

## SQUARE FUNCTION ESTIMATES IN SPACES OF HOMOGENEOUS TYPE AND ON UNIFORMLY RECTIFIABLE EUCLIDEAN SETS

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ABSTRACT. We announce a local  $T(b)$  theorem, an inductive scheme, and  $L^p$  extrapolation results for  $L^2$  square function estimates related to the analysis of integral operators that act on Ahlfors–David regular sets of arbitrary codimension in ambient quasi-metric spaces. The inductive scheme is a natural application of the local  $T(b)$  theorem and it implies the stability of  $L^2$  square function estimates under the so-called big pieces functor. In particular, this analysis implies  $L^p$  and Hardy space square function estimates for integral operators on uniformly rectifiable subsets of the Euclidean space.

### 1. INTRODUCTION

This work is motivated by the  $L^2$  square function estimates proved by G. David and S. Semmes in [10], [11], for convolution type integral operators associated with the Riesz kernels  $x_j/|x|^{n+1}$ ,  $1 \leq j \leq n+1$ , on uniformly rectifiable sets in  $\mathbb{R}^{n+1}$ . Our main aim here is to extend their results in a number of directions, including the consideration of a larger class of kernels, and establishing  $L^p$  as well as Hardy type square function estimates. A substantial portion of our analysis is valid in the general setting of abstract quasi-metric spaces (which automatically forces the consideration of non-convolution type kernels), and we succeed in dealing with Ahlfors–David regular sets of arbitrary codimension in that general setting. We announce three results obtained in this setting. The starting point is a local  $T(b)$  theorem for square functions which is then used to prove an inductive scheme whereby square function estimates are shown to be stable under the so-called big pieces functor. The third result is an extrapolation principle whereby an  $L^p$  (or weak- $L^p$ , or Hardy space  $H^p$ ) square function estimate for one  $p$  yields a full range of square function bounds. We also indicate how the inductive scheme can be applied to obtain square function estimates for a large class of integral operators that act on uniformly rectifiable sets of codimension one in Euclidean space.

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To formulate the latter result, we need to introduce some notation and recall some definitions. Following [10], [11], a closed set  $\Sigma \subseteq \mathbb{R}^{n+1}$  is called *uniformly rectifiable* if it is  $n$ -dimensional Ahlfors-David regular and has big pieces of Lipschitz images. The property of being  *$n$ -dimensional Ahlfors-David regular* (ADR for short) means that there exists a constant  $C \in [1, \infty)$  such that

$$C^{-1} r^n \leq \mathcal{H}^n(\Sigma \cap B(x, r)) \leq C r^n, \quad \forall x \in \Sigma, \forall \text{ finite } r \in (0, \text{diam}(\Sigma)], \quad (1.1)$$

where

$$\begin{aligned} B(x, r) &:= \{y \in \mathbb{R}^{n+1} : |x - y| < r\}, \\ \text{diam}(\Sigma) &:= \sup \{|x - y| : x, y \in \Sigma\} \in (0, \infty], \end{aligned}$$

and  $\mathcal{H}^n$  denotes the  $n$ -dimensional Hausdorff measure in  $\mathbb{R}^{n+1}$ . The property of having *big pieces of Lipschitz images* means that there exist constants  $\eta, C \in (0, \infty)$  such that for each  $x \in \Sigma$  and each finite  $r \in (0, \text{diam}(\Sigma)]$ , there is a ball  $B_r^n$  of radius  $r$  in  $\mathbb{R}^n$  and a Lipschitz map  $\varphi : B_r^n \rightarrow \mathbb{R}^{n+1}$  with Lipschitz constant at most equal to  $C$ , such that

$$\mathcal{H}^n(\Sigma \cap B(x, r) \cap \varphi(B_r^n)) \geq \eta r^n. \quad (1.2)$$

We now state our principal result in the Euclidean context, dealing with Hardy and  $L^p$  square function estimates on uniformly rectifiable sets.

**Theorem 1.1.** *Suppose that  $K$  is a real-valued function with the following properties:*

$$\begin{aligned} K &\in C^2(\mathbb{R}^{n+1} \setminus \{0\}), \\ K &\text{ is odd,} \\ K(\lambda x) &= \lambda^{-n} K(x) \quad \forall \lambda > 0, \forall x \in \mathbb{R}^{n+1} \setminus \{0\}. \end{aligned} \quad (1.3)$$

Let  $\Sigma$  be a uniformly rectifiable subset of  $\mathbb{R}^{n+1}$  and let  $\sigma := \mathcal{H}^n \llcorner \Sigma$  denote the restriction of the  $n$ -dimensional Hausdorff measure in  $\mathbb{R}^{n+1}$  to  $\Sigma$ . For each  $p \in (\frac{n}{n+1}, \infty)$  let  $H^p(\Sigma, \sigma)$  stand for the Lebesgue scale  $L^p(\Sigma, \sigma)$  if  $p \in (1, \infty)$ , and the Coifman-Weiss scale of Hardy spaces on the space of homogeneous type  $(\Sigma, |\cdot - \cdot|, \sigma)$  if  $p \in (\frac{n}{n+1}, 1]$ . Finally, consider the integral operator  $\mathcal{T}$  acting on functions  $f \in L^p(\Sigma, \sigma)$ ,  $p \in (1, \infty)$ , according to

$$\mathcal{T}f(x) := \int_{\Sigma} K(x - y)f(y) d\sigma(y), \quad \forall x \in \mathbb{R}^{n+1} \setminus \Sigma. \quad (1.4)$$

Then the operator  $\mathcal{T}$  extends to  $H^p(\Sigma, \sigma)$  and there exists  $C \in (0, \infty)$  such that

$$\left\| \left( \int_{\Gamma_{\kappa}(x)} |(\nabla \mathcal{T}f)(y)|^2 \frac{dy}{\text{dist}(y, \Sigma)^{n-1}} \right)^{\frac{1}{2}} \right\|_{L_x^p(\Sigma, \sigma)} \leq C \|f\|_{H^p(\Sigma, \sigma)}, \quad \forall f \in H^p(\Sigma, \sigma), \quad (1.5)$$

where  $\Gamma_{\kappa}(x) := \{y \in \mathbb{R}^{n+1} \setminus \Sigma : |x - y| < (1 + \kappa) \text{dist}(y, \Sigma)\}$ , for each  $x \in \Sigma$ .

As a corollary, the following  $L^2$  square function estimate holds

$$\int_{\mathbb{R}^{n+1} \setminus \Sigma} |(\nabla \mathcal{T}f)(x)|^2 \text{dist}(x, \Sigma) dx \leq C \int_{\Sigma} |f(x)|^2 d\sigma(x), \quad \forall f \in L^2(\Sigma, \sigma). \quad (1.6)$$

The fact that (1.6) is a consequence of (1.5) with  $p = 2$  follows from Fubini's theorem and the ADR property of  $\Sigma$ . This being said, our proof of (1.5) begins by first establishing (1.6) and then deriving (1.5) using arguments akin to Calderón-Zygmund theory which, in effect, indicates that deriving Hardy and  $L^p$  square-function estimates is not unlike proving boundedness results for singular integral operators. That (1.6) holds in the case when  $\mathcal{T}$  is associated as in (1.4) with each of the Riesz kernels  $K_j(x) := x_j/|x|^{n+1}$ ,  $1 \leq j \leq n+1$ , is due to David and Semmes [11]. Here we generalize their result in two basic aspects, by considering any kernel satisfying the purely real-variable conditions in (1.3) (which are not tied to any particular partial differential operator, in the manner that the kernels  $K_j(x) := x_j/|x|^{n+1}$ ,  $1 \leq j \leq n+1$ , are related to the Laplacian), and by considering the entire Hardy-Lebesgue scale (and not just  $L^2$ ). In fact, it is also possible to obtain a version of Theorem 1.1 for variable coefficient kernels, which ultimately applies to integral operators on manifolds that are associated with the Schwartz kernels of certain classes of pseudodifferential operators acting between vector bundles, although we shall not discuss this further here. We sketch the proof of Theorem 1.1 in Sections 2.2-2.3.

It is both useful and instructive to separate the portion of the proof of Theorem 1.1 which makes essential use of Euclidean tools from the portion which works in the general context of quasi-metric spaces. As regards the first portion, two ingredients are essential for our approach, namely the fact that the square function estimate (1.6) holds when  $\Sigma$  is a Lipschitz graph in  $\mathbb{R}^{n+1}$ , and the recent characterization of uniformly rectifiable subsets of  $\mathbb{R}^{n+1}$  proved by J. Azzam and R. Schul in [3] in terms of the two-fold iteration of the so-called big pieces functor starting with Lipschitz graphs. Concerning the second portion of the proof, mentioned earlier, it is remarkable that a large number of significant results, including a local  $T(b)$  theorem for square functions, a geometrically inductive scheme showing that  $L^2$  square function estimates are stable under the action of the big pieces functor, and extrapolation results for square function estimates, make no specific use of the vector space structure of the ambient space (as well as any other beneficial aspect which such a setting entails). As such, the aforementioned results can be developed abstractly in the general context of quasi-metric spaces. This point of view is systematically pursued in Section 2.

## 2. MAIN RESULTS

We begin by discussing the abstract setting which constitutes the foundation of most of our results in this paper. The following notation and assumptions are fixed and used without specific reference hereafter, unless otherwise specified.

*Dimensions  $d$  and  $m$ .* Let  $d$  and  $m$  be two real numbers such that  $0 < d < m < \infty$ .

*Ambient Space  $\mathcal{X}$ .* Let  $(\mathcal{X}, \rho, \mu)$  denote an  $m$ -dimensional Ahlfors-David regular ( $m$ -ADR) space. This is defined to mean that  $\mathcal{X}$  is a set of cardinality at least two, equipped with a quasi-distance  $\rho$ , and a Borel measure  $\mu$  with the property that all  $\rho$ -balls are  $\mu$ -measurable, and for which there exists a constant  $C \in [1, \infty)$  such that

$$C^{-1} r^m \leq \mu(B_\rho(x, r)) \leq C r^m, \quad \forall x \in \mathcal{X}, \forall \text{finite } r \in (0, \text{diam}_\rho(\mathcal{X})], \quad (2.7)$$

where

$$\begin{aligned} B_\rho(x, r) &:= \{y \in \mathcal{X} : \rho(x, y) < r\}, \\ \text{diam}_\rho(\mathcal{X}) &:= \sup \{\rho(x, y) : x, y \in \mathcal{X}\} \in (0, \infty]. \end{aligned}$$

The constant  $C$  in (2.7) will be referred to as the ADR constant of  $\mathcal{X}$ . The quasi-distance  $\rho : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$  has the property that there exist two constants  $\tilde{C}_\rho, C_\rho \in [1, \infty)$  such that

$$\begin{aligned} \rho(x, y) &= 0 \text{ if and only if } x = y, \\ \rho(y, x) &\leq \tilde{C}_\rho \rho(x, y), \\ \rho(x, y) &\leq C_\rho \max\{\rho(x, z), \rho(z, y)\}, \end{aligned} \tag{2.8}$$

for all  $x, y, z \in \mathcal{X}$ , and we introduce the related constant

$$\alpha_\rho := \frac{1}{\log_2 C_\rho} \in (0, \infty]. \tag{2.9}$$

The measure  $\mu$  is Borel with respect to topology  $\tau_\rho$  canonically induced by  $\rho$ , which is defined to be the largest topology on  $\mathcal{X}$  with the property that for each point  $x \in \mathcal{X}$  the family  $\{B_\rho(x, r)\}_{r>0}$  is a fundamental system of neighborhoods of  $x$ .

*Subspace  $E \subseteq \mathcal{X}$ .* Let  $(E, \sigma)$  consist of a closed subset  $E$  of  $(\mathcal{X}, \tau_\rho)$  and a Borel regular measure  $\sigma$  on  $(E, \tau_{\rho|_E})$  with the property that  $(E, \rho|_E, \sigma)$  is a  $d$ -dimensional ADR space, where  $\rho|_E$  denotes the restriction of  $\rho$  to  $E \times E$ . From [21] (which extends work in [18]) we know that, in this context, there exists a symmetric quasi-distance  $\rho_\#$  on  $\mathcal{X}$ , called the regularization of  $\rho$ , such that  $\rho_\#$  and  $\rho$  are equivalent quasi-metrics, the topology  $\tau_{\rho_\#} = \tau_\rho$ , and the regularized distance

$$\delta_E(x) := \inf \{\rho_\#(x, y) : y \in E\}, \quad \forall x \in \mathcal{X}, \tag{2.10}$$

is continuous on  $(\mathcal{X}, \tau_\rho)$ .

It is also well-known (see [5], [9]) that in this context there exists a dyadic cube structure on  $E$ . In particular, fix  $\kappa_E$  in  $\mathbb{Z} \cup \{-\infty\}$  with the property that  $2^{-\kappa_E - 1} < \text{diam}_\rho(E) \leq 2^{-\kappa_E}$ . For each integer  $k \geq \kappa_E$ , a collection  $\mathbb{D}_k(E) := \{Q_\alpha^k\}_{\alpha \in I_k}$  of subsets  $Q_\alpha^k$  of  $E$ , indexed by a nonempty and at most countable set  $I_k$  of indices  $\alpha$ , is fixed such that the entire collection

$$\mathbb{D}(E) := \bigcup_{k \in \mathbb{Z}, k \geq \kappa_E} \mathbb{D}_k(E) \tag{2.11}$$

has properties analogous to the ordinary dyadic cube structure of  $\mathbb{R}^n$ . We refer to the sets  $Q$  in  $\mathbb{D}_k(E)$  as dyadic cubes with side length  $\ell(Q) := 2^{-k}$ . We stress that all quantitative aspects pertaining to  $\mathbb{D}(E)$  are controlled in terms of the ADR constants of  $E$  (as well as on  $\text{diam}_\rho(E)$  when  $E$  is bounded).

There also exists a Whitney covering of  $\mathcal{X} \setminus E$  (see for instance [21]) since  $(\mathcal{X}, \rho)$  is geometrically doubling and  $E$  is closed in  $(\mathcal{X}, \tau_\rho)$ . We use this covering to associate to each cube  $Q$  in  $\mathbb{D}(E)$  the region  $\mathcal{U}_Q \subseteq \mathcal{X} \setminus E$  which is the union of all Whitney cubes of size comparable to  $\ell(Q)$  located at a distance less than or equal to  $C\ell(Q)$  from  $Q$ . Intuitively, the reader should think of these as being analogous to the upper halves of ordinary Carleson regions in the Euclidean upper half-space. In particular, the dyadic Carleson tent  $T_E(Q)$  over  $Q$  (relative to the set  $E$ ) may

now be defined as

$$T_E(Q) := \bigcup_{Q' \in \mathbb{D}(E), Q' \subseteq Q} \mathcal{U}_{Q'}. \quad (2.12)$$

*Integral Operator  $\Theta_E$ .* Let  $\theta : (\mathcal{X} \times \mathcal{X}) \setminus \{(x, x) : x \in \mathcal{X}\} \rightarrow \mathbb{R}$  denote a Borel measurable function, with respect to the product topology  $\tau_\rho \times \tau_\rho$ , for which there exist finite positive constants  $C_\theta, \alpha, v$  such that for all  $x, y \in \mathcal{X}$  with  $x \neq y$  the following hold:

$$\text{Decay condition:} \quad |\theta(x, y)| \leq \frac{C_\theta}{\rho(x, y)^{d+v}}, \quad (2.13)$$

$$\text{H\"older regularity:} \quad |\theta(x, y) - \theta(x, \tilde{y})| \leq C_\theta \frac{\rho(y, \tilde{y})^\alpha}{\rho(x, y)^{d+v+\alpha}}, \quad (2.14)$$

$$\forall \tilde{y} \in \mathcal{X} \setminus \{x\} \text{ with } \rho(y, \tilde{y}) \leq \frac{1}{2}\rho(x, y).$$

The integral operator  $\Theta_E$  is then defined for all functions  $f \in L^p(E, \sigma)$ ,  $1 \leq p \leq \infty$ , by

$$(\Theta_E f)(x) := \int_E \theta(x, y) f(y) d\sigma(y), \quad \forall x \in \mathcal{X} \setminus E. \quad (2.15)$$

We proceed to describe our main tools in the treatment of square function estimates.

### 2.1. An arbitrary codimension local $T(b)$ theorem for square functions.

The local  $T(b)$  theorem below states that a global square function estimate for the integral operator  $\Theta_E$  holds if there exists a family of suitably non-degenerate and normalized functions  $\{b_Q\}_{Q \in \mathbb{D}(E)}$  with the property that, for each  $Q \in \mathbb{D}(E)$ , a uniform, scale-invariant, local version of the  $L^2$  square function estimate on the dyadic Carleson tent  $T_E(Q)$  holds for  $\Theta_E$  acting on  $b_Q$ . Naturally, the formulation of the aforementioned square function estimates takes into account both the codimension  $m - d$  of  $E$  in  $\mathcal{X}$ , and the exponent  $v$  intervening in the decay condition (2.13).

Here is the formal statement of our first main result which generalizes a Euclidean codimension one version that was implicit in the solution of the Kato problem in [2], [14], [16], and later formulated explicitly in [1], [13], [17].

**Theorem 2.1.** *If there exist two constants  $C_0 \in [1, \infty)$ ,  $c_0 \in (0, 1]$  and a collection  $\{b_Q\}_{Q \in \mathbb{D}(E)}$  of  $\sigma$ -measurable functions  $b_Q : E \rightarrow \mathbb{C}$  such that for each  $Q \in \mathbb{D}(E)$  the following hold:*

- (1)  $\int_E |b_Q|^2 d\sigma \leq C_0 \sigma(Q)$ ;
- (2) *there exists  $\tilde{Q} \in \mathbb{D}(E)$ ,  $\tilde{Q} \subseteq Q$ ,  $\ell(\tilde{Q}) \geq c_0 \ell(Q)$ , and  $\left| \int_{\tilde{Q}} b_Q d\sigma \right| \geq \frac{1}{C_0} \sigma(\tilde{Q})$ ;*
- (3)  $\int_{T_E(Q)} |(\Theta b_Q)(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x) \leq C_0 \sigma(Q)$ ,

*then there exists a constant  $C \in (0, \infty)$ , depending only on  $C_0, C_\theta$ , and the ADR constants of  $E$  and  $\mathcal{X}$ , as well as on  $\text{diam}_\rho(E)$  when  $E$  is bounded, such that*

$$\int_{\mathcal{X} \setminus E} |(\Theta_E f)(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x) \leq C \int_E |f(x)|^2 d\sigma(x), \quad (2.16)$$

*for all  $f \in L^2(E, \sigma)$ .*

**2.2. An inductive scheme for square function estimates.** The inductive scheme in the theorem below shows that the integral operator  $\Theta_E$  satisfies square function estimates whenever the set  $E$  contains (uniformly, at all scales and locations) so-called big pieces of sets on which square function estimates hold. In short, we say that big pieces of square function estimates (BPSFE) imply square function estimates (SFE). We sketch the proof of this result, since it is a natural application of our local  $T(b)$  theorem, and then indicate how the inductive scheme can be applied to uniformly rectifiable sets by proving (1.6) in Theorem 1.1.

The formulation of this result requires the notion of Hausdorff measure in the quasi-metric setting. Specifically, let  $\mathcal{H}_{\mathcal{X}, \rho_{\#}}^d$  denote the  $d$ -dimensional Hausdorff measure on  $(\mathcal{X}, \rho_{\#})$  (see [21, Definition 4.70]), and for any closed subset  $A$  of  $(\mathcal{X}, \tau_{\rho})$ , let  $\mathcal{H}_{\mathcal{X}, \rho_{\#}}^d \llcorner A$  denote the measure given by the restriction of  $\mathcal{H}_{\mathcal{X}, \rho_{\#}}^d$  to  $A$ . We begin by defining what it means for the set  $E$  to have big pieces of square function estimate.

**Definition 2.2.** *The set  $E \subseteq \mathcal{X}$  is said to have big pieces of square function estimate (BPSFE) relative to the kernel  $\theta$  if there exist three constants  $\eta, C_1, C_2 \in (0, \infty)$  with the property that for each  $Q \in \mathbb{D}(E)$  there exists a closed subset  $E_Q$  of  $(\mathcal{X}, \tau_{\rho})$  such that  $(E_Q, \rho|_{E_Q}, \mathcal{H}_{\mathcal{X}, \rho_{\#}}^d \llcorner E_Q)$  is a  $d$ -dimensional ADR space with ADR constant less than or equal to  $C_1$ , and which satisfies*

$$\mathcal{H}_{\mathcal{X}, \rho_{\#}}^d(E_Q \cap Q) \geq \eta \mathcal{H}_{\mathcal{X}, \rho_{\#}}^d(Q) \quad (2.17)$$

as well as

$$\int_{\mathcal{X} \setminus E_Q} |\Theta_{E_Q} f(x)|^2 \text{dist}_{\rho_{\#}}(x, E_Q)^{2v-(m-d)} d\mu(x) \leq C_2 \int_{E_Q} |f|^2 d\mathcal{H}_{\mathcal{X}, \rho_{\#}}^d \llcorner E_Q, \quad (2.18)$$

for all  $f \in L^2(E_Q, \mathcal{H}_{\mathcal{X}, \rho_{\#}}^d \llcorner E_Q)$ , where  $\Theta_{E_Q}$  is the operator associated with  $E_Q$  as in (2.15). The constants  $\eta, C_1, C_2$  will collectively be referred to as the BPSFE character of the set  $E$ .

We now state and sketch the proof of our second main result.

**Theorem 2.3.** *If the set  $E \subseteq \mathcal{X}$  has BPSFE relative to the kernel  $\theta$ , then there exists a constant  $C \in (0, \infty)$ , depending only on  $\rho, m, d, v, C_{\theta}$ , the BPSFE character of  $E$ , and the ADR constants of  $E$  and  $\mathcal{X}$ , such that*

$$\int_{\mathcal{X} \setminus E} |\Theta_E f(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x) \leq C \int_E |f|^2 d\mathcal{H}_{\mathcal{X}, \rho_{\#}}^d, \quad (2.19)$$

for all  $f \in L^2(E, \mathcal{H}_{\mathcal{X}, \rho_{\#}}^d \llcorner E)$ .

*Proof.* For each  $Q \in \mathbb{D}(E)$  there exists  $E_Q \subseteq \mathcal{X}$  satisfying (2.17)-(2.18), since  $E$  has BPSFE relative to the kernel  $\theta$ . We then define the function  $b_Q : E \rightarrow \mathbb{R}$  by setting

$$b_Q(y) := \mathbf{1}_{Q \cap E_Q}(y), \quad \forall y \in E. \quad (2.20)$$

The strategy for proving (2.19) is to invoke Theorem 2.1 for the family  $\{b_Q\}_{Q \in \mathbb{D}(E)}$ , and as such, it suffices to verify conditions (1)-(3) in Theorem 2.1. Condition (1) is immediate, and condition (2) with  $\tilde{Q} := Q$  is a consequence of (2.17). To verify condition (3), we introduce a constant  $C_A \in (1, \infty)$  to be chosen later, and the set

$$A := \{x \in \mathcal{X} : C_A^{-1} \delta_E(x) \leq \delta_{E_Q}(x) \leq C_A \delta_E(x)\}, \quad (2.21)$$

in order to write

$$\int_{T_E(Q)} |\Theta_E b_Q(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x) = I_{\mathcal{X} \setminus A} + I_A, \quad (2.22)$$

where

$$I_{\mathcal{X} \setminus A} := \int_{T_E(Q) \setminus A} |\Theta_E b_Q(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x), \quad (2.23)$$

$$I_A := \int_{T_E(Q) \cap A} |\Theta_E b_Q(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x). \quad (2.24)$$

The estimate for  $I_{\mathcal{X} \setminus A}$  requires choosing  $C_A$  sufficiently large, in accordance with the ADR geometry of  $E$  and the decay of the kernel  $\theta$ . The idea is to rely on a pointwise bound for  $\Theta_E b_Q$  and Carleson measure estimates of a purely geometric nature. More specifically, we prove that a judicious choice of  $C_A$  gives

$$\int_{\{x \in T_E(Q) : \delta_{E_Q}(x) > C_A \delta_E(x)\}} \delta_{E_Q}(x)^{-2v} \delta_E(x)^{2v-(m-d)} d\mu(x) \leq C\sigma(Q), \quad (2.25)$$

plus a Carleson measure estimate similar in nature (involving a suitable choice of the powers of the distance functions) on the piece  $\{x \in T_E(Q) : \delta_{E_Q}(x) < C_A^{-1} \delta_E(x)\}$ . As regards  $I_A$ , we use (2.18) to obtain (with  $\mathbf{1}_A$  denoting the characteristic function of  $A$ )

$$\begin{aligned} I_A &= \int_{T_E(Q) \setminus E_Q} |\Theta_{E_Q} b_Q(x)|^2 \mathbf{1}_A(x) \delta_E(x)^{2v-(m-d)} d\mu(x) \\ &\lesssim \int_{\mathcal{X} \setminus E_Q} |\Theta_{E_Q} b_Q(x)|^2 \delta_{E_Q}(x)^{2v-(m-d)} d\mu(x) \\ &\lesssim \int_{E_Q} |b_Q|^2 d\mathcal{H}_{\mathcal{X}, \rho\#}^d \llcorner E_Q \lesssim \mathcal{H}_{\mathcal{X}, \rho\#}^d(Q), \end{aligned} \quad (2.26)$$

as desired.  $\square$

We now prove (1.6) in Theorem 1.1 by combining Theorem 2.3 with a characterization of uniform rectifiability obtained recently by J. Azzam and R. Schul in [3].

*Proof of (1.6) in Theorem 1.1.* To get started recall that the set  $\Sigma$  is uniformly rectifiable, so by the characterization of such sets in [3, Corollary 1.7], it follows that  $\Sigma$  has big pieces of big pieces of Lipschitz graphs (BP)<sup>2</sup>LG. The next step is to prove estimate (1.6) in the case when  $E$  is a Lipschitz graph in  $\mathbb{R}^{n+1}$  and this is achieved by building on earlier work in [6], [12], [15]. We conclude that  $\Sigma$  has big pieces of big pieces of square function estimates, i.e., (BP)<sup>2</sup>SFE. We shall not define (BP)<sup>2</sup>LG nor (BP)<sup>2</sup>SFE here, but both should be understood in a natural manner. Theorem 2.3 can be iterated to show that (BP)<sup>2</sup>SFE implies BPSFE, which in turn implies that the square function estimate in (2.19) holds, so (1.6) follows by applying (2.19) with  $(d, m, \mathcal{X}, E, \theta, \Theta) = (n, n+1, \mathbb{R}^{n+1}, \Sigma, \nabla K, \nabla T)$ .  $\square$

**2.3.  $L^p$  square function estimates.** We now consider  $L^p$  versions of the  $L^2$  square function estimates considered above for the integral operator  $\Theta_E$ . We list three extrapolation theorems and show how they can be applied to uniformly rectifiable sets by proving (1.5) in Theorem 1.1. The first states that  $L^2$  square function estimates follow automatically from weak- $L^p$  square function estimates for any

$p \in (0, \infty)$ . The second and third provide a range of sufficient conditions for  $L^p$ , weak- $L^p$ , and Hardy space  $H^p$  square function estimates to hold.

For any  $\kappa > 0$ , consider the nontangential approach region

$$\Gamma_\kappa(x) := \{y \in \mathcal{X} \setminus E : \rho_\#(x, y) < (1 + \kappa) \delta_E(y)\}, \quad \forall x \in E. \quad (2.27)$$

The following is the first extrapolation theorem.

**Theorem 2.4.** *If  $\kappa, p, C_o$  are finite positive constants such that for every  $z \in E$  and  $r > 0$  the surface ball  $\Delta := E \cap B_{\rho_\#}(z, r)$  satisfies*

$$\sigma\left(\left\{x \in E : \int_{\Gamma_\kappa(x)} |(\Theta_E \mathbf{1}_\Delta)(y)|^2 \delta_E(y)^{2v-m} d\mu(y) > \lambda^2\right\}\right) \leq C_o \lambda^{-p} \sigma(\Delta), \quad \forall \lambda > 0, \quad (2.28)$$

then there exists  $C \in (0, \infty)$  which depends only on  $\kappa, p, C_o$  and finite positive background constants (including  $\text{diam}_\rho(E)$  in the case when  $E$  is bounded) such that

$$\int_{\mathcal{X} \setminus E} |(\Theta_E f)(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x) \leq C \int_E |f(x)|^2 d\sigma(x), \quad \forall f \in L^2(E, \sigma). \quad (2.29)$$

The requirement in (2.28) is less restrictive than the standard weak- $L^p$  estimate

$$\sup_{\lambda > 0} \left[ \lambda \cdot \sigma\left(\left\{x \in E : \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^2 \delta_E(y)^{2v-m} d\mu(y) > \lambda^2\right\}\right)^{1/p} \right] \leq C_o \|f\|_{L^p(E, \sigma)} \quad (2.30)$$

for every  $f \in L^p(E, \sigma)$ , since (2.28) follows by substituting  $f = \mathbf{1}_\Delta$  in (2.30).

The next extrapolation theorem shows that a weak- $L^q$  square function estimate for any  $q \in (0, \infty)$  implies that square functions are bounded from  $H^p$  into  $L^p$  for all  $p \in (\frac{d}{d+\gamma}, \infty)$ , where  $d$  is the dimension of  $E$  and (recalling  $\alpha_\rho$  from (2.9) and  $\alpha$  from (2.14)) the constant

$$\gamma := \min\{\alpha_\rho, \alpha\}. \quad (2.31)$$

The theory of Hardy-Lebesgue spaces  $H^p = H^p(E, \rho|_E, \sigma)$  for ADR subsets of a quasi-metric space has been developed in [21] for  $p$  belonging to an interval containing  $(\frac{d}{d+\alpha_\rho}, \infty)$ . These spaces become  $L^p(E, \sigma)$  when  $p \in (1, \infty)$ , and in the case when  $p \in (\frac{d}{d+\alpha_\rho}, 1]$  they have an atomic characterization as in the work of R. Coifman and G. Weiss in [8], as well as a grand maximal function characterization akin to that established by R. Macías and C. Segovia in [19].

**Theorem 2.5.** *Fix  $\kappa > 0$ . Given  $q \in (1, \infty)$  and  $p \in (\frac{d}{d+\gamma}, \infty)$ , consider the estimate*

$$\left\| \left( \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-qv}} \right)^{\frac{1}{q}} \right\|_{L_x^p(E, \sigma)} \leq C \|f\|_{H^p(E, \rho|_E, \sigma)}, \quad \forall f \in H^p(E, \rho|_E, \sigma), \quad (2.32)$$

for some constant  $C \in (0, \infty)$ .

(I) Assume  $q \in (1, \infty)$  has the property that, for some constant  $C_o \in (0, \infty)$ , either

$$\left\| \left( \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-qv}} \right)^{\frac{1}{q}} \right\|_{L_x^q(E, \sigma)} \leq C_o \|f\|_{L^q(E, \sigma)}, \quad \forall f \in L^q(E, \sigma), \quad (2.33)$$

or there exists  $p_o \in (q, \infty)$  such that for every  $f \in L^{p_o}(E, \sigma)$  there holds

$$\sup_{\lambda > 0} \left[ \lambda \cdot \sigma \left( \left\{ x \in E : \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-qv}} > \lambda^q \right\} \right)^{1/p_o} \right] \leq C_o \|f\|_{L^{p_o}(E, \sigma)}. \quad (2.34)$$

Then (2.32) holds for every  $p \in (\frac{d}{d+\gamma}, \infty)$ .

(II) Assume that  $q \in (1, \infty)$  is such that there exist  $p_o \in (1, \infty)$  and a constant  $C_o \in (0, \infty)$  such that (2.34) holds for every  $f \in L^{p_o}(E, \sigma)$ . Then (2.32) holds for every  $p \in (1, p_o)$  and, in addition, for every  $f \in L^1(E, \sigma)$  one has

$$\sup_{\lambda > 0} \left[ \lambda \cdot \sigma \left( \left\{ x \in E : \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^q \frac{d\mu(y)}{\delta_E(y)^{m-qv}} > \lambda^q \right\} \right) \right] \leq C_o \|f\|_{L^1(E, \sigma)}. \quad (2.35)$$

The conclusion (2.32) in Theorem 2.5 may be conveniently re-phrased by saying that the operator

$$\delta_E^{v-m/q} \Theta_E : H^p(E, \rho|_E, \sigma) \rightarrow L^{(p, q)}(\mathcal{X}, E) \quad (2.36)$$

is well-defined, linear and bounded, where  $L^{(p, q)}(\mathcal{X}, E)$  is a mixed norm space in the quasi-metric setting introduced in [20] as a generalization of the tent spaces  $T_q^p$  in  $\mathbb{R}_+^{n+1}$  that originated with R. Coifman, Y. Meyer and E. Stein in [7] (see also [4] for related matters).

Estimate (1.5) in Theorem 1.1 now readily follows by combining Theorem 2.5 with (1.6).

*Proof of (1.5) in Theorem 1.1.* The ADR geometry of  $E$  implies that

$$\left\| \left( \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^2 \frac{d\mu(y)}{\delta_E(y)^{m-2v}} \right)^{\frac{1}{2}} \right\|_{L_x^2(E, \sigma)}^2 \approx \int_{\mathcal{X} \setminus E} |(\Theta_E f)(x)|^2 \delta_E(x)^{2v-(m-d)} d\mu(x), \quad (2.37)$$

uniformly for  $f \in L^2(E, \sigma)$ . Based on this and (1.6) (which has been established earlier) we conclude that the hypotheses in part (I) of Theorem 2.5 hold with  $q = 2$  in the setting in which  $(d, m, \mathcal{X}, E, \theta, \Theta) = (n, n+1, \mathbb{R}^{n+1}, \Sigma, \nabla K, \nabla \mathcal{T})$ . In turn, this yields (1.5), as wanted.  $\square$

We conclude with an extrapolation theorem that combines Theorems 2.4 and 2.5.

**Theorem 2.6.** Fix  $\kappa > 0$ . If there exist  $p_o \in (0, \infty)$  and a constant  $C_o \in (0, \infty)$  such that

$$\sup_{\lambda > 0} \left[ \lambda \cdot \sigma \left( \left\{ x \in E : \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^2 \frac{d\mu(y)}{\delta_E(y)^{m-2v}} > \lambda^2 \right\} \right)^{1/p_o} \right] \leq C_o \|f\|_{L^{p_o}(E, \sigma)}, \quad (2.38)$$

for all  $f \in L^{p_0}(E, \sigma)$ , then for each  $p \in (\frac{d}{d+\gamma}, \infty)$  there holds

$$\left\| \left( \int_{\Gamma_\kappa(x)} |(\Theta_E f)(y)|^2 \frac{d\mu(y)}{\delta_E(y)^{m-2\nu}} \right)^{\frac{1}{2}} \right\|_{L_x^p(E, \sigma)} \leq C \|f\|_{H^p(E, \rho|_E, \sigma)}, \quad \forall f \in H^p(E, \rho|_E, \sigma), \quad (2.39)$$

where  $C \in (0, \infty)$  is a constant that is allowed to depend only on  $p, C_o, \kappa, C_\theta$ , and geometry.

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