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UNBOUNDEDNESS OF THE LAGRANGIAN HOFER DISTANCE IN THE EUCLIDEAN BALL

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Abstract. Let $\mathcal{L}$ denote the space of Lagrangians Hamiltonian isotopic to the standard Lagrangian in the unit ball in $\mathbb{R}^{2n}$. We prove that the Lagrangian Hofer distance on $\mathcal{L}$ is unbounded.

1. Introduction

Let (M, ω) be a symplectic manifold and denote by $C_c^\infty([0,1] \times M)$ the set of compactly supported Hamiltonians on M. Each Hamiltonian $H \in C_c^\infty([0,1] \times M)$ generates a Hamiltonian flow ϕ_H^t. The group of (compactly supported) Hamiltonian diffeomorphisms of M, denoted by $Ham(M)$, is defined to be the set of time-1 maps of such flows. The Hofer norm of $\psi \in Ham(M)$ is defined by the expression $\|\psi\| = \inf\{\|H\|_{(1,\infty)} : \psi = \phi_H^1\}$, where

$$\|H\|_{(1,\infty)} = \int_0^1 (\max_M H(t, \cdot) - \min_M H(t, \cdot))\, dt.$$

Let $B^{2n} \subset \mathbb{R}^{2n}$ denote the open unit ball equipped with the symplectic structure $\omega_0 = \frac{1}{\pi} \sum_{i=1}^n dx_i \wedge dy_i$. Denote by $L_0 = \{(x_i, y_i) \in B : y_i = 0\} \cap B^{2n}$ the standard Lagrangian in the unit ball, and let $\mathcal{L} = \{\phi(L_0) : \phi \in Ham(B^{2n})\}$. We endow $\mathcal{L}$ with the Lagrangian Hofer distance d, defined by:

$$\forall\ L_1, L_2 \in \mathcal{L}, \quad d(L_1, L_2) = \inf\{\|\phi\| : \phi \in Ham(B^{2n}),\ \phi(L_1) = L_2\}.$$

In [7], Khanevsky proved that, in dimension 2, the metric space $(\mathcal{L}, d)$ is unbounded. In higher dimensions, this seemingly basic question has remained open despite the fact that much progress has been made in the field of Lagrangian Hofer geometry; see for example [6, 12, 14]. Our main goal here is to prove the unboundedness of $(\mathcal{L}, d)$ in full generality. Let $C_c^\infty([0,1])$ denote the space of smooth functions

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with support compactly contained in the interior of $[0,1]$. For $f \in C_c^\infty([0,1])$ define $\|f\|_\infty = \max_{[0,1]} |f|$. Here is our main result.

Theorem 1. *There exist a map $\Psi : C_c^\infty([0,1]) \to \mathcal{L}$ and positive constants $C, D \in \mathbb{R}$ such that*

$$\frac{\|f-g\|_\infty - C}{D} \leqslant d(\Psi(f), \Psi(g)) \leqslant \|f-g\|_\infty.$$

In particular, the metric space $(\mathcal{L}, d)$ is unbounded.

Remark. *I have recently been informed that Michael Khanevsky and Frol Zapolsky have jointly obtained a different proof of the unboundedness of $(\mathcal{L}, d)$. Their proof relies on a forthcoming work of Rémi Leclercq and Frol Zapolsky on Lagrangian spectral invariants [8].*

2. Proof of Theorem 1

This paper, like Khanevsky's, relies heavily on Entov and Polterovich's work on the theory of quasimorphisms [2, 1, 3, 4, 5]. A quasimorphism on a group G is a function $\mu : G \to \mathbb{R}$ such that $|\mu(ab) - \mu(a) - \mu(b)| \leqslant D, \ \forall\ a, b \in G$, where D is a constant which is usually referred to as the defect of the quasimorphism μ. If $\mu(a^k) = k\mu(a)$ for all $a \in G$ and $k \in \mathbb{Z}$, then μ is called a homogeneous quasimorphism.

Let $B^{2n}(\frac{1}{\sqrt{\pi}})$ denote the open Euclidean ball of radius $\frac{1}{\sqrt{\pi}}$. For $0 < r < \frac{1}{n\pi}$, let $T(r) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^2 = \cdots = |z_n|^2 = r\}$. Note that $T(r)$ is a Lagrangian torus in $B^{2n}(\frac{1}{\sqrt{\pi}})$. In Section 3, we will construct a family of homogenous quasimorphisms $\eta_\delta : Ham(B^{2n}(\frac{1}{\sqrt{\pi}})) \to \mathbb{R}$; the parameter δ will range over the interval $(\frac{n}{n+1}, 1]$. Note that for values of $\delta > \frac{n}{n+1}$, the Lagrangian torus $T(\frac{1}{\delta\pi(n+1)})$ is contained in $B^{2n}(\frac{1}{\sqrt{\pi}})$.

Theorem 2. *The homogeneous quasimorphisms $\eta_\delta : Ham(B^{2n}(\frac{1}{\sqrt{\pi}})) \to \mathbb{R}$ have the following properties:*

(1) There exists a constant $C > 0$, independent of δ, such that $|\eta_\delta(\phi)| \leqslant C\|\phi\|$.
(2) If $F : [0,1] \times B^{2n}(\frac{1}{\sqrt{\pi}}) \to \mathbb{R}$ is a compactly supported Hamiltonian such that $F(t,x) \leqslant c$ (resp., $F(t,x) \geqslant c$) for all $(t,x) \in [0,1] \times T(\frac{1}{\delta\pi(n+1)})$, then $\eta_\delta(\phi_F^1) \leqslant c$ (resp., $\eta_\delta(\phi_F^1) \geqslant c$).
(3) There exists a constant $D > 0$, independent of δ, such that if $\phi(L_0) = \psi(L_0)$, then $|\eta_\delta(\phi) - \eta_\delta(\psi)| \leqslant D$.

We will next use the above theorem to prove Theorem 1. The fact that Theorem 1 follows from the existence of quasimorphisms with the above list of properties is implicitly present in Khanevsky's paper [7]. Various versions of the quasimorphisms employed in this note have appeared in the work of Entov, Polterovich and their collaborators; see for example [2, 1, 3, 4, 5]. This close connection to Entov and Polterovich's work allows us to prove the first two properties by standard techniques. The non-standard part is establishing Property (3), which in a sense states that, up to a bounded error, $\eta_\delta(\phi)$ is an invariant of the Lagrangian $\phi(L_0)$. In [7], Khanevsky proves this property using two dimensional methods. We will use a different approach which is applicable in all dimensions.

Proof of Theorem 1. We will prove the theorem for $B^{2n}(\frac{1}{\sqrt{\pi}})$ rather than B^{2n}. Of course, this is sufficient as $B^{2n}(\frac{1}{\sqrt{\pi}})$ and B^{2n} are conformally symplectomorphic. We will continue to denote by L_0 the restriction of $\{(x_i, y_i) \in B : y_i = 0\}$ to $B^{2n}(\frac{1}{\sqrt{\pi}})$ and by $\mathcal{L}$ the space of Lagrangians which are Hamiltonian isotopic to L_0 inside $B^{2n}(\frac{1}{\sqrt{\pi}})$.

Take $\phi \in Ham(B^{2n}(\frac{1}{\sqrt{\pi}}))$ and let ψ denote any other Hamiltonian diffeomorphism such that $\phi(L_0) = \psi(L_0)$. Using Properties (1) and (3) we obtain the following for all $\delta \in (\frac{n}{n+1}, 1]$:

$$|\eta_\delta(\phi)| - D \leqslant |\eta_\delta(\psi)| \leqslant C\|\psi\|.$$

The above implies that $\forall\ \phi \in Ham(B^{2n}(\frac{1}{\sqrt{\pi}}))$ and $\forall\ \delta \in (\frac{n}{n+1}, 1]$ we have

$$\frac{|\eta_\delta(\phi)| - D}{C} \leqslant d(L_0, \phi(L_0)). \tag{1}$$

Let $J = [\frac{n}{n+1}, 1]$ and denote by $C_c^\infty(J)$ the set of functions whose support is compactly contained in the interior of J. We will construct a map $\Psi : C_c^\infty(J) \to \mathcal{L}$ such that $\frac{\|f-g\|_\infty - C}{D} \leqslant d(\Psi(f), \Psi(g)) \leqslant \|f - g\|_\infty$. Of course, this establishes Theorem 1 as J is diffeomorphic to $[0, 1]$.

For any $f \in C_c^\infty(J)$, set $\tilde{f}(z) = f(\pi|z|^2)$, where $z \in \mathbb{C}^n$. Note that the Hamiltonian $\tilde{f}$ is compactly supported in $B^{2n}(\frac{1}{\sqrt{\pi}})$. We define $\Psi : C_c^\infty(J) \to \mathcal{L}$ by the following expression:

$$\Psi(f) = \phi^1_{\tilde{f}}(L_0).$$

We will first show that $\frac{\|f-g\|_\infty - D}{C} \leqslant d(\Psi(f), \Psi(g))$. Apply Inequality (1) to $\phi^{-1}_{\tilde{g}}\phi^1_{\tilde{f}}$, where $\phi^{-1}_{\tilde{g}}$ denotes the inverse of $\phi^1_{\tilde{g}}$, to obtain

$$\frac{|\eta_\delta(\phi^{-1}_{\tilde{g}}\phi^1_{\tilde{f}})| - D}{C} \leqslant d(L_0, \phi^{-1}_{\tilde{g}}\phi^1_{\tilde{f}}(L_0)),\ \forall\ \delta \in (\tfrac{n}{n+1}, 1].$$

Observe that $\tilde{f}$ and $\tilde{g}$ Poisson commute and hence $\phi^{-1}_{\tilde{g}}\phi^1_{\tilde{f}} = \phi^1_{\tilde{f}-\tilde{g}}$. Also, by symplectic invariance of Hofer's distance $d(L_0, \phi^{-1}_{\tilde{g}}\phi^1_{\tilde{f}}(L_0)) = d(\phi^1_{\tilde{g}}(L_0), \phi^1_{\tilde{f}}(L_0))$. Therefore, the previous inequality is equivalent to

$$\frac{|\eta_\delta(\phi^1_{\tilde{f}-\tilde{g}})| - D}{C} \leqslant d(\phi^1_{\tilde{g}}(L_0), \phi^1_{\tilde{f}}(L_0)),\ \forall\ \delta \in (\tfrac{n}{n+1}, 1].$$

Note that $\tilde{f}$ and $\tilde{g}$ are constant on each of the tori $T(\frac{1}{\delta\pi(n+1)})$. Hence, using Property (2) from Theorem 2, we conclude that $\eta_\delta(\phi^1_{\tilde{f}-\tilde{g}}) = f(\frac{n}{\delta(n+1)}) - g(\frac{n}{\delta(n+1)})$. Picking δ so that $f - g$ attains its maximum at $\frac{n}{\delta(n+1)}$ yields

$$\frac{\|f - g\|_\infty - D}{C} \leqslant d(\phi^1_{\tilde{g}}(L_0), \phi^1_{\tilde{f}}(L_0)).$$

It remains to prove that $d(\Psi(f), \Psi(g)) \leqslant \|f - g\|_\infty$, for any $f, g \in C_c^\infty(J)$. Indeed,

$$\begin{aligned} d(\Psi(f), \Psi(g)) &= d(\phi^1_{\tilde{f}}(L_0), \phi^1_{\tilde{g}}(L_0)) = d(L_0, (\phi^1_{\tilde{f}})^{-1}\phi^1_{\tilde{g}}(L_0)) \\ &\leqslant \|(\phi^1_{\tilde{f}})^{-1}\phi^1_{\tilde{g}}\| \leqslant \|\tilde{f} - \tilde{g}\|_\infty = \|f - g\|_\infty. \end{aligned}$$

This completes our proof. □

3. Construction of the Quasimorphisms

In this section, following Biran, Entov, and Polterovich [1], we construct the quasimorphisms η_δ.

3.1. Spectral invariants on $\mathbb{C}P^n$. Let ω_{FS} denote the Fubini-Study symplectic form on $\mathbb{C}P^n$ normalized so that $Vol(\mathbb{C}P^n) = 1$. Throughout the rest of this article we assume that $\mathbb{C}P^n$ is equipped with the symplectic structure induced by ω_{FS}.

It follows from the works of C. Viterbo, M. Schwarz, and Y.-G. Oh [13, 10, 9] that one can associate to each Hamiltonian $H \in C^\infty([0,1] \times \mathbb{C}P^n)$ and each non-zero quantum homology class $a \in QH_*(\mathbb{C}P^n)) \setminus \{0\}$ a so-called spectral invariant $c(a, H) \in \mathbb{R}$. These invariants are constructed via the machinery of Hamiltonian Floer theory and, in fact, they can be defined on a very large class of symplectic manifolds; however, we will only be concerned with $\mathbb{C}P^n$. The only quantum homology class that we will be dealing with is the fundamental class $[\mathbb{C}P^n]$; for brevity we will denote $c(H) = c([\mathbb{C}P^n], H)$.

For our purposes we must introduce the asymptotic version of the spectral invariant c. This is defined as follows: for $H, G \in C^\infty([0,1] \times \mathbb{C}P^n)$ define

$$H\#G(t,x) = H(t,x) + G(t, (\phi_H^t)^{-1}(x));$$

the flow of $H\#G$ is $\phi_H^t \circ \phi_G^t$. Following Entov and Polterovich [2] we define the asymptotic spectral invariant of a Hamiltonian H by

$$\zeta(H) = \lim_{k \to \infty} \frac{c(H^{\#k})}{k}.$$

Our asymptotic spectral invariant, ζ, is defined for all time-dependent Hamiltonians. The restriction of ζ to the set of time-independent Hamiltonians, i.e., $C^\infty(\mathbb{C}P^n)$, is what Entov and Polterovich refer to as a symplectic quasi-state on $\mathbb{C}P^n$. In fact, the restriction of ζ to $C^\infty(\mathbb{C}P^n)$ is precisely the quasi-state constructed on $\mathbb{C}P^n$ in [3].

According to Entov and Polterovich [4], a closed subset $X \subset \mathbb{C}P^n$ is said to be superheavy with respect to ζ if

$$\inf_X H \leqslant \zeta(H) \leqslant \sup_X H \ \forall \ H \in C^\infty(\mathbb{C}P^n).$$

The Clifford torus and $\mathbb{R}P^n$ are two examples of superheavy subsets of $\mathbb{C}P^n$. Superheaviness of these two sets follows from Theorems 1.6 and 1.15 of [4], respectively. In [4], the notion of superheaviness is defined only for autonomous Hamiltonians. Using standard properties of spectral invariants one can easily show that if $X \subset \mathbb{C}P^n$ is superheavy, then

$$\inf_X H \leqslant \zeta(H) \leqslant \sup_X H \ \forall \ H \in C^\infty([0,1] \times \mathbb{C}P^n). \tag{2}$$

3.2. Entov and Polterovich's Calabi quasi-morphism. In [2], Entov and Polterovich construct a quasimorphism $\mu : Ham(\mathbb{C}P^n) \to \mathbb{R}$ defined by the following expression:

$$\mu(\phi_F^1) = -\zeta(F) + \int_0^1 \int_{\mathbb{C}P^n} F(t,x)\, \omega_{FS}^n \, dt. \tag{3}$$

In [2], it is proven that μ does not depend on the choice of the generating Hamiltonian F and hence is a well-defined map from $Ham(\mathbb{C}P^n)$ to $\mathbb{R}$. Furthermore, μ is a homogeneous quasimorphism. The relationship between this quasimorphism and the aforementioned quasi-state ζ is discussed in detail in [3]. For example, the above formula relating ζ and μ can be found in Section 6 of [3].

Recall that a subset U of a symplectic manifold is said to be displaceable if there exists a Hamiltonian diffeomorphism ψ such that $U \cap \psi(U) = \emptyset$. If the support of a Hamiltonian F is displaceable, then

$$\zeta(F) = 0, \tag{4}$$

and hence, $\mu(\phi_F^1) = \int_0^1 \int_{\mathbb{C}P^n} F(t,x)\, \omega_{FS}^n \, dt$. This is referred to as the Calabi property of μ and for this reason μ is often referred to as a Calabi quasimorphism; for further details see [2, 3].

3.3. **Constructing η_δ.** In this section we closely follow Section 4 of [1].
For $0 < \delta \leqslant 1$, define embeddings $\theta_\delta : B^{2n}(\frac{1}{\sqrt{\pi}}) \to \mathbb{C}P^n$ by the formula

$$(z_1, \cdots, z_n) \mapsto \left[\sqrt{\frac{1}{\pi} - \Sigma_{i=1}^n \delta |z_i|^2} : \sqrt{\delta} z_1 : \cdots : \sqrt{\delta} z_n \right],$$

where $z_i's$ denote the standard complex coordinates on $\mathbb{C}^n$. The maps θ_δ pull ω_{FS} back to $\delta\, \omega_0$, and so these embeddings are all conformally symplectic. Clearly, θ_1 is a genuine symplectic embedding.

As displayed in [1], for $\delta > \frac{n}{n+1}$ the torus $T(\frac{1}{\delta\pi(n+1)}) \subset B^{2n}(\frac{1}{\sqrt{\pi}})$ is mapped by θ_δ onto the Clifford torus. It is also clear that $\theta_\delta(L_0)$ is contained inside $\mathbb{R}P^n \subset \mathbb{C}P^n$.

For $\delta > \frac{n}{n+1}$, define $\eta_\delta : Ham(B^{2n}(\frac{1}{\sqrt{\pi}})) \to \mathbb{R}$ as so: Given $\phi \in Ham(B^{2n}(\frac{1}{\sqrt{\pi}}))$ pick a Hamiltonian $F \in C_c^\infty([0,1] \times B^{2n}(\frac{1}{\sqrt{\pi}}))$ such that $\phi_F^1 = \phi$. Define

$$\eta_\delta(\phi) = \delta^{-1} \zeta(\delta F \circ \theta_\delta^{-1}). \tag{5}$$

We must show that η_δ does not depend on the choice of the generating Hamiltonian F. One can easily check that the time-1 map of the Hamiltonian $\delta F \circ \theta_\delta^{-1}$: $[0,1] \times \mathbb{C}P^n \to \mathbb{R}$ is the Hamiltonian diffeomorphism $\theta_\delta \phi \theta_\delta^{-1}$. This combined with Equation (3) yields

$$\delta^{-1} \zeta(\delta F \circ \theta_\delta^{-1}) = -\delta^{-1} \mu(\theta_\delta \phi \theta_\delta^{-1}) + \delta^{-1} \int_0^1 \int_{\mathbb{C}P^n} \delta F(t, \theta_\delta^{-1}(x))\, \omega_{FS}^n.$$

A simple computation shows that

$$\int_0^1 \int_{\mathbb{C}P^n} F(t, \theta_\delta^{-1}(x))\, \omega_{FS}^n = \delta^n \int_0^1 \int_{B^{2n}(\frac{1}{\sqrt{\pi}})} F(t,x)\, \omega_0^n = \delta^n Cal(\phi),$$

where $Cal : Ham(B^{2n}(\frac{1}{\sqrt{\pi}})) \to \mathbb{R}$ denotes the Calabi homomorphism:

$$Cal(\phi_F^1) = \int_0^1 \int_{B^{2n}(\frac{1}{\sqrt{\pi}})} F(t,x)\, \omega_0^n \, dt.$$

Hence, we have obtained the following alternative definition for η_δ:

$$\eta_\delta(\phi) = -\delta^{-1} \mu(\theta_\delta \phi \theta_\delta^{-1}) + \delta^n Cal(\phi). \tag{6}$$

It is clear from the above formula that η_δ is well-defined and does not depend on the choice of the generating Hamiltonian F. Furthermore, η_δ is a quasimorphism as it is a linear combination of quasimorphisms. Results from [11] on descent of asymptotic spectral invariants provide an alternative method for proving that η_δ is a well-defined quasimorphism. Additionally, one can use Formula (6) to show that the defects of the quasimorphisms η_δ are bounded uniformly with respect to δ. Indeed, Formula (6), combined with the fact the Cal is a homomorphism, implies that

$$\eta_\delta(\phi\psi) - \eta_\delta(\phi) - \eta_\delta(\psi) = -\delta^{-1}(\mu(\theta_\delta\phi\psi\theta_\delta^{-1}) - \mu(\theta_\delta\phi\theta_\delta^{-1}) - \mu(\theta_\delta\psi\theta_\delta^{-1})).$$

From this we obtain that

$$|\eta_\delta(\phi\psi) - \eta_\delta(\phi) - \eta_\delta(\psi)| \leqslant \delta^{-1}Def(\mu) \leqslant \frac{n+1}{n}Def(\mu), \tag{7}$$

where $Def(\mu)$ is the defect of μ. The last inequality holds because $\delta \in (\frac{n}{n+1}, 1]$.

3.4. **Proof of Theorem 2.** Part (1) of the theorem follows from Formula (6) and the fact that both μ and Cal are Lipschitz continuous with respect to Hofer's norm. Part (2) follows from Formula (5): Indeed, if $F \leqslant c$ on $T(\frac{1}{\pi\delta(n+1)})$, then $\delta F \circ \theta_\delta^{-1} \leqslant \delta c$ on the Clifford torus. Since the Clifford torus is superheavy for ζ we see that $\delta^{-1}\zeta(\delta F \circ \theta_\delta^{-1}) \leqslant c$.

It remains to prove Part (3) of Theorem 2. Let $D = \frac{n+1}{n}Def(\mu)$. Inequality (7) implies that $|\eta_\delta(\phi^{-1}\psi) + \eta_\delta(\phi) - \eta_\delta(\psi)| \leqslant D$. The Hamiltonian diffeomorphism $\phi^{-1}\psi$ preserves L_0 and hence it is sufficient to prove that η_δ vanishes on the set of Hamiltonian diffeomorphisms which preserve L_0.

Suppose that $\phi_F^1(L_0) = L_0$. Pick a positive number ϵ such that $B^{2n}(\epsilon)$, the ball of radius ϵ, is displaceable inside $B^{2n}(\frac{1}{\sqrt{\pi}})$. For each $s \in [\epsilon, 1]$ we define a Hamiltonian diffeomorphism ψ_s as follows:

$$\psi_s(x) = \begin{cases} s\phi_F^1(\frac{x}{s}) & \text{if } |x| \leqslant s, \\ x & \text{if } |x| \geqslant s. \end{cases}$$

A simple computation shows that $\psi_s \in Ham(B^{2n}(\frac{1}{\sqrt{\pi}}))$, and in fact, ψ_s is the time-1 map of the flow of the following Hamiltonian:

$$F_s(t,x) = \begin{cases} s^2F(t, \frac{x}{s}) & \text{if } |x| \leqslant s, \\ 0 & \text{if } |x| \geqslant s. \end{cases}$$

Furthermore, the Hamiltonian diffeomorphisms ψ_s all preserve L_0 because $\phi_F^1(L_0) = L_0$. Now, consider the path of Hamiltonian diffeomorphisms $[0,1] \to Ham(B^{2n}(\frac{1}{\sqrt{\pi}}))$ defined by

$$t \mapsto \psi_\epsilon^{-1}\psi_{t(1-\epsilon)+\epsilon}.$$

Pick a Hamiltonian $H \in C_c^\infty([0,1] \times B^{2n}(\frac{1}{\sqrt{\pi}}))$ such that

$$\phi_H^t = \psi_\epsilon^{-1}\psi_{t(1-\epsilon)+\epsilon}.$$

Because $\phi_H^t(L_0) = L_0$ for each $t \in [0,1]$, the Hamiltonian vector fields $\{X_{H_t}\}_{t\in[0,1]}$ must all be tangential to L_0. Since L_0 is a Lagrangian, we conclude that $dH_t = \omega_0(X_{H_t}, \cdot)$ vanishes on the tangent space to L_0. This implies that for each $t \in [0,1]$ the restriction of H_t to L_0 is constant. Now, H_t is compactly supported in the interior of $B^{2n}(\frac{1}{\sqrt{\pi}})$ and thus we conclude $H_t|_{L_0} = 0 \ \forall \ t \in [0,1]$. As a consequence, the Hamiltonian $\delta H \circ \theta_\delta^{-1}$ vanishes on $\mathbb{R}P^n \subset \mathbb{C}P^n$; this is because $\theta_\delta(L_0) \subset \mathbb{R}P^n$. Since $\mathbb{R}P^n$ is superheavy, we conclude, by Inequality (2) that

$$\eta_\delta(\phi_H^1) = 0.$$

Next, note that ψ_ϵ is supported in B^{2n}_ϵ, which is displaceable inside $B^{2n}(\frac{1}{\sqrt{\pi}})$. Therefore, Equation (4) yields that

$$\eta_\delta(\psi_\epsilon) = 0.$$

Since $\phi^1_F = \psi_\epsilon \phi^1_H$ and η_δ is a quasimorphism, it follows that

$$|\eta_\delta(\phi^1_F)| = |\eta_\delta(\phi^1_F) - \eta_\delta(\psi_\epsilon) - \eta_\delta(\phi^1_H)| \leqslant D.$$

Thus far, we have proven that if $\phi^1_F(L_0) = L_0$, then $|\eta_\delta(\phi^1_F)| \leqslant D$. Of course, if $\phi^1_F(L_0) = L_0$, then $(\phi^1_F)^k(L_0) = L_0$ for any $k \in \mathbb{Z}$ and therefore $|\eta_\delta((\phi^1_F)^k)| \leqslant D$. But η_δ is homogeneous and so $\eta_\delta((\phi^1_F)^k) = k\eta_\delta(\phi^1_F)$. It follows that $|k\eta_\delta(\phi^1_F)| \leqslant D$ for all $k \in \mathbb{Z}$ and so $\eta_\delta(\phi^1_F) = 0$. This completes our proof.

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