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ON DEGENERATIONS OF MODULI OF HITCHIN PAIRS

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ABSTRACT. The purpose of this note is to announce certain basic results on the construction of a degeneration of $\mathscr{M}^H_{X_k}(n,d)$ as the smooth curve X_k degenerates to an irreducible nodal curve with a single node.

Let X_k be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic zero and let $\mathcal{L}$ be a line bundle on X_k. A Hitchin pair (E, θ) is comprised of a torsion-free $\mathcal{O}_{X_k}$-module E together with a $\mathcal{O}_{X_k}$-morphism $\theta : E \to E \otimes \mathcal{L}$ called the Higgs structure. Let $\mathscr{M}^H_{X_k}(n,d)$ denote the moduli space of semistable Hitchin pairs on X_k with Higgs structure given by the line bundle $\mathcal{L}$. The geometry of Hitchin pairs or Higgs bundles has been extensively studied for over twenty-five years beginning with Hitchin ([4], [5]), Nitsure ([8]), and Simpson ([11], [12], [13]).

More precisely, let R be a discrete valuation ring with quotient field K and residue field an algebraically closed field k, for instance $R = k[[t]]$. Let $S = \text{Spec } R$, and $\text{Spec } K$ the generic point and let s be the closed point of S. Let $X \to S$ be a proper, flat family with generic fibre X_K a smooth projective curve of genus $g \geq 2$ and with closed fibre X_s a irreducible nodal curve C with a single node $p \in C$. Assume that X is regular as a scheme over k. Let $\mathcal{L}$ be a relative line bundle on X and assume that $deg(\mathcal{L}|_C) > deg(\omega_C)$, where ω_C is the dualizing sheaf on C. Let (n, d) be a pair of integers such that $gcd(n, d) = 1$.

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We now make the key definitions (motivated by the constructions in Gieseker [3] and Nagaraj-Seshadri [7]) before we state our principal results. Let $\tilde{C}$ be its normalization and let $\nu : \tilde{C} \to C$ be the normalization map and let $\nu^{-1}(p) = \{p_1, p_2\}$.

Definition 1. *A scheme* $R^{(m)}$ *is called a* chain of projective lines *if* $R^{(m)} = \bigcup_{i=1}^{m} R_i$, *with* $R_i \simeq \mathbb{P}^1$, *and if* $i \neq j$,

$$R_i \cap R_j = \begin{cases} \text{singleton} & \text{if } \mid i - j \mid = 1 \\ \emptyset & \text{otherwise} \end{cases} \qquad (1)$$

Definition 2. *Let* E *be a vector bundle of rank* n *on a chain* $R^{(m)}$. *Let* $E|_{R_i} = \bigoplus_{j=1}^{n} \mathcal{O}(a_{ij})$. *Say that* E *is* standard *if* $0 \leq a_{ij} \leq 1, \forall i, j$. *Say that* E *is* strictly standard *if moreover, for every* i *there is an index* j *such that* $a_{ij} = 1$.

Definition 3. *Let* $C^{(m)}$ *denote the semi-stable curve which is semistably equivalent to* C, *which is obtained as follows: the normalization* $\tilde{C}$ *is a component of* $C^{(m)}$ *and further, if* $\nu : C^{(m)} \to C$ *is the canonical morphism, the fibre* $\nu^{-1}(p)$ *is a chain* $R^{(m)}$ *of projective lines of length* m *cutting* $\tilde{C}$ *in* p_1 *and* p_2.

Let $p : X \to S$ be as before a family of smooth curves degenerating to the singular curve C. For an S-scheme T, let $X_T := X \times_S T$.

Definition 4. *(cf.* [6, Definition 3.8]*) For every* S*-scheme* T, *a* modification *is a diagram:*

$$\begin{array}{ccc} X_T^{(mod)} & \xrightarrow{\quad \nu \quad} & X_T \\ & {\scriptstyle p_T} \searrow \quad \swarrow {\scriptstyle p} & \\ & T & \end{array} \qquad (2)$$

(1) $p_T : X_T^{(mod)} \to T$ *is* flat,
(2) *the* T*-morphism* ν *is finitely presented which is an isomorphism when* $(X_T)_t$ *is* smooth,
(3) *over each closed point* $t \in T$ *over* $s \in S$, *we have* $(X_T^{(mod)})_t = C^{(m)}$ *for some* m *and* ν *restricts to the morphism which contracts the* $\mathbb{P}^1$*'s on* $C^{(m)}$.

Definition 5. *(see* [7] *and* [10]*) A vector bundle* V *on* $C^{(m)}$ *of rank* n *is called a* Gieseker vector bundle *if it satisfies the following conditions:*

(1) *for* $m \geq 1$, *the restriction* $V|_{R^{(m)}}$ *is* strictly standard,
(2) *the direct image* $\nu_*(V)$ *to be a torsion-free* $\mathcal{O}_C$*-module.*

A Gieseker vector bundle on a modification $X_T^{(mod)}$ is a vector bundle such that its restriction to each $C^{(m)}$ in it is a Gieseker vector bundle.

Let $\mathcal{L}_{\text{mod}}$ be the line bundle on $X_T^{(\text{mod})}$ defined by $\mathcal{L}_{mod} := \nu^*(\mathcal{L})$. In particular, $\mathcal{L}_{\text{mod}}|_{R^{(m)}} = \mathcal{O}_{R^{(m)}}$ on the chain $R^{(m)}$ in $C^{(m)}$.

Definition 6. *A Gieseker-Hitchin pair on $X_T^{(mod)}$ is a locally free Hitchin pair (V_T, ϕ_T), with an element*

$$\phi_T \in H^0(T, (p_T)_*(\mathcal{L}_{mod} \otimes \mathscr{E}nd(V_T)),$$

i.e., a morphism $\phi_T : V_T \to V_T \otimes\ \mathcal{L}_{mod}$ satisfying the following:

1. *V_T is a Gieseker vector bundle on $X_T^{(mod)}$ (Definition 5).*
2. *For each closed point $t \in T$ over $s \in S$, the direct image $\nu_*(V_t, \phi_t)$ is a torsion-free Hitchin pair on $X_t = C$.*

A Gieseker-Hitchin pair (V_T, ϕ_T) is called stable *if the direct image $(\nu)_*(V_T, \phi_T)$ is a family of* stable *Hitchin pairs on X_T over T (for the notion of (semi)stability of torsion-free Hitchin pairs, see [12], [13] and [1]).*

Definition 7. *Two families (V_T, ϕ_T) and (V'_T, ϕ'_T) parametrized by T are called* equivalent *if there exists a X_T-automorphism σ, i.e.,*

$$\begin{array}{ccc} X_T^{(mod)} & \xrightarrow{\ \sigma\ } & X_T^{(mod)} \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle\nu} & \\ & X_T & \end{array} \tag{3}$$

and a line bundle $\mathscr{D}_T$ on the parameter space T such that

$$\sigma^*((V_T, \phi_T) \otimes \mathscr{D}_T \simeq (V'_T, \phi'_T). \tag{4}$$

Equivalently, for each closed point $t \in T$ over $s \in S$, there exists an automorphism g of $C^{(m)}$ which is the identity automorphism on the normalization $\tilde{C}$, with the property that $g^*(V_t, \phi_t) \simeq (V'_t, \phi'_t)$.

Let $\underline{\mathscr{M}}_S^H(n,d)$ be the functor which associates to every S-scheme T, the set $\underline{\mathscr{M}}_S^H(n,d)(T)$ of the equivalence classes of families of `p`-*semistable* torsion-free Hitchin pairs (E, θ) on $X_T := X \times_S T$ with Hilbert polynomial P given by n and d, where $(E_T, \theta_T) \sim (E'_T, \theta'_T)$ if there exists a line bundle L_T on T such that $E_T \simeq E'_T \otimes p_T^*(L_T)$ which sends θ_T to $\theta'_T \otimes id$.

Definition 8. *The Gieseker-Hitchin functor $\underline{\mathscr{G}}_S^H(n,d)(T)$ is defined as follows: for every S-scheme T,*

$$\underline{\mathscr{G}}_S^H(n,d)(T) := \left[X_T^{(mod)}, (V_T, \phi_T)\right], \tag{5}$$

i.e., equivalence *classes such that* (V_T, ϕ_T) *is a stable Gieseker-Hitchin pair on* $X_T^{(mod)}$ *and* $\nu_*(V_T, \phi_T) \in \underline{\mathscr{M}}_S^H(n,d)(T)$.

Our principal results are the following:

Theorem 1.

(1) *There is a quasi-projective* S*-scheme* $\mathscr{G}_S^H(n,d)$ *of Gieseker-Hitchin pairs which coarsely represents the functor* $\underline{\mathscr{G}}_S^H(n,d)$*; the* S*-scheme* $\mathscr{G}_S^H(n,d)$ *is flat over* S *and regular over* k*, with the closed fibre a divisor with (analytic) normal crossing singularities.*
(2) *The generic fibre is isomorphic to the classical Hitchin space* $\mathscr{M}_{X_K}^H(n,d)$.

Theorem 2. *We have a Hitchin morphism of* S*-schemes*

$$\mathsf{g}_S : \mathscr{G}_S^H(n,d) \to \mathcal{A}_S \tag{6}$$

to an affine space $\mathcal{A}_S$ *over* S *which extends the classical Hitchin map on* $\mathscr{M}_{X_K}^H(n,d)$*. Furthermore,* g_S *is* proper *and has the following properties:*

(1) *To a* general *section* $\xi : S \to \mathcal{A}_S$ *we can associate a spectral fibered surface* Y_ξ *over* S *with smooth projective generic fibre* $Y_{\xi,K}$ *and whose closed fibre* $Y_{\xi,s}$ *is an irreducible* vine curve *with* n*-nodes (cf. [2]).*
(2) *Let* $\delta = d + deg(\mathcal{L})\frac{n(n-1)}{2}$ *and let* P_{δ,Y_ξ} *denote the compactified relative Picard* S*-scheme of the spectral fibered surface* Y_ξ *over* S *(see [2]). Then we have a proper birational morphism*

$$\nu_* : \mathsf{g}_S^{-1}(\xi) \to P_{\delta,Y_\xi} \tag{7}$$

which is an isomorphism over the generic fibre and this map coincides with the classical Hitchin isomorphism of the Hitchin fibre with the Jacobian of $Y_{\xi,K}$.
(3) *The* S*-scheme* $\mathsf{g}_S^{-1}(\xi)$ *gives a new compactification of the Picard variety, whose fibre over* s *is a divisor with* analytic normal crossing singularities.

The compactified Picard variety $P_{\delta,Y_{\xi,s}}$ of the irreducible vine curve $Y_{\xi,s}$ with n-nodes, has a stratification in terms of the complexity of the torsion-freeness of the sheaves. This can be given as follows:

$$P_{\delta,Y_{\xi,s}} = \bigsqcup P_{\delta,Y_{\xi,s}}(j), \tag{8}$$

where

$$P_{\delta,Y_{\xi,s}}(j) := \{\eta \mid \eta \text{ is non-free at exactly } j \text{ nodes}\}. \tag{9}$$

In this description the stratum $P_{\delta,Y_{\xi,s}}(0)$ corresponds to the open subset of line bundles on $Y_{\xi,s}$ of degree δ. The fibres of the morphism ν_* to the compactified Picard variety of the *vine curve* $Y_{\xi,s}$ gets the following description:

THEOREM 3. *The morphism ν_* is an isomorphism over the subscheme of locally free sheaves of rank* 1 *and for each j, over the stratum $P_{\delta,Y_{\xi,s}}(j)$ the fibres are canonical toric subvarieties of the wonderful compactification $\overline{PGL(j)}$ obtained from the closures of the maximal tori of $PGL(j)$. These are toric varieties associated to the Weyl chamber of $PGL(j)$ (see* [9]*).*

For the details of this announcement see [1].

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