \leq m \leq l(0) \}, \]
where \( Q(x, r) := \bigcap_{i=1}^{d} [x_i - r, x_i + r] \) is the cube with radius \( r \) centered at \( x \).
Since \( \mathcal{S}_0 \) has finitely many elements, there exists some \( S_0^* \in \mathcal{S}_0 \) such that for infinitely many \( K_n, S_0^* \) is the minimal element of \( \mathcal{S}_0 \) (with respect to inclusion) containing \( K_n \). We denote this subsequence by \( K_n' \) and set \( K_1^* := K_1' \).

Now we define a finer grid
\[ \Delta_1 := \left\{ x_{1}^{(1)}, \ldots, x_{l(1)}^{(1)} \right\} := S_0^* \cap 2^{-1}Z \]
on \( S_0^* \) and a system of sets
\[ \mathcal{S}_1 := \left\{ \bigcup_{j=1}^{m} Q(x_{i_j}^{(1)}, 2^{-2}) : \right\} \]
\[ i_j \in \{1, \ldots, l(1)\}, i_j \neq i_j' \quad \text{for} \quad j \neq j', \ 1 \leq n \leq l(m) \}. \]
Since \( \mathcal{S}_1 \) has finitely many elements, there exists some \( S_1^* \in \mathcal{S}_1 \) such that for infinitely many \( K_n', S_1^* \) is the minimal element of \( \mathcal{S}_1 \) containing \( K_n' \). We denote this subsequence by \( K_n'' \) and set \( K_2^* := K_2'' \).

Iterating this procedure, we obtain a subsequence \( K_n^* \) of the original \( K_n \) and a nested sequence \( S_n^* \) of nonempty compact sets such that \( K_n^* \subseteq S_n^* \). It is well-known that the intersection \( K^* := \cap_{n \in \mathbb{N}} S_n^* \) is a nonempty compact set.

We have \( \text{dist}(S_n^*, K^*) \to 0 \) as \( n \to \infty \), because otherwise there exist \( \varepsilon > 0 \) and a sequence of elements \( x_{n_j} \in S_n^* \) such that \( \text{dist}(x_{n_j}, K^*) > \varepsilon \). Since \( K \) is compact, we can extract a subsequence (without changing notation) such that \( \lim_{j \to \infty} x_{n_j} =: \bar{x} \in K \). But then \( \bar{x} \in S_n^* \) for every \( n \in \mathbb{N} \), and the resulting \( \bar{x} \in K^* = \cap_{n \in \mathbb{N}} S_n^* \) is a contradiction. Consequently,
\[ \text{dist}(K_n^*, K^*) \leq \text{dist}(K_n^*, S_n^*) + \text{dist}(S_n^*, K^*) \to 0 \quad \text{as} \quad n \to \infty. \]

On the other hand, we have \( \text{dist}(S_n^*, K_n^*) \leq 2^{-n-1} \) by construction. Hence,
\[ \text{dist}(K_n^*, K_n^*) \leq \text{dist}(K_n^*, S_n^*) + \text{dist}(S_n^*, K_n^*) \to 0 \quad \text{as} \quad n \to \infty, \]
which proves that \( \lim_{n \to \infty} \text{dist}_H(K_n^*, K^*) = 0 \). \( \square \)

Proof of Lemma [B.2.2] Suppose the contrary. Then for any \( k \geq 0 \), there exist set trajectories \( X^{(k)}, \tilde{X}^{(k)} \in \text{STr}_{\pm k}(F, K) \) satisfying
for some \( x \) limits \( X \) \( x \) \( x \) \( x \) there exists a sequence \( I \) we can assume that on any finite interval \( X \) of the subsequences generated in the previous steps, in order to establish the Dorff distance.

By Lemma B.2.3, the sequence \( \{ X_0^{(k)} \} \) \( k \in \mathbb{N} \) contains a subsequence \( \{ X_0^{(k)} \} \) \( j \in \mathbb{N} \) which converges to some nonempty compact set \( X_0^* \) with respect to the Hausdorff distance.

Now we apply this argument to \( X_n^{(k)} \), \( n = \pm 1, \pm 2, \ldots \), taking subsequences of the subsequences generated in the previous steps, in order to establish the existence of a limit sequence \( X^* := \{ X_n^* \} \) \( n \in \mathbb{Z} \). Note that by construction, we can assume that on any finite interval \( I \subseteq \mathbb{Z} \), the limits \( X_n^* \), \( n \in I \), are generated by the same subsequence.

Let us check that \( X^* \in \text{STr}_{\pm \infty}(F, K) \). For arbitrary \( x_{n+1}^* \in X_{n+1}^* \), there exists a sequence \( x_{n+1}^{(k)} \in X_{n+1}^{(k)} \) with \( \lim_{j \to \infty} x_{n+1}^{(k)} = x_{n+1}^* \). By definition, \( x_{n+1}^{(k)} \in F(x_{n+1}^{(k)}) \) for some \( x_{n+1}^{(k)} \in X_{n+1}^{(k)} \). By compactness of \( K \), and since the limits \( X_n^* \) and \( X_{n+1}^* \) are generated by the same subsequence, we can once more extract a subsequence (without changing notation) so that \( \lim_{j \to \infty} x_{n+1}^{(k)} = x_n^* \) for some \( x_n^* \in X_n^* \). Hence

\[
\text{dist}(x_{n+1}^*, F(x_n^*)) \\
\leq |x_{n+1}^* - x_{n+1}^{(k)}| + \text{dist}(x_{n+1}^{(k)}, F(x_{n+1}^{(k)})) + \text{dist}(F(x_{n+1}^{(k)}), F(x_n^*)) \to 0
\]
as \( j \to \infty \). Since the images of \( F \) are compact, \( x_{n+1}^* \in F(x_n^*) \) and thus \( X_{n+1}^* \subseteq F(X_n^*) \).

On the other hand, let \( x_n^* \in X_n^* \) and \( x_{n+1}^* \in F(x_n^*) \) be given. Again, there exists a sequence \( x_{n+1}^{(k)} \in X_{n+1}^{(k)} \) such that \( \lim_{j \to \infty} x_{n+1}^{(k)} = x_n^* \). But then,

\[
\text{dist}(x_{n+1}^*, X_{n+1}^*) \\
\leq \text{dist}(x_{n+1}^*, F(x_n^*)) + \text{dist}(F(x_n^*), F(x_{n+1}^{(k)})) + \text{dist}(F(x_{n+1}^{(k)}), X_{n+1}^*) \to 0
\]
as \( j \to \infty \). By compactness of \( X_{n+1}^* \), we have \( x_{n+1}^* \in X_{n+1}^* \) and thus \( F(X_n^*) \subseteq X_{n+1}^* \).

We can repeat the whole construction with \( \tilde{X}^{(k)} \) obtaining a limiting set trajectory \( \tilde{X}^* \). But then \( X^* \) and \( \tilde{X}^* \) are set trajectories such that

\[
\text{dist}_H(X_0^*, \tilde{X}_0^*) \geq \varepsilon \quad \text{and} \quad \text{dist}_H(X_n^*, \tilde{X}_n^*) \leq \theta < \xi,
\]
which contradicts the initial assumption that \( F \) is expansive. \( \square \)

It is still unclear at present how semi-hyperbolicity is related to expansivity in the set-valued context. The following lemma only guarantees the existence of individual trajectories which satisfy the classical expansivity condition, but not expansivity in the sense of Definition B.2.1.

**Lemma B.2.4.** Let \( K \subseteq \mathbb{R}^d \) be a compact set. If \( F : K \to \mathcal{C}(K) \) is weakly \( s \)-semi-hyperbolic and \( K \) is invariant under \( F \), then the generated set-valued
A dynamical system is exponentially expansive in the following sense: There exist an exponent $r > 1$ and constants $\xi, c > 0$ such that for any finite trajectory
\[
\mathbf{x} = \{x_{-n}, \ldots, x_0, \ldots, x_{n+}\}
\]
and any $y_0 \in K$, there exists some trajectory
\[
\mathbf{y} = \{y_{-n}, \ldots, y_0, \ldots, y_{n+}\}
\]
such that at least one of the following inequalities holds:
\[
\|x_n - y_n\| \geq cr^n \|x_0 - y_0\|, \quad n = 1, 2, \ldots, n+
\]
(B.10)
or
\[
\|x_n - y_n\| \geq cr^{-n} \|x_0 - y_0\|, \quad n = -1, -2, \ldots, -n-.
\]
(B.11)

**Proof.** By Definition B.1.3,
\[
x_{n+1} \in F(x_n) = f(x_n, U), \quad n \in [-n-, n_+ - 1],
\]
and thus there exists a sequence $\{u_n\} \subseteq U$ such that
\[
x_{n+1} = f(x_n, u_n), \quad n \in [-n-, n_+ - 1].
\]

Since $K$ is invariant under $F$, it is also invariant under the sequence of mappings $\{f(\cdot, u_n)\}$. In particular, there exists a sequence $\{y_n\}$ satisfying
\[
y_{n+1} = f(y_n, u_n), \quad n \in [-n-, n_+ - 1].
\]

As the selections $f(\cdot, u_n)$ of $F$ are semi-hyperbolic according to the same split and splitting, the results from Chap. 6 remain valid for the sequence $\{f(\cdot, u_n)\}$. In particular, Theorem 6.1.5 applies and guarantees that either estimate B.10 or B.11 holds. $\Box$

**B.3 Bi-Shadowing**

The setting of weak semi-hyperbolicity and individual trajectories allows to prove fairly general shadowing and inverse shadowing theorems. Since strongly semi-hyperbolic mappings are weakly semi-hyperbolic (see Lemma B.1.4), a similar result can be shown for set-trajectories. A complete and self-contained coverage of the topic can be found in [117].

**Definition B.3.1.** A $\gamma$-pseudo-trajectory of a set-valued dynamical system generated by a mapping $F : X \to \mathcal{C}(X)$ is a sequence $\mathbf{y} = \{y_n\} \subseteq X$ on some finite or infinite interval $\mathbb{I} \in \mathbb{Z}$ satisfying
\[
\text{dist}(y_{n+1}, F(y_n)) \leq \gamma
\]
whenever $n, n + 1 \in \mathbb{I}$.
Let $\text{Tr}(F, K, \gamma)$ denote the set of all infinite $\gamma$-pseudo-trajectories remaining in $K$, and let $\text{Tr}(F, K) = \text{Tr}(F, K, 0)$ denote the set of all individual trajectories remaining in $K$.

**Definition B.3.2.** An individual trajectory $x = \{x_n\} \subseteq X$ is said to $\varepsilon$-shadow a $\gamma$-pseudo-trajectory $y = \{y_n\}$ defined on some finite or infinite interval $I \subseteq \mathbb{Z}$ if
\[
\|x_n - y_n\| \leq \varepsilon, \quad n \in I.
\]

Let us formulate a bi-shadowing result similar to Theorem 6.3.9. If we do not want to impose restrictive conditions on the geometry of the set-valued mappings, we have to measure their distance in terms of the Hausdorff distance between their relevant selections rather than the Hausdorff distance of their images.

**Definition B.3.3.** Let $F, \Phi : X \rightarrow \mathcal{C}(X)$ be set-valued mappings with parametrizations $F(x) = f(x,U)$ and $\Phi(x) = \phi(x,V)$ for all $x \in X$, where $U$ and $V$ are arbitrary fixed index sets. The distance between $F$ and $\Phi$ with respect to $f$ and $\phi$ is defined by
\[
\text{dist}_{f,\phi}(F, \Phi) = \max \left\{ \sup_{u \in U} \inf_{v \in V} \|\phi(\cdot, v) - f(\cdot, u)\|_{\infty}, \sup_{v \in V} \inf_{u \in U} \|\phi(\cdot, v) - f(\cdot, u)\|_{\infty} \right\}.
\]

**Theorem B.3.4.** A set-valued dynamical system generated by a weakly s-semi-hyperbolic mapping $F : X \rightarrow \mathcal{C}(X)$ is bi-shadowing on a compact subset $K$ of $X$ with positive parameters $\alpha = \alpha(s, h)$ and $\beta = \beta(s, h, \delta)$ specified in (6.44) and (6.45) in the following sense:

Let $x = \{x_n\} \in \text{Tr}(F, K, \gamma)$ be any given infinite pseudo-trajectory with $0 \leq \gamma \leq \beta$ and let $\Phi : X \rightarrow \mathcal{C}(\mathbb{R}^d)$ be any mapping that can be parametrized by a function $\phi : X \times V \rightarrow X$ such that
\begin{itemize}
  \item[(i)] $V$ is an arbitrary index set;
  \item[(ii)] $\Phi(x) = \phi(x,V) := \bigcup_{v \in V} \phi(x,v)$ for any $x \in X$;
  \item[(iii)] $\phi(\cdot, v)$ is continuous for every $v \in V$;
  \item[(iv)] $\text{dist}_{f,\phi}(F, \Phi) \leq \beta - \gamma$.
\end{itemize}

Then there exists a trajectory $y = \{y_n\} \in \text{Tr}(\Phi, K)$ such that
\[
\|x_n - y_n\| \leq \alpha(\gamma + \text{dist}_{f,\phi}(F, \Phi)), \quad n \in \mathbb{Z}.
\]

**Proof.** An individual trajectory of the set-valued system is defined by
\[
x_{n+1} \in F(x_n) = f(x_n, U), \quad n \in \mathbb{Z}.
\]

If we restrict our attention to this trajectory, we may select a sequence $\{u_n\} \subseteq U$ such that
\[
x_{n+1} = f(x_n, u_n), \quad n \in \mathbb{Z}.
\]
By definition, all selections \( f(\cdot, u_n) \) of \( F \) are semi-hyperbolic according to the same split and splitting, so that the results developed in Chap. 6 remain valid for the non-autonomous dynamical system (B.12). Moreover, we have \( \text{dist}_{f,\phi}(F, \Phi) \leq \beta - \gamma \) by assumption, so that for any \( n \in \mathbb{N} \) we find some \( v_n \in V \) such that \( \|f(\cdot, u_n) - \phi(\cdot, v_n)\|_{\infty} \leq \beta - \gamma \).

Now a repetition of the chain of arguments proving Theorem 6.3.9 yields the existence of a sequence \( \{y_n\} \) satisfying
\[
y_{n+1} = \phi(y_n, v_n) \in \Phi(y_n), \quad n \in \mathbb{Z},
\]
which is the desired trajectory of the set-valued mapping \( \Phi \).

It was shown in [115, Remark 3] that even under stronger assumptions, the trajectory \( \{y_n\} \) of \( \Phi \) cannot be expected to be unique.

Combining Lemma B.1.4 and Theorem B.3.4 yields a shadowing theorem for strongly semi-hyperbolic mappings.

**Theorem B.3.5.** Consider a set-valued dynamical system generated by a mapping
\[
F(x) = f(x) + G(x), \quad x \in X,
\]
which is strongly \( s \)-semi-hyperbolic on a compact subset \( K \) of \( X \). Then \( F \) is bi-shadowing on \( K \) with a modified split \( \tilde{s} \) and positive parameters \( \alpha = \alpha(\tilde{s}, h) \) and \( \beta = \beta(\tilde{s}, h, \delta) \) specified in Theorem 6.3.3 in the following sense:

Let \( x = \{X_n\} \in \text{STr}(F, K, \gamma) \) be any given infinite \( \gamma \)-pseudo-trajectory with \( 0 \leq \gamma \leq \beta \) and let \( \Phi : X \rightarrow C(\mathbb{R}^d) \) be any mapping that can be parametrized by a function \( \phi : X \times V \rightarrow X \) such that
(i) \( V \) is an arbitrary index set;
(ii) \( \Phi(x) = \phi(x, V) := \bigcup_{v \in V} \phi(x, v) \) for any \( x \in X \);
(iii) \( \phi(\cdot, v) \) is continuous for every \( v \in V \);
(iv) \( \text{dist}_{f,\phi}(F, \Phi) \leq \beta - \gamma \).

Then there exists a set trajectory \( y = \{Y_n\} \in \text{STr}(\Phi, K) \) such that
\[
\text{dist}_H(X_n, Y_n) \leq \alpha(\gamma + \text{dist}_{f,\phi}(F, \Phi)), \quad n \in \mathbb{Z}.
\]

The following lemma is useful for the proof of Theorem B.3.5.

**Lemma B.3.6.** Let \( X \) be an open subset of \( \mathbb{R}^d \) and let \( K \subseteq X \) be compact. Let \( \gamma \geq 0 \) and let \( x^{(k)} \in \text{Tr}(F, K, \gamma), \ k \in \mathbb{N}, \) be a sequence of \( \gamma \)-pseudo-trajectories. If there exists some accumulation point \( x \in X \) of \( \{x_n^{(k)} : k \in \mathbb{N}\} \) for some \( n_0 \in \mathbb{Z}, \) then there exists some \( x \in \text{Tr}(F, K, \gamma) \) such that \( x_{n_0} = x \) and \( x_n \) is an accumulation point of \( \{x_n^{(k)} : k \in \mathbb{N}\} \) for all \( n \in \mathbb{Z}. \)

**Proof.** The construction of the limiting trajectory is the same as the one in the proof of Lemma B.2.2. Since \( K \) is closed, all limits are in \( K, \) and the distances are not affected by taking limits. \( \square \)
Proof of Theorem B.3.5 Since $x \in \text{STr}(F,K,\gamma)$, it is clear that there exists a collection $P_x \subseteq \text{Tr}(F,K,\gamma)$ such that

$$X_n = \bigcup_{x \in P_x} x_n, \quad n \in \mathbb{Z}.\]

By Lemma B.1.4, $F$ is weakly $\tilde{s}$-semi-hyperbolic with respect to a modified split $\tilde{s}$ and the same splitting, so that for every $x \in P_x$, Theorem B.3.4 guarantees the existence of some $y(x) = \{y_n(x)\} \in \text{Tr}(\Phi,K)$ such that

$$\|y(x) - x\|_{\infty} \leq \alpha(\gamma + \text{dist}_{f,\phi}(F,\Phi)).$$

Let us denote $Y_n := \{y_n(x) : x \in P_x\}$. According to Lemma B.3.6, for every accumulation point $y$ of $Y_{n_0}$ there exists some $y = \{y_n\} \in \text{Tr}(\Phi,K)$ such that $y_{n_0} = y$ and $y_n$ is an accumulation point of $Y_n$ for all $n \in \mathbb{Z}$. Thus, the closures $Y_n^*$ of $Y_n$ form a set-trajectory $Y = \{Y_n^*\} \in \text{STr}(\Phi,K)$ with

$$\text{dist}(X_n, Y_n^*)_H \leq \alpha(\gamma + \text{dist}_{f,\phi}(F,\Phi)), \quad n \in \mathbb{Z}.\]

$\square$
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Semi-hyperbolicity extends the classical concept of hyperbolicity for diffeomorphisms to noninvertible Lipschitz mappings on sets that need not be invariant. Moreover, the stable and unstable manifolds need to be mapped only approximately onto their counterparts.

Semi-hyperbolic mappings are bi-shadowing, i.e., they also satisfy a converse form of shadowing in which the pseudo-orbits are not arbitrary but are the true orbits of admissible classes of approximating mappings.

The theory of these concepts is developed systematically in this monograph and illustrated through applications of nonsmooth systems including those with delays and hysteresis, as well as numerical approximations. Set-valued mappings are also considered.