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REMARKS ON 5-DIMENSIONAL COMPLETE INTERSECTIONS

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Abstract. This paper will give some examples of diffeomorphic complex 5-dimensional complete intersections and remarks on these examples. Then a result on the existence of diffeomorphic complete intersections that belong to components of the moduli space of different dimensions will be given as a supplement to the results of P.Brückmann (J. reine angew. Math. **476** (1996), 209–215; **525** (2000), 213–217).

1. Introduction

Let $X_n(\underline{d}) \subset \mathbb{C}P^{n+r}$ be a smooth complete intersection of multidegree $\underline{d} = (d_1, \cdots, d_r)$, i.e., the transversal intersections of hypersurfaces of degrees $d_1, \cdots, d_r$ respectively. We call the product $d_1 d_2 \cdots d_r$ the total degree, denoted by d. It is well known that all complete intersections of fixed multidegree are diffeomorphic. On the other hand, there exist diffeomorphic complete intersections with different multidegrees. For lower dimensions, such as complex dimensions $2, 3, 4$, the diffeomorphic examples can be found in [1, 2, 8]. But for higher dimensions, there is no evident examples. W. Ebeling ([3]) and A.S. Libgober-J. Wood ([10]) independently found examples of homeomorphic complex 2-dimensional complete intersections but not diffeomorphic. In [6], F.Q. Fang and the author proved that, in dimensions $n = 5, 6, 7$, two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ are homeomorphic if and only if they have the same total degree, Pontrjagin classes and Euler characteristics. Particularly, to the prime factorization of total degree $d = \prod_{p \text{ primes}} p^{\nu_p(d)}$, under the assumption that $\nu_p(d) \geqslant \frac{2n+1}{2(p-1)} + 1$ for all primes p with $p(p-1) \leqslant n+1$, C. Traving proved the following result (see [7, Theorem A] or [11]).

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Theorem (Traving)**.** *Two complete intersections $X_n(\underline{d})$ and $X_n(\underline{d}')$ of complex dimension $n > 2$ fulfilling the assumption above for the total degree are diffeomorphic if and only if the total degrees, the Pontrjagin classes and the Euler characteristics agree.*

The first purpose of this paper is to give examples of diffeomorphic complex 5-dimensional complete intersections with different multidegrees. These examples, which are easy to check but hard to happen upon, were found by computer search. From these examples, we can deduce some interesting remarks about complete intersections.

Libgober and Wood ([9]) showed the existence of homeomorphic complete intersections of dimension 2 and diffeomorphic ones of dimension 3 which belong to components of the moduli space having different dimensions. In fact it was shown that there is a procedure which allows one to produce from a pair of homeomorphic complete intersections an arbitrarily long family, all members of which are homeomorphic. P. Brückmann ([1]) shows that the construction mentioned yields families of arbitrary length t of complete intersections in $\mathbb{C}P^{4t-2}$ (resp. $\mathbb{C}P^{5t-2}$) consisting of homeomorphic complete intersections of dimension 2 (resp. diffeomorphic ones of dimension 3) but that belong to components of the moduli space of different dimensions. Furthermore, under Theorem 1 of [5], Brückmann also proves the similar result for the complete intersections of dimension 4 in $\mathbb{C}P^{6t-2}$([2]).

Another purpose of this paper is to give the following theorem, which is a supplement to the results of Brückmann [1, 2].

Theorem 1.1. *For each integer $t > 1$, there exist t diffeomorphic complex 5-dimensional complete intersections in $\mathbb{C}P^{7t-2}$ isomorphism class of which lie in different dimensional components of the moduli space.*

This paper is organized as follows: After presenting the basic formulas of characteristic classes of complete intersections in Section 2, we will give examples of diffeomorphic complex 5-dimensional complete intersections in Section 3. Section 4 proves Theorem 1.1. The last section will be devoted to the code of computer program to evaluate an inequality, which is a key to prove Theorem 1.1.

2. Characteristic classes of complete intersections

For a complete intersection $X_n(\underline{d})$, let H be the restriction of the dual bundle of the canonical line bundle over $\mathbb{C}P^{n+r}$ to $X_n(\underline{d})$, and $x = c_1(H) \in H^2(X_n(\underline{d});\mathbb{Z})$. Associate the multidegree $\underline{d} = (d_1, d_2, \dots, d_r)$, define the power sums $s_i = \sum_{j=1}^{r} d_j^i$ for $1 \leqslant i \leqslant n$. Let $g_k (k \geqslant 1)$ be the polynomials that can be iteratively computed from the Newton formula:

$$s_k - g_1(s_1)s_{k-1} + \frac{1}{2}g_2(s_1, s_2)s_{k-2} + \cdots + (-1)^k \frac{1}{k!} g_k(s_1, s_2, \dots, s_k)k = 0.$$

For example, the first six are

$$\begin{aligned}
g_1(s_1) &= s_1,\\
g_2(s_1,s_2) &= s_1^2 - s_2,\\
g_3(s_1,s_2,s_3) &= s_1^3 - 3s_1s_2 + 2s_3,\\
g_4(s_1,\dots,s_4) &= s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4,\\
g_5(s_1,\dots,s_5) &= s_1^5 - 10s_1^3s_2 + 20s_1^2s_3 - 30s_1s_4 + 15s_1s_2^2 - 20s_2s_3 + 24s_5,\\
g_6(s_1,\dots,s_6) &= s_1^6 - 15s_1^4s_2 + 40s_1^3s_3 - 90s_1^2s_4 + 45s_1^2s_2^2 - 120s_1s_2s_3 + 144s_1s_5\\
&\quad - 15s_2^3 + 90s_2s_4 + 40s_3^2 - 120s_6.
\end{aligned}$$

Then the Chern classes and Pontrjagin classes are presented as follows ([8]):

$$\begin{aligned}
c_k &= \frac{1}{k!}g_k(n+r+1-s_1,\dots,n+r+1-s_k)x^k, 1 \leqslant k \leqslant n,\\
p_k &= \frac{1}{k!}g_k(n+r+1-s_2,\dots,n+r+1-s_{2k})x^{2k}, 1 \leqslant k \leqslant \left[\frac{n}{2}\right].
\end{aligned}$$

The Euler characteristic is (where $x^n \cap [X_n(\underline{d})] = d = d_1 \cdots d_r$)

$$e(X_n(\underline{d})) = c_n(X_n(\underline{d})) \cap [X_n(\underline{d})] = d\frac{1}{n!}g_n(n+r+1-s_1,\dots,n+r+1-s_n).$$

Note that the k^{th} Pontrjagin class p_k is an integral multiple of x^{2k}, where x generates the second cohomology of the complete intersection. Thus we can compare this invariant for different complete intersections. For convenience, throughout the rest of the paper, we view the Pontrjagin class p_k of $X_n(\underline{d})$ as the multiple of x^{2k}.

3. Examples of diffeomorphic 5-dimensional complete intersections

For complex 5-dimensional complete intersection $X_5(d_1,\dots,d_r)$, its total degree, Pontrjagin classes and Euler characteristic are as follows:

$$d = d_1 \times \cdots \times d_r, \tag{3.1}$$

$$p_1 = 6 + r - s_2, \tag{3.2}$$

$$p_2 = \frac{1}{2}\big[(6+r-s_2)^2 - (6+r-s_4)\big], \tag{3.3}$$

$$\begin{aligned}
e = \frac{1}{5!}d\big[&(6+r-s_1)^5 - 10(6+r-s_1)^3(6+r-s_2)\\
&+ 20(6+r-s_1)^2(6+r-s_3) - 30(6+r-s_1)(6+r-s_4)\\
&+ 15(6+r-s_1)(6+r-s_2)^2 - 20(6+r-s_2)(6+r-s_3)\\
&+ 24(6+r-s_5)\big].
\end{aligned} \tag{3.4}$$

Here, p_1 and p_2 denote the Pontrjagin classes in the sense described in the end of Section 2.

By Theorem 1.1 of [6], to find homeomorphic complex 5-dimensional complete intersections, we only need to find different multidegrees, such that (3.1)-(3.4) all agree respectively. Additionally, by [7, Theorem A], for the total degree $d = \prod_{p \text{ primes}} p^{\nu_p(d)}$, if $\nu_2(d) \geqslant 7$ and $\nu_3(d) \geqslant 4$, the homeomorphic 5-dimensional complete intersections are diffeomorphic. This searching can completely be done by computer. According to [8, Proposition 7.3], if $X_n(\underline{d}) \subset \mathbb{C}P^{n+r}$ be a complete intersection of given codimension r with $n > 2$ and $2r \leqslant n+2$, then the total degree and Pontrjagin classes of $X_n(\underline{d})$ determine the multidegree. Thus, it is impossible to find out such a homeomorphic or diffeomorphic example with different multidegrees in which one of the complete intersections has codimension 2 or 3 for complex dimension 5. Theoretically, there should exist a lot of homeomorphic complete intersections with codimension $\geqslant 4$. However, with the codimension becoming smaller, it will become more difficult to find out such examples. In fact, we can offer such diffeomorphic examples with codimension 7 (See Section 4).

Example 3.1. (1), Take complete intersections $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$, it is easy to check that they have the same total degree, Pontrjagin classes and Euler characteristic:

d	p_1	p_2	e/d
37362124800	−11578	84696853	−31485015068

Since $d = 37362124800 = 2^{11} \times 3^6 \times 5^2 \times 7 \times 11 \times 13$, so $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$ are diffeomorphic.

(2), By deleting the last degree 3 from the multidegrees in (1), we take complete intersections $X_5(66, 56, 45, 39, 16, 15, 8)$ and $X_5(64, 60, 42, 39, 20, 11, 9)$.

$X_5(\underline{d})$	d	e/d
$X_5(66, 56, 45, 39, 16, 15, 8)$	12454041600	−30762573120
$X_5(64, 60, 42, 39, 20, 11, 9)$	12454041600	−30762561840

The different Euler characteristics imply that $X_5(66, 56, 45, 39, 16, 15, 8)$ is not homotopy equivalent to $X_5(64, 60, 42, 39, 20, 11, 9)$.

(3), By appending a degree 7 into the multidegrees in (1), we find that complete intersections $X_5(66, 56, 45, 39, 16, 15, 8, 7, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 7, 3)$ have different Euler characteristics,

$X_5(\underline{d})$	d	e/d
$X_5(66, 56, 45, 39, 16, 15, 8, 7, 3)$	261534873600	−33795490160
$X_5(64, 60, 42, 39, 20, 11, 9, 7, 3)$	261534873600	−33795524864

So $X_5(66, 56, 45, 39, 16, 15, 8, 7, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 7, 3)$ are not homotopy equivalent.

Example 3.2. For complex 4-dimensional complete intersection $X_4(d_1, \ldots, d_r)$, its Euler characteristic is as follows:

$$e = \frac{d}{4!}\big[(5 + r - s_1)^4 - 6(5 + r - s_1)^2(5 + r - s_2) + 8(5 + r - s_1)(5 + r - s_3) + 3(5 + r - s_2)^2 - 6(5 + r - s_4)\big].$$

Let $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42, 39, 20, 11, 9, 3)$, which are diffeomorphic by Example 3.1 (1), simultaneously make transversal intersection with hypersurface of homogeneous degree 2, we can construct two complex 4-dimensional complete intersections $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ and $X_4(64, 60, 42, 39, 20, 11, 9, 3, 2)$. They have different Euler characteristics,

$X_4(\underline{d})$	d	e/d
$X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$	74724249600	365019422
$X_4(64, 60, 42, 39, 20, 11, 9, 3, 2)$	74724249600	365025086

So $X_4(66, 56, 45, 39, 16, 15, 8, 3, 2)$ and $X_4(64, 60, 42, 39, 20, 11, 9, 3, 2)$ are not homotopy equivalent. That is, although $X_5(66, 56, 45, 39, 16, 15, 8, 3)$ and $X_5(64, 60, 42,$

39, 20, 11, 9, 3) are diffeomorphic, $X_4(66,56,45,39,16,15,8,3,2)$ and $X_4(64,60,42,39,20,11,9,3,2)$, which are the transversal intersection of diffeomorphic complex 5-dimensional complete intersections with the same hypersurface of homogeneous degree 2, do not have the same homotopy type.

Example 3.3. For complex 6-dim complete intersection $X_6(d_1,\dots,d_r)$, its Euler characteristic is as follows:

$$\begin{aligned} e = \frac{d}{6!}\big[&(7+r-s_1)^6 - 15(7+r-s_1)^4(7+r-s_2) + 40(7+r-s_1)^3(7+r-s_3) \\ &- 90(7+r-s_1)^2(7+r-s_4) + 45(7+r-s_1)^2(7+r-s_2)^2 \\ &- 120(7+r-s_1)(7+r-s_2)(7+r-s_3) + 144(7+r-s_1)(7+r-s_5) \\ &- 15(7+r-s_2)^3 + 90(7+r-s_2)(7+r-s_4) \\ &+ 40(7+r-s_3)^2 - 120(7+r-s_6)\big]. \end{aligned}$$

Take $X_6(66,56,45,16,15,8,3)$, $X_6(64,60,42,20,11,9,3)$, it is easy to check that

$X_6(\underline{d})$	d	e/d
$X_6(66,56,45,16,15,8,3)$	958003200	1370218430570
$X_6(64,60,42,20,11,9,3)$	958003200	1369971514442

The different Euler characteristics imply that $X_6(66,56,45,16,15,8,3)$ is not homotopy equivalent to $X_6(64,60,42,20,11,9,3)$. However, $X_5(66,56,45,39,16,15,8,3)$ and $X_5(64,60,42,39,20,11,9,3)$, which are the transversal intersection of the above two non-homotopy equivalent complex 6-dimensional complete intersections with the same hypersurface of homogeneous degree 39, are diffeomorphic by Example 3.1 (1).

Compare the above Examples 3.1, 3.2 and 3.3, we can obtain the following interesting remarks.

Remark 3.4. $X_n(d_1,\dots,d_{r-1},c)$ is homeomorphic (diffeomorphic, homotopy equivalent) to $X_n(d'_1,\dots,d'_{r-1},c)$, however, $X_n(d_1,\dots,d_{r-1})$ is not necessarily homeomorphic (diffeomorphic, homotopy equivalent) to $X_n(d'_1,\dots,d'_{r-1})$, and $X_n(d_1,\dots,d_{r-1},c,c')$ is not necessarily homeomorphic (diffeomorphic, homotopy equivalent) to $X_n(d'_1,\dots,d'_{r-1},c,c')$ (See Example 3.1 (1),(2),(3)).

Note that, in [4], Fang asked the following question: If $X_n(\underline{d})$ and $X_n(\underline{d}')$ are diffeomorphic/or homeomorphic/or homotopy equivalent, is $X_n(\underline{d};a)$ diffeomorphic to $X_n(\underline{d}';a)$ for a natural number a? Here $X_n(\underline{d};a)$ is the complete intersection with multidegree $(d_1,d_2,\dots,d_r,a)$. Now, Remark 3.4 partially gives a negative answer to Fang's question.

Remark 3.5. $X_{n+1}(d_1,\dots,d_{r-1})$ is diffeomorphic to $X_{n+1}(d'_1,\dots,d'_{r-1})$, but it may not be true to $X_n(d_1,\dots,d_{r-1},c)$ and $X_n(d'_1,\dots,d'_{r-1},c)$ for some c (See Example 3.2).

Remark 3.6. Even if $X_{n+1}(d_1,\dots,d_{r-1})$ is not diffeomorphic to $X_{n+1}(d'_1,\dots,d'_{r-1})$, $X_n(d_1,\dots,d_{r-1},c)$ can be diffeomorphic to $X_n(d'_1,\dots,d'_{r-1},c)$ for some c (See Example 3.3).

4. Moduli spaces of complete intersections

In this section, we will prove Theorem 1.1.

Let $X_n(\underline{d}) \subset \mathbb{C}P^N$, where $n \geqslant 2, \underline{d} = (d_1, \ldots, d_r), d_i \geqslant 2$ and $r = N - n$. Assume that $X_n(\underline{d})$ is not a **K**3-surface or a quadratic hypersurface, then from [1, Lemma 3], the explicit formula for moduli space dimension is

$$m(\underline{d}) := m(X_n(\underline{d})) = 1 - (N+1)^2 + \sum_{i=1}^{r} \binom{N+d_i}{N} + \sum_{i=1}^{r}\sum_{j=1}^{r}(-1)^j \sum_{1\leqslant k_1 < \cdots < k_j \leqslant r} \binom{N+d_i-d_{k_1}-\cdots-d_{k_j}}{N}. \tag{4.1}$$

Where $\binom{m}{N} = 0$ for $m < N (m \in \mathbb{Z})$.

Proof of Theorem 1.1. Consider the following two multidegrees

$$\underline{d} = (88, 77, 72, 54, 48, 31, 29), \ \underline{d}' = (87, 81, 64, 62, 44, 33, 28).$$

They have the same power sums, so the corresponding complete intersections have the same total degree, Pontrjagin classes and Euler characteristic:

s_1	s_2	s_3	s_4	s_5
399	25879	1833489	137438707	10682130249

d	p_1	p_2	e/d
1136843237376	−25866	403244325	−296492615140

The total degree is $1136843237376 = 2^{11} \times 3^6 \times 7 \times 11^2 \times 29 \times 31$, so the two complete intersections $X_5(\underline{d})$ and $X_5(\underline{d}')$ are diffeomorphic but have different moduli space dimensions:

$$m(\underline{d}) = 1\ 382\ 270\ 197\ 857\ 128,$$
$$m(\underline{d}') = 1\ 370\ 693\ 416\ 581\ 393.$$

There is a way to generate larger sets of diffeomorphic complete intersections from the above pairs $\underline{d}$ and $\underline{d}'$, which arose from [9] and had an application in [1, 2].

Denote the composed multidegree

$$d_{\lambda,\mu} = (\underbrace{\underline{d}, \ldots, \underline{d}}_{\lambda}, \underbrace{\underline{d}', \ldots, \underline{d}'}_{\mu}), \lambda + \mu = s \geqslant 1\ .$$

Then the composed multidegrees $d_{0,s}, d_{1,s-1}, \ldots, d_{s,0}$ have the same power sums $s_1, s_2, \ldots, s_5$ respectively, so the corresponding complete intersections $X_5(d_{0,s})$, $X_5(d_{1,s-1})$, $\ldots$, $X_5(d_{s,0})$ are diffeomorphic to each other. We can prove the following inequality:

$$m(d_{\lambda+1,\mu-1}) - m(d_{\lambda,\mu}) > 0, 0 \leqslant \lambda < s = \lambda + \mu.$$

This inequality will be proved in the coming Proposition.

Now, the sequence $m(d_{\lambda,s-\lambda})|_{\lambda=0,1,\ldots,s-1}$ is strictly monotonously increasing. Let $t = s+1$, there exist t five-dimensional complete intersections $X_5(d_{0,s})$, $X_5(d_{1,s-1})$, $\ldots$, $X_5(d_{s,0})$ in $\mathbb{C}P^{7s+5} = \mathbb{C}P^{7t-2}$ with the desired properties. The proof is finished. □

Proposition 4.1.

$$m(d_{\lambda+1,s-\lambda-1}) - m(d_{\lambda,s-\lambda}) > 0, 0 \leqslant \lambda < s.$$

Proof. For the chosen multidegrees $\underline{d}$ and $\underline{d}'$,

$$m(d_{\lambda,s-\lambda}) = 1-(N+1)^2 + \Big[\lambda \sum_{d_i\in \underline{d}} + (s-\lambda)\sum_{d_i\in\underline{d}'}\Big]\binom{N+d_i}{N} \tag{4.2}$$
$$+\Big[\lambda \sum_{d_i\in \underline{d}} + (s-\lambda)\sum_{d_i\in\underline{d}'}\Big]\sum_{j=1}^{3}(-1)^j \sum_{\substack{1\leqslant k_1<\cdots<k_j\leqslant 7s\\ d_{k_1},\ldots,d_{k_j}\in d_{\lambda,s-\lambda}}}\binom{N+d_i-d_{k_1}-\cdots-d_{k_j}}{N},$$

Where, the index j is maximally 3 that is determined by $\max(\underline{d},\underline{d}') = 88$ and $\min(\underline{d},\underline{d}') = 28$. So,

$$m(d_{\lambda+1,s-\lambda-1}) - m(d_{\lambda,s-\lambda}) \tag{4.3}$$
$$= \Big[\sum_{d_i\in\underline{d}} - \sum_{d_i\in\underline{d}'}\Big]\binom{N+d_i}{N}$$
$$+\Bigg\{\Big[(\lambda+1)\sum_{d_i\in\underline{d}} + (s-\lambda-1)\sum_{d_i\in\underline{d}'}\Big]\sum_{j=1}^{3}(-1)^j\sum_{\substack{1\leqslant k_1<\cdots<k_j\leqslant 7s\\ d_{k_1},\ldots,d_{k_j}\in d_{\lambda+1,s-\lambda-1}}}$$
$$-\Big[\lambda\sum_{d_i\in\underline{d}} + (s-\lambda)\sum_{d_i\in\underline{d}'}\Big]\sum_{j=1}^{3}(-1)^j\sum_{\substack{1\leqslant k_1<\cdots<k_j\leqslant 7s\\ d_{k_1},\ldots,d_{k_j}\in d_{\lambda,s-\lambda}}}\Bigg\}\binom{N+d_i-d_{k_1}-\cdots-d_{k_j}}{N}$$
$$:= \sum_{j=0}^{3} M_j(\lambda,s),$$

To show that expression given by formula (4.3) is positive, let us decompose (4.3) into the sum of $M_j(\lambda,s), j = 0,1,2,3$. In the following, we will describe $M_j(\lambda,s)$ as polynomials of invariants s and λ ($N = 7s+5$).

Firstly,

$$M_0(\lambda,s) := \Big[\sum_{d_i\in\underline{d}} - \sum_{d_i\in\underline{d}'}\Big]\binom{N+d_i}{N}, \tag{4.4}$$
$$M_1(\lambda,s) := \Big[-(\lambda+1)\sum_{d_i\in\underline{d}}\sum_{d_k\in d_{\lambda+1,s-\lambda-1}} - (s-\lambda-1)\sum_{d_i\in\underline{d}'}\sum_{d_k\in d_{\lambda+1,s-\lambda-1}}$$
$$+\lambda\sum_{d_i\in\underline{d}}\sum_{d_k\in d_{\lambda,s-\lambda}} + (s-\lambda)\sum_{d_i\in\underline{d}'}\sum_{d_k\in d_{\lambda,s-\lambda}}\Big]\binom{N+d_i-d_k}{N}$$
$$= \Big[(-2\lambda-1)\sum_{d_i\in\underline{d}}\sum_{d_k\in\underline{d}} + (1+2\lambda-s)\sum_{d_i\in\underline{d}}\sum_{d_k\in\underline{d}'} \tag{4.5}$$
$$+(1+2\lambda-s)\sum_{d_i\in\underline{d}'}\sum_{d_k\in\underline{d}} + (2s-2\lambda-1)\sum_{d_i\in\underline{d}'}\sum_{d_k\in\underline{d}'}\Big]\binom{N+d_i-d_k}{N}.$$

There are four summations in the third part $M_2(\lambda, s)$,

$$M_2(\lambda,s) := \Big[(\lambda+1)\sum_{d_i\in\underline{d}}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7s\\ d_{k_1},d_{k_2}\in d_{\lambda+1,s-\lambda-1}}} + (s-\lambda-1)\sum_{d_i\in\underline{d}'}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7s\\ d_{k_1},d_{k_2}\in d_{\lambda+1,s-\lambda-1}}} - \lambda\sum_{d_i\in\underline{d}}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7s\\ d_{k_1},d_{k_2}\in d_{\lambda,s-\lambda}}} - (s-\lambda)\sum_{d_i\in\underline{d}'}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7s\\ d_{k_1},d_{k_2}\in d_{\lambda,s-\lambda}}}\Big]\binom{N+d_i-d_{k_1}-d_{k_2}}{N},$$

For the simplification of summations, let's define

$$\Gamma_{\underline{d}\underline{d}'\underline{d}} = \sum_{d_i\in\underline{d}}\sum_{d_j\in\underline{d}'}\sum_{d_k\in\underline{d}}\binom{N+d_i-d_j-d_k}{N} = \Gamma_{\underline{d}\underline{d}\underline{d}'},$$

$$\Gamma_{\underline{d}'\underline{d}'\underline{d}} = \sum_{d_i\in\underline{d}'}\sum_{d_j\in\underline{d}'}\sum_{d_k\in\underline{d}}\binom{N+d_i-d_j-d_k}{N} = \Gamma_{\underline{d}'\underline{d}\underline{d}'}.$$

$$\Gamma_{\underline{d}\underline{d}_<} = \sum_{d_i\in\underline{d}}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7\\ d_{k_1},d_{k_2}\in\underline{d}}}\binom{N+d_i-d_{k_1}-d_{k_2}}{N},$$

$$\Gamma_{\underline{d}'\underline{d}_<} = \sum_{d_i\in\underline{d}'}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7\\ d_{k_1},d_{k_2}\in\underline{d}}}\binom{N+d_i-d_{k_1}-d_{k_2}}{N}.$$

Quantities $\Gamma_{\underline{d}\underline{d}\underline{d}}, \Gamma_{\underline{d}'\underline{d}'\underline{d}'}, \Gamma_{\underline{d}\underline{d}'\underline{d}'}, \Gamma_{\underline{d}'\underline{d}\underline{d}}, \Gamma_{\underline{d}\underline{d}'_<}$ and $\Gamma_{\underline{d}'\underline{d}'_<}$ can be defined similarly. By induction, it is easy to see that:

$$\begin{aligned}\sum_{d_i\in\underline{d}}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7s\\ d_{k_1},d_{k_2}\in d_{\lambda,s-\lambda}}}\binom{N+d_i-d_{k_1}-d_{k_2}}{N} &= \lambda\Gamma_{\underline{d}\underline{d}_<} + \frac{\lambda(\lambda-1)}{2}\Gamma_{\underline{d}\underline{d}\underline{d}} + \lambda(s-\lambda)\Gamma_{\underline{d}\underline{d}\underline{d}'}\\ &\quad + (s-\lambda)\Gamma_{\underline{d}\underline{d}'_<} + \frac{(s-\lambda)(s-\lambda-1)}{2}\Gamma_{\underline{d}\underline{d}'\underline{d}'},\end{aligned}$$

$$\begin{aligned}\sum_{d_i\in\underline{d}'}\sum_{\substack{1\leqslant k_1<k_2\leqslant 7s\\ d_{k_1},d_{k_2}\in d_{\lambda,s-\lambda}}}\binom{N+d_i-d_{k_1}-d_{k_2}}{N} &= \lambda\Gamma_{\underline{d}'\underline{d}_<} + \frac{\lambda(\lambda-1)}{2}\Gamma_{\underline{d}'\underline{d}\underline{d}} + \lambda(s-\lambda)\Gamma_{\underline{d}'\underline{d}\underline{d}'}\\ &\quad + (s-\lambda)\Gamma_{\underline{d}'\underline{d}'_<} + \frac{(s-\lambda)(s-\lambda-1)}{2}\Gamma_{\underline{d}'\underline{d}'\underline{d}'}.\end{aligned}$$

Then,

$$\begin{aligned}M_2(\lambda,s) =& (\lambda+1)\Big[(\lambda+1)\Gamma_{\underline{d}\underline{d}_<} + \frac{(\lambda+1)\lambda}{2}\Gamma_{\underline{d}\underline{d}\underline{d}} + (\lambda+1)(s-\lambda-1)\Gamma_{\underline{d}\underline{d}\underline{d}'}\\ &+ (s-\lambda-1)\Gamma_{\underline{d}\underline{d}'_<} + \frac{(s-\lambda-1)(s-\lambda-2)}{2}\Gamma_{\underline{d}\underline{d}'\underline{d}'}\Big]\\ &+ (s-\lambda-1)\Big[(\lambda+1)\Gamma_{\underline{d}'\underline{d}_<} + \frac{(\lambda+1)\lambda}{2}\Gamma_{\underline{d}'\underline{d}\underline{d}} + (\lambda+1)(s-\lambda-1)\Gamma_{\underline{d}'\underline{d}\underline{d}'}\\ &+ (s-\lambda-1)\Gamma_{\underline{d}'\underline{d}'_<} + \frac{(s-\lambda-1)(s-\lambda-2)}{2}\Gamma_{\underline{d}'\underline{d}'\underline{d}'}\Big]\end{aligned}$$

$$
\begin{aligned}
&- \lambda\Big[\lambda\Gamma_{\underline{d}\underline{d}_<} + \frac{\lambda(\lambda-1)}{2}\Gamma_{\underline{d}\underline{d}\underline{d}} + \lambda(s-\lambda)\Gamma_{\underline{d}\underline{d}\underline{d'}} \\
&\qquad\qquad + (s-\lambda)\Gamma_{\underline{d}\underline{d'}_<} + \frac{(s-\lambda)(s-\lambda-1)}{2}\Gamma_{\underline{d}\underline{d'}\underline{d'}}\Big] \\
&- (s-\lambda)\Big[\lambda\Gamma_{\underline{d'}\underline{d}_<} + \frac{\lambda(\lambda-1)}{2}\Gamma_{\underline{d'}\underline{d}\underline{d}} + \lambda(s-\lambda)\Gamma_{\underline{d'}\underline{d}\underline{d'}} \\
&\qquad\qquad + (s-\lambda)\Gamma_{\underline{d'}\underline{d'}_<} + \frac{(s-\lambda)(s-\lambda-1)}{2}\Gamma_{\underline{d'}\underline{d'}\underline{d'}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
M_2(\lambda,s) = &(2\lambda+1)\Gamma_{\underline{d}\underline{d}_<} + \frac{\lambda(3\lambda+1)}{2}\Gamma_{\underline{d}\underline{d}\underline{d}} \\
&+ \big[(\lambda+1)^2(s-\lambda-1) - \lambda^2(s-\lambda)\big]\Gamma_{\underline{d}\underline{d}\underline{d'}} \\
&+ (s-2\lambda-1)\big(\Gamma_{\underline{d}\underline{d'}_<} + \Gamma_{\underline{d'}\underline{d}_<}\big) + \frac{(s-\lambda-1)(s-3\lambda-2)}{2}\Gamma_{\underline{d}\underline{d'}\underline{d'}} \\
&+ \frac{\lambda(2s-3\lambda-1)}{2}\Gamma_{\underline{d'}\underline{d}\underline{d}} + \big[(\lambda+1)(s-\lambda-1)^2 - \lambda(s-\lambda)^2\big]\Gamma_{\underline{d'}\underline{d}\underline{d'}} \\
&+ (1-2s+2\lambda)\Gamma_{\underline{d'}\underline{d'}_<} + \frac{(s-\lambda-1)(2-3s+3\lambda)}{2}\Gamma_{\underline{d'}\underline{d'}\underline{d'}}.
\end{aligned} \tag{4.6}
$$

For the last part $M_3(\lambda,s)$,

$$
\begin{aligned}
M_3(\lambda,s) \coloneqq \Big[&- (\lambda+1)\sum_{d_i\in\underline{d}}\ \sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda+1,s-\lambda-1}}} \\
&- (s-\lambda-1)\sum_{d_i\in\underline{d'}}\ \sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda+1,s-\lambda-1}}} \\
&+ \lambda\sum_{d_i\in\underline{d}}\ \sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda,s-\lambda}}} \\
&+ (s-\lambda)\sum_{d_i\in\underline{d'}}\ \sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda,s-\lambda}}}\Big]\binom{N+d_i-d_{k_1}-d_{k_2}-d_{k_3}}{N}.
\end{aligned}
$$

If $\underline{d} = (88,77,72,54,48,31,29)$ and $\underline{d'} = (87,81,64,62,44,33,28)$, to make sure that $\binom{N+d_i-d_{k_1}-d_{k_2}-d_{k_3}}{N}$ is nontrivial, d_i can only be chosen from 88 or 87, and d_{k_1},d_{k_2},d_{k_3} are chosen from $31, 29$ or 28. So

$$
\begin{aligned}
M_3(\lambda,s) = &\Big[-(\lambda+1)\sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda+1,s-\lambda-1}}} + \lambda\sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda,s-\lambda}}}\Big]\binom{N+88-d_{k_1}-d_{k_2}-d_{k_3}}{N} \\
+ &\Big[-(s-\lambda-1)\sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda+1,s-\lambda-1}}} + (s-\lambda)\sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda,s-\lambda}}}\Big]\binom{N+87-d_{k_1}-d_{k_2}-d_{k_3}}{N}.
\end{aligned}
$$

By induction, it is easy to see that

$$\begin{aligned}
&\sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda,s-\lambda}}}\binom{N+88-d_{k_1}-d_{k_2}-d_{k_3}}{N}\\
&\quad=\lambda^2(s-\lambda)+\Big[\lambda\binom{s-\lambda}{2}+\frac{1}{6}(\lambda-2)(\lambda-1)\lambda\Big]\binom{N+1}{N}\\
&\qquad+\frac{\lambda(\lambda-1)(s-\lambda)}{2}\binom{N+2}{N}+\lambda\binom{s-\lambda}{2}\binom{N+3}{N}\\
&\qquad+\frac{1}{6}(s-\lambda-2)(s-\lambda-1)(s-\lambda)\binom{N+4}{N},\\
&\sum_{\substack{1\leqslant k_1<k_2<k_3\leqslant 7s\\ d_{k_1},d_{k_2},d_{k_3}\in d_{\lambda,s-\lambda}}}\binom{N+87-d_{k_1}-d_{k_2}-d_{k_3}}{N}\\
&\quad=\lambda\binom{s-\lambda}{2}+\frac{1}{6}(\lambda-2)(\lambda-1)\lambda+\frac{\lambda(\lambda-1)(s-\lambda)}{2}\binom{N+1}{N}\\
&\qquad+\lambda\binom{s-\lambda}{2}\binom{N+2}{N}+\frac{1}{6}(s-\lambda-2)(s-\lambda-1)(s-\lambda)\binom{N+3}{N}.
\end{aligned}$$

Note that $M_3(\lambda,s)$ will non-trivially appear only when $s\geqslant 2$. Thus

$$\begin{aligned}
M_3(\lambda,s)=&\frac{1}{6}(12-21s+12s^2-3s^3+44\lambda-54s\lambda\\
&\qquad+18s^2\lambda+60\lambda^2-48s\lambda^2+40\lambda^3)\\
&+\frac{1}{6}(-6+9s-3s^2-23\lambda+30s\lambda\\
&\qquad-12s^2\lambda-33\lambda^2+36s\lambda^2-28\lambda^3)\binom{N+1}{N}\\
&-\frac{1}{2}(-1+s-2\lambda)(2-3s+s^2+4\lambda-4s\lambda+4\lambda^2)\binom{N+2}{N}\\
&+\frac{1}{3}(-1+s-\lambda)(6-7s+2s^2+13\lambda-7s\lambda+8\lambda^2)\binom{N+3}{N}\\
&+\frac{1}{6}(1-s+\lambda)(2-s+\lambda)(3-s+4\lambda)\binom{N+4}{N}.
\end{aligned}\tag{4.7}$$

Summarizing (4.4)-(4.7), we see that (4.3) is exactly a polynomial of s,λ with complicated coefficients and higher degree. Fortunately, using the technical computational software ***Mathematica***, (4.4)-(4.7) can all be computed by executable programs. Finally, we calculate the following results:

$$\begin{aligned}
m(d_{1,0})-m(d_{0,1})&=11\ 576\ 781\ 275\ 735,\\
m(d_{2,0})-m(d_{1,1})&=34\ 356\ 628\ 415\ 559\ 239\ 284,\\
m(d_{1,1})-m(d_{0,2})&=34\ 347\ 842\ 980\ 758\ 828\ 832.
\end{aligned}$$

More generally,

$$m(d_{\lambda+1,s-\lambda-1})-m(d_{\lambda,s-\lambda})>\begin{cases}3148, & 0\leqslant\lambda<s,\\ 4\times10^{24}, & 0\leqslant\lambda<s,s\geqslant 3.\end{cases}$$

Thus, it is clear that, with any fixed $s\geqslant 1, s>\lambda\geqslant 0$, $m(d_{\lambda,s-\lambda})$ form a strictly monotonously increasing sequence for λ. □

Remark 4.2. We don't have a general way to prove Proposition 4.1 for different multidegrees $\underline{d}$ and $\underline{d}'$ (cf. [9, page 192]). However, for the fixed two different multidegrees $\underline{d}$ and $\underline{d}'$, note that the maximum value of index j in $m(d_{\lambda,s-\lambda})$ (please see (4.2)) is determined by $\left[\dfrac{\max(\underline{d},\underline{d}')}{\min(\underline{d},\underline{d}')}\right]$. So using the method in the proof of Proposition 4.1, we can still deal with more general multidegrees as long as $1 \leqslant \left[\dfrac{\max(\underline{d},\underline{d}')}{\min(\underline{d},\underline{d}')}\right] \leqslant 3$.

5. Mathematica code and outputs

In this section, the ***Mathematica*** code and outputs, which are designed to evaluate the inequality in Proposition 4.1, are attached in the following.

```
A1={88,77,72,54,48,31,29};

A2={87,81,64,62,44,33,28};

\[Alpha][x_,a_]:=Sum[Binomial[7x+5+a[[k]],a[[k]]],{k,
   Length[a]}];

\[Beta][x_,a_,b_]:=Sum[If[a[[k]]-b[[l]]>=0,Binomial[7x+5+
   a[[k]]-b[[l]],a[[k]]-b[[l]]],0],{k,Length[a]},{l,
   Length[b]}];

\[Gamma][x_,a_,b_,c_]:=Sum[If[a[[k]]-b[[l]]-c[[t]]>=0,
   Binomial[7x+5+a[[k]]-b[[l]]-c[[t]],a[[k]]-b[[l]]-c[[t
   ]]],0],{k,Length[a]},{l,Length[b]},{t,Length[c]}];

\[Delta][x_,a_,b_]:=Sum[If[a[[k]]-b[[l]]-b[[t]]>=0,
   Binomial[7x+5+a[[k]]-b[[l]]-b[[t]],a[[k]]-b[[l]]-b[[t
   ]]],0],{k,Length[a]},{l,Length[b]-1},{t,l+1,Length[b
   ]}];

Subscript[M,0][y_,x_]:=\[Alpha][x,A1]-\[Alpha][x,A2];

Subscript[M,1][y_,x_]:=(-2y-1)\[Beta][x,A1,A1]+(1+2y-x)\[
   Beta][x,A1,A2]+(1+2y-x)\[Beta][x,A2,A1]+(2x-2y-1)\[
   Beta][x,A2,A2];

Subscript[M,2][y_,x_]:=(2y+1)*\[Delta][x,A1,A1]+(y(3y+1))
   /2*\[Gamma][x,A1,A1,A1]+((y+1)^2(x-y-1)-y^2(x-y))*\[
   Gamma][x,A1,A2,A1]+(x-2y-1)*(\[Delta][x,A1,A2]+\[
   Delta][x,A2,A1])+((x-y-1)(x-3y-2))/2*\[Gamma][x,A1,A2
   ,A2]+(y(2x-3y-1))/2*\[Gamma][x,A2,A1,A1]+((y+1)(x-y
   -1)^2-y(x-y)^2)*\[Gamma][x,A2,A2,A1]+(1-2x+2y)*\[
   Delta][x,A2,A2]+((x-y-1)(2-3x+3y))/2*\[Gamma][x,A2,A2
   ,A2];
```

```
Subscript[M,3][y_,x_]:=1/6(12-21x+12x^2-3x^3+44y-54xy+18x
   ^2y+60y^2-48xy^2+40y^3)+1/6(-6+9x-3x^2-23y+30xy-12x^2
   y-33y^2+36xy^2-28y^3)*Binomial[7x+6,1]-1/2(-1+x-2y)
   (2-3x+x^2+4 y-4xy+4y^2)*Binomial[7x+7,2]+1/3(-1+x-y)
   (6-7x+2x^2+13y-7xy+8y^2)*Binomial[7x+8,3]+1/6(1-x+y)
   (2-x+y)(3-x+4y)*Binomial[7x+9,4];

Print["The case s=x=1,2"]

Print["m(A1)=",1-13^2+\[Alpha][1,A1]-\[Beta][1,A1,A1]+\[
   Delta][1,A1,A1]]

Print["m(A2)=",1-13^2+\[Alpha][1,A2]-\[Beta][1,A2,A2]+\[
   Delta][1,A2,A2]]

Print["m(A1)-m(A2)=",Subscript[M,0][0,1]+Subscript[M
   ,1][0,1]+Subscript[M,2][0,1]]

Print["m(A1,A1)-m(A1,A2)=",Subscript[M,0][1,2]+Subscript[
   M,1][1,2]+Subscript[M,2][1,2]+Subscript[M,3][1,2]]

Print["m(A1,A2)-m(A2,A2)=",Subscript[M,0][0,2]+Subscript[
   M,1][0,2]+Subscript[M,2][0,2]+Subscript[M,3][0,2]]

Print["Solution of Subscript[M,0][y,x]+Subscript[M,1][y,x
   ]+Subscript[M,2][y,x]+Subscript[M,3][y,x]>4*10^24&x
   >=3&x>y&y>=0"]

Reduce[FunctionExpand[Subscript[M,0][y,x]+Subscript[M,1][
   y,x]+Subscript[M,2][y,x]+Subscript[M,3][y,x
   ]]>4*10^24&&x>=3&&x>y&&y>=0,{x,y}]

Print["Solution of Subscript[M,0][y,x]+Subscript[M,1][y,x
   ]+Subscript[M,2][y,x]+Subscript[M,3][y,x]>3148&x>y&y
   >=0"]

Reduce[FunctionExpand[Subscript[M,0][y,x]+Subscript[M,1][
   y,x]+Subscript[M,2][y,x]+Subscript[M,3][y,x]]>3148&&x
   >y&&y>=0,{x,y}]

The case s=x=1,2

m(A1)=1382270197857128

m(A2)=1370693416581393

m(A1)-m(A2)=11576781275735
```

```
m(A1,A1)-m(A1,A2)=343566284155559239284

m(A1,A2)-m(A2,A2)=343478429807588288832

Solution of Subscript[M,0][y,x]+Subscript[M,1][y,x]+
    Subscript[M,2][y,x]+Subscript[M,3][y,x]>4*10^24&x>=3&
    x>y&y>=0

x>=3&&0<=y<x

Solution of Subscript[M,0][y,x]+Subscript[M,1][y,x]+
    Subscript[M,2][y,x]+Subscript[M,3][y,x]>3148&x>y&y>=0

x>0&&0<=y<x
```

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