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ON THE DYNAMICS OF SOCIAL CONFLICTS: LOOKING FOR THE BLACK SWAN

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Abstract. This paper deals with the modeling of social competition, possibly resulting in the onset of extreme conflicts. More precisely, we discuss models describing the interplay between individual competition for wealth distribution that, when coupled with political stances coming from support or opposition to a Government, may give rise to strongly self-enhanced effects. The latter may be thought of as the early stages of massive unpredictable events known as Black Swans, although no analysis of any fully-developed Black Swan is provided here. Our approach makes use of the framework of the kinetic theory for active particles, where nonlinear interactions among subjects are modeled according to game-theoretical principles.

1. Introduction.

The dynamics of social and economic systems rest ultimately on individual behaviors, by which subjects express, either consciously or unconsciously, a particular strategy. Often the latter is based not only on their own individual purposes but also on those they attribute to other agents. However, the sheer complexity of such systems makes it difficult to ascertain the impact of personal decisions on the resulting collective dynamics. In particular, interactions among individuals need not have an additive linear character. As a consequence, the global effect of a given number of entities (*field entities*) over a single one (*test entity*) cannot be assumed to merely consist in the linear superposition of every single field entity action. This nonlinear feature represents a serious conceptual difficulty to the derivation, and subsequent analysis, of mathematical models for that type of systems. These concepts are well understood e.g., in the interpretation of swarming phenomena from the point of view of both Physics [10] and Mathematics [13, 14, 19].

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In the last few years, a radical philosophical change has been undertaken in social and economic disciplines. An interplay among Economics, Psychology, and Sociology has taken place, thanks to a new cognitive approach no longer grounded on the traditional assumption of rational socio-economic behavior. Starting from the concept of bounded rationality [26], the idea of Economics as a subject highly affected by individual (either rational or irrational) behaviors, reactions, and interactions has begun to spread out. In this frame, the contribution of mathematical methods to a deeper understanding of the relationships between individual behaviors and collective outcomes may be fundamental.

More in general, in fields ranging from Economics to Sociology and Ecology, the last decades have witnessed an increasing interest for the introduction of quantitative mathematical methods that could account for individual, not necessarily rational, behaviors. Terms as game theory, bounded rationality, evolutionary dynamics are often used in that contexts, which clearly illustrates the continuous search for techniques able to provide mathematical models that can describe, and predict, living behaviors, see [17, 27, 28], as also documented in the bibliography on evolutionary game theory cited in the following. As a result, a picture of social and biological sciences as evolutionary complex systems is unfolding [9, 14]. A key empirical feature of such systems is that interactions among heterogeneous individuals often produce unexpected outcomes, which were absent at the individual level, and which are commonly termed *emergent behaviors* [1].

In this context, setting up a mathematical description able to capture the evolving features of socio-economic systems is a challenging, however difficult, task, which calls for a proper interaction between mathematics and social sciences. In this paper we attempt a preliminary step in this direction. We outline a mathematical framework suitable to incorporate some of the main complexity features of socio-economic systems. Out of it, we derive specific mathematical models with an eye towards the prediction of the so called *Black Swan*. The latter is defined to be a rare event, coming out as an irrational collective trend generated by possibly rational individual behaviors [29].

To achieve our goal, we use mathematical tools based on a development of the kinetic theory for active particles, see e.g., [8, 18], suitable to include nonlinear interactions and learning phenomena. This mathematical approach has been applied to various fields of Life Sciences, such as social systems [15], opinion formation [16], and has been revisited in [4, 5] with reference to Behavioral Sciences including Politics and Economics. Moreover, it has also been applied in fields different from the above-mentioned ones, for instance propagation of epidemics under virus mutations [20] and complex systems in general [18]. The conceptual link between methods of statistical mechanics and game theory was also introduced by Helbing [22]; on the other hand, methods of the mean-field kinetic theory have been used to model socio-economic systems, see e.g., [21, 30]. The specialized literature offers a great variety of different approaches, such as population dynamics with structure [32] or macroscopic dynamical systems [23]. In all cases, the challenging goal to be met consists in capturing the relevant features of living complex systems.

After the above general overview, the plan of the paper can now be illustrated in more detail. The contents are distributed into three more sections. Section 2 analyzes the complexity aspects of socio-economic systems and introduces the mathematical structures which offer the basis for the derivation of specific models. Section 3 opens with two illustrative applications focused on social conflicts: the first one shows that a social competition, if not properly controlled, may induce an unbalanced distribution of wealth with a clustering of the population in two extreme classes (a large class of poor people and a small oligarchic class of wealthy ones); the second one exemplifies, in connection with the aforesaid dynamics, how such a clustering can lead to a growing opposition against a Government. Subsequently, the investigation moves on toward the identification of premonitory signals, that can provide preliminary insights into the emergence of a Black Swan viewed as a large deviation from some heuristically expected trend. Section 4 finally proposes a critical analysis and focuses on research perspectives.

As a general remark, it is worth pointing out that a partly phenomenological approach cannot be avoided, due to the lack of a well-consolidated background theory for living complex systems. Nevertheless, a rigorous modeling method can still be developed by relying on mathematical structures consistent with the main complexity features of the systems at hand (cf. Section 2).

2. **From complexity features to mathematical structures.** Socio-economic systems can be described as ensembles of living entities able to develop *behavioral strategies*, by which they interact with each other. For this reason, such entities are also called *active particles*. Typically, their strategies are *heterogeneously* distributed and *change in time* in consequence of the interactions, since active particles can update them by *learning* from past experiences. Therefore the mathematical representation calls for *distribution functions* over the strategies, whose time evolution depends on the dynamics of interactions. According to the approach under consideration, the latter are regarded as *stochastic games*, whose payoff is the new strategy that each player will use in the next interaction. Remarkably, payoffs are supposed to be known only in probability because irrational behaviors in individual choices cannot be excluded. Deterministic payoffs would imply instead a kind of rational behavior of active particles, as if they reacted in the same manner when placed in the same conditions.

Bearing all above in mind, we detail now a mathematical framework suitable to derive specific models of socio-economic dynamics. Particularly, we consider systems of interacting individuals whose number is constant in time; namely, we disregard birth and death processes or inlets from an outer environment.

Let u be a variable denoting the strategy expressed by active particles, that in the present context can be identified with their wealth status. As shown in [16], see also the references therein, for the kind of applications we are interested in it is customary to speak of *wealth classes*, which implies that the strategy u is a discrete variable taking values in a lattice

$$I_u = \{u_1, \dots, u_i, \dots, u_n\}.$$

The number of individuals that, at a certain time t, are expressing the strategy u_i, or, in other words, that belong to the i-th wealth class, is given by a distribution function

$$f_i = f_i(t) : [0, T_{\max}] \to \mathbb{R}_+, \quad i = 1, \dots, n,$$

$T_{\max} > 0$ being a certain final time (possibly $+\infty$). Notice that, up to normalizing with respect to the total number of active particles, $\mathbf{f} = (f_1, \dots, f_i, \dots, f_n)$ is a time-evolving discrete probability distribution over the lattice I_u.

As previously anticipated, the time evolution of each f_i is determined by game-type interactions with stochastic payoffs. When describing qualitatively interactions among active particles, it is useful to distinguish three main actors named test, candidate, and field particles. This terminology is indeed standard in the generalized kinetic theory for active particles.

- The *test* particle, with strategy, say, u_i, is a generic representative entity of the system. Studying interactions means studying how the test particle can loose its strategy or other particles can gain it.
- *Candidate* particles, with strategy, say, u_h, are the particles which can gain the test strategy u_i in consequence of the interactions.
- *Field* particles, with strategy, say, u_k, are the particles whose presence triggers the interactions of the candidate particles.

The quantitative modeling of the interactions is further based on the derivation of the following two terms:

- The *interaction rate*, denoted by η_{hk}, modeling the frequency of interaction between a candidate particle expressing the strategy u_h and a field particle expressing the strategy u_k.
- The *transition probabilities*, denoted by $\mathcal{B}_{hk}(i)$, expressing the probability that a candidate particle with strategy u_h changes the latter to u_i after interacting with a field particle with strategy u_k. The following probability density property has to hold:

$$\sum_{i=1}^{n} \mathcal{B}_{hk}(i) = 1, \quad \forall\, h,\, k = 1, \ldots, n. \tag{1}$$

By letting all particles play together for a time $\Delta t > 0$, the expected number of particles shifting to and losing, respectively, the strategy u_i during the time interval $[t,\, t + \Delta t]$ can be estimated, at first order, as:

$$\Delta t \sum_{h,\,k=1}^{n} \eta_{hk}\mathcal{B}_{hk}(i) f_h(t) f_k(t) + o(\Delta t), \qquad \Delta t f_i(t) \sum_{k=1}^{n} \eta_{ik} f_k(t) + o(\Delta t).$$

The variation $f_i(t + \Delta t) - f_i(t)$ of the number of active particles expressing the strategy u_i can then be equated with the net balance of these two terms, which, in the limit $\Delta t \to 0^+$, leads to the equation:

$$\frac{df_i}{dt} = \sum_{h,\,k=1}^{n} \eta_{hk}\mathcal{B}_{hk}(i) f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k, \quad i = 1, \ldots, n. \tag{2}$$

2.1. **Games with multiple strategies.** The mathematical framework just presented does not account for a possible interplay between different interaction dynamics due to concomitant social and economic issues. To be more specific, and to anticipate the application we will be concerned with in the next section, we consider the case of a multiple strategy $(u,\, v)$, with u representing, like before, the wealth status of the active particles and v their level of support/opposition to a Government policy. We assume, for simplicity, that also v is a discrete variable taking values in a lattice

$$I_v = \{v_1, \ldots, v_r, \ldots, v_m\},$$

each v_r standing for an *opinion class* about the Government's doings. It is plain that, in order to obtain an accurate picture of the interconnected socio-economic dynamics, the description of game-type interactions has to take into account that each active particle now plays with the multiple strategy (u_i, v_r). To this end, Eq. (2) can be technically generalized as:

$$\frac{df_i^r}{dt} = \sum_{p,\,q=1}^{m} \sum_{h,\,k=1}^{n} \eta_{hk}^{pq} \mathcal{B}_{hk}^{pq}(i,\,r) f_h^p f_k^q - f_i^r \sum_{q=1}^{m} \sum_{k=1}^{n} \eta_{ik}^{rq} f_k^q, \tag{3}$$

for $i = 1, \ldots, n$ and $r = 1, \ldots, m$, where

$$f_i^r = f_i^r(t) : [0,\, T_{\max}] \to \mathbb{R}_+$$

is the number of individuals that, at time t, are expressing the strategy (u_i, v_r), or, in other words, that belong to the wealth class u_i and to the opinion class v_r. In addition, η_{hk}^{pq} is the interaction rate between candidate particles with strategy (u_h, v_p) and field particles with strategy (u_k, v_q), while $\mathcal{B}_{hk}^{pq}(i, r)$ is the probability that a candidate particle changes its strategy to the test one (u_i, v_r) after interacting with a field particle. Analogously to Eq. (1), it is required to fulfill the probability density property:

$$\sum_{r=1}^{m} \sum_{i=1}^{n} \mathcal{B}_{hk}^{pq}(i,\,r) = 1, \quad \forall\, h,\, k = 1, \ldots, n, \quad \forall\, p,\, q = 1, \ldots, m.$$

3. **On the interplay between socio-economic dynamics and political conflicts.** The mathematical frameworks developed in the previous section provide a background for tackling some illustrative applications concerned with social competition problems. Particularly, our main interest here lies in phenomena such as unbalanced wealth distributions possibly leading to strong dissensus against Governments, which can be classified as a Black Swan.

The contents of the following subsections pursue this goal through three steps. The first one is the modeling of welfare dynamics in terms of cooperation and competition among economic classes. The second one is then the modeling of dynamics of support and opposition to a certain regime triggered by the welfare distribution. Finally, the third one is a preliminary approach to the identification of premonitory signals possibly implying the onset of a Black Swan, here understood as an exceptional growth of opposition to the regime fostered by the synergy with socio-economics dynamics.

In order to describe the above-mentioned cooperation/competition interactions, we adopt the following qualitative paradigm of consensus/dissensus dynamics:

- *Consensus* – The candidate particle sees its state either increased, by profiting from a field particle with a higher state, or decreased, by pandering to a field particle with a lower state. After mutual interaction, the states of the particles get closer than before the interaction.
- *Dissensus* – The candidate particle sees its state either further decreased, by facing a field particle with a higher state, or further increased, by facing a field particle with a lower state. After mutual interaction, the states of the particles get farther than before the interaction.

Once formalized at a quantitative level, this paradigm can act as a basis for proposing the transition probabilities introduced in Section 2.

3.1. **Modeling socio-economic competition.** In this section we consider the modeling of socio-economic interactions based on the previously discussed consensus/dissensus dynamics. This problem has been first addressed in [15] for a large community of individuals divided into different social classes. The model proposed

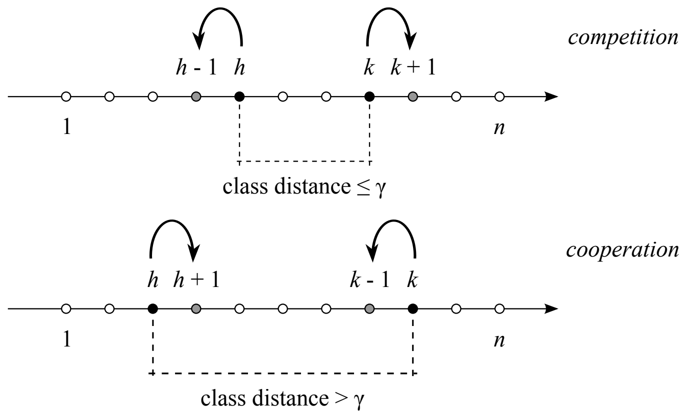

FIGURE 1. Dynamics of competition (top) and cooperation (bottom) between pairs of candidate (index h) and field (index k) active particles. The critical distance γ, which triggers either behavior depending on the actual distance between the interacting classes, may evolve in time according to the global evolution of the system.

there introduces a *critical distance*, which triggers either cooperation or competition among classes. In more detail, if the actual distance between the interacting classes is lower than the critical one then a competition takes place, which causes a further enrichment of the wealthier class and a further impoverishment of the poorer one. Conversely, if the actual distance is greater than the critical one then the social organization forces cooperation, namely the richer class has to contribute to the wealth of the poorer one.

In the above-cited paper, linearly additive interactions are used along with a constant critical distance. Such an approach is here revisited by introducing nonlinearly additive interactions, particularly by considering a critical distance which evolves in time depending on the global wealth distribution. In more detail, the characteristics of the present framework are summarized as follows:

- *Interaction rate.* Two different rates of interaction are considered, corresponding to competitive and cooperative interactions, respectively.
- *Strategy leading to the transition probabilities.* When interacting with other particles, each active particle plays a game with stochastic output. If the difference of wealth class between the interacting particles is lower than a critical distance γ then the particles compete in such a way that those with higher wealth increase their state against those with lower wealth. Conversely, if the difference of wealth class is higher than γ then the opposite occurs (see Fig. 1). The critical distance evolves in time according to the global wealth distribution over wealthy and poor particles.

Remark 1. According to the assumptions above, the threshold separating the two types of games depends on the probability distribution $\mathbf{f}$: $\gamma = \gamma[\mathbf{f}]$. As we will see in a moment, this implies that also the interaction rate and the transition probability density depend on $\mathbf{f}$ (through γ), which gives rise to *nonlinearly additive interactions*: $\eta_{hk} = \eta_{hk}[\mathbf{f}]$, $\mathcal{B}_{hk}(i) = \mathcal{B}_{hk}(i)[\mathbf{f}]$. However, in order to avoid cumbersome notations in the equations, we will henceforth systematically refrain from writing the explicit dependence of these two quantities on $\mathbf{f}$.

The reference mathematical structure is Eq. (2). Among the possible choices, we select a uniformly spaced wealth grid in the interval $[-1, 1]$ with odd number n of classes:

$$I_u = \{u_1 = -1, \ldots, u_{\frac{n+1}{2}} = 0, \ldots, u_n = 1\},$$
$$u_i = \frac{2}{n-1}i - \frac{n+1}{n-1}, \quad i = 1, \ldots, n, \tag{4}$$

agreeing that $u_i < 0$ identifies a poor class whereas $u_i > 0$ a wealthy one.

We next assume that the **interaction rate** $\eta_{hk} \geq 0$ is piecewise constant over the wealth classes:

$$\eta_{hk} = \begin{cases} \eta_0 & \text{if } |k-h| \leq \gamma[\mathbf{f}] \text{ (competition)}, \\ \mu\eta_0 & \text{if } |k-h| > \gamma[\mathbf{f}] \text{ (cooperation)}, \end{cases} \tag{5}$$

where $\eta_0 > 0$ is a constant to be hidden in the time scale and $0 < \mu \leq 1$.

The **transition probabilities** $\mathcal{B}_{hk}(i) \in [0, 1]$ are required to satisfy Eq. (1) plus an additional condition ensuring the conservation of the average wealth status of the population:

$$\sum_{i=1}^{n} u_i f_i(t) = \text{constant}, \quad \forall\, t \geq 0. \tag{6}$$

This means that the interaction dynamics causes globally neither production nor loss of wealth, but simply its redistribution among the classes. We will denote by U_0 the average wealth status as fixed at the initial time:

$$U_0 := \sum_{i=1}^{n} u_i f_i(0).$$

By computing on Eq. (2), it turns out that sufficient conditions for the fulfillment of (6) are:

- symmetric encounter rate, i.e., $\eta_{hk} = \eta_{kh}$, $\forall\, h, k = 1, \ldots, n$;
- transition probabilities such that

$$\sum_{i=1}^{n} u_i \mathcal{B}_{hk}(i) = u_h + \sigma_{hk}, \qquad \forall\, h, k = 1, \ldots, n, \tag{7}$$

  where σ_{hk} is an antisymmetric tensor, i.e., $\sigma_{hk} = -\sigma_{kh}$, $\forall\, h, k = 1, \ldots, n$.

Notice that the encounter rate (5) is indeed symmetric. In order to explain condition (7), let us consider preliminarily the particular case $\sigma_{hk} = 0$ for all h, k. Then (7) reduces to $\sum_{i=1}^{n} u_i \mathcal{B}_{hk}(i) = u_h$, which says that the expected payoff of a candidate particle after an interaction coincides with its wealth class before the interaction. Namely, interactions do not cause, in average, either enrichment or impoverishment, pretty much like a fair game. In the general case, however, Eq. (7) allows for fluctuations of the expected payoff over the pre-interaction wealth class, such that they globally balance: $\sum_{h,k=1}^{n} \sigma_{hk} = 0$.

Transition probabilities describing cooperation/competition according to the distance between the interacting classes, and complying with Eqs. (1), (6), can be inspired by those proposed in [15], with suitable modification also related to the introduction of nonlinear interactions. We need to distinguish three cases:

1. Interaction between equally wealthy particles, viz. $h = k$:

$$\begin{cases} \mathcal{B}_{hh}(h) = 1 \\ \mathcal{B}_{hh}(i) = 0, \ \forall\, i \neq h \end{cases} \tag{8}$$

2. Competitive interaction, viz. $h \neq k$ and $|h-k| \leq \gamma[\mathbf{f}]$:

$$h = 1,\, n \begin{cases} \mathcal{B}_{hk}(h) = 1 \\ \mathcal{B}_{hk}(i) = 0, \ \forall\, i \neq h \end{cases}$$

$$h \neq 1,\, n \begin{cases} h < k \begin{cases} k < n \begin{cases} \mathcal{B}_{hk}(h-1) = \alpha_{hk} \\ \mathcal{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathcal{B}_{hk}(i) = 0, \ \forall\, i \neq h-1,\, h \end{cases} \\ k = n \begin{cases} \mathcal{B}_{hn}(h) = 1 \\ \mathcal{B}_{hn}(i) = 0, \ \forall\, i \neq h \end{cases} \end{cases} \\ h > k \begin{cases} k = 1 \begin{cases} \mathcal{B}_{h1}(h) = 1 \\ \mathcal{B}_{h1}(i) = 0, \ \forall\, i \neq h \end{cases} \\ k > 1 \begin{cases} \mathcal{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathcal{B}_{hk}(h+1) = \alpha_{hk} \\ \mathcal{B}_{hk}(i) = 0, \ \forall\, i \neq h,\, h+1 \end{cases} \end{cases} \end{cases} \tag{9}$$

3. Cooperative interaction, viz. $h \neq k$ and $|h-k| > \gamma[\mathbf{f}]$:

$$\begin{aligned} & h < k \begin{cases} \mathcal{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathcal{B}_{hk}(h+1) = \alpha_{hk} \\ \mathcal{B}_{hk}(i) = 0, \ \forall\, i \neq h,\, h+1 \end{cases} \\ & h > k \begin{cases} \mathcal{B}_{hk}(h-1) = \alpha_{hk} \\ \mathcal{B}_{hk}(h) = 1 - \alpha_{hk} \\ \mathcal{B}_{hk}(i) = 0, \ \forall\, i \neq h-1,\, h, \end{cases} \end{aligned} \tag{10}$$

where it is assumed that interactions within the same class produce no effect.

The parameter $\alpha_{hk} \in [0,\, 1]$ appearing in Eqs. (9), (10) has the following meaning:

- in case of competition, cf. Eq. (9), it is the probability that the candidate particle further increases or decreases its wealth if it is, respectively, richer or poorer than the field particle;
- in case of cooperation, cf. Eq. (10), it is the probability that the candidate particle gains or transfers part of its wealth if it is, respectively, poorer or richer than the field particle.

This probability may be constant, like in the already cited work [15], or, as we will assume throughout the remaining part of this paper, may depend on the wealth classes e.g.,

$$\alpha_{hk} = \frac{|k-h|}{n-1}, \tag{11}$$

in such a way that the larger the distance between the interacting classes the more stressed the effect of cooperation or competition. Any proportionality constant can be transferred into a scaling of the time variable.

Remark 2. For wealth conservation purposes, the transition probabilities (8)–(10) are such that the extreme classes never take part nor trigger social competition. Namely, a candidate particle in the class $h = 1$ or $h = n$ can only stay in the same class after any interaction with whatever field particle. Correspondingly, a field particle in the class $k = 1$ or $k = n$ can only cause a candidate particle to remain in its pre-interaction class, no matter what the latter is.

The **critical distance** $\gamma[\mathbf{f}]$, taken constant in [15], is here assumed to depend on the instantaneous distribution of the active particles over the wealth classes. In more detail, we conceive the time evolution of $\gamma[\mathbf{f}]$ so as to translate the following (heuristic) phenomenology of social competition:

- in general, $\gamma[\mathbf{f}]$ grows with the number of poor active particles, thus causing larger and larger gaps of social competition. Few wealthy active particles insist on maintaining, and possibly improving, their benefits;
- in a population constituted almost exclusively by poor active particles $\gamma[\mathbf{f}]$ attains a value such that cooperation is inhibited, for individuals tend to be involved in a "battle of the have-nots";
- conversely, in a population constituted almost exclusively by wealthy active particles $\gamma[\mathbf{f}]$ attains a value such that competition is inhibited, because individuals tend preferentially to cooperate for preserving their common benefits.

Bearing these ideas in mind, we introduce the number of poor and wealthy active particles at time t:

$$N^-(t) = \sum_{i=1}^{\frac{n-1}{2}} f_i(t), \qquad N^+(t) = \sum_{i=\frac{n+3}{2}}^{n} f_i(t).$$

Notice that, by excluding the middle class $u_{\frac{n+1}{2}} = 0$ from both N^- and N^+, we implicitly regard it as economically "neutral". Up to normalization over the total number of active particles, we have $0 \leq N^\pm \leq 1$ with also $N^- + N^+ \leq 1$, hence the quantity

$$S[\mathbf{f}] := N^- - N^+,$$

which provides a macroscopic measure of the *social gap* in the population, is bounded between -1 and 1. Given that, we now look for a quadratic polynomial dependence of $\gamma[\mathbf{f}]$ on $S[\mathbf{f}]$ taking into account the following conditions, which bring to a quantitative level the previous qualitative arguments:

- $S[\mathbf{f}] = S_0 \Rightarrow \gamma[\mathbf{f}] = \gamma_0$, where S_0, γ_0 are a reference social gap and the corresponding reference critical distance, respectively;
- $S[\mathbf{f}] = 1 \Rightarrow \gamma[\mathbf{f}] = n$, which implies that when the population is composed by poor particles only ($N^- = 1$, $N^+ = 0$) the socio-economic dynamics are of full competition;
- $S[\mathbf{f}] = -1 \Rightarrow \gamma[\mathbf{f}] = 0$, which implies that, conversely, when the population is composed by wealthy particles only ($N^- = 0$, $N^+ = 1$) the socio-economic dynamics are of full cooperation.

Considering further that only integer values of $\gamma[\mathbf{f}]$ are meaningful, for so are the distances between pairs of indexes of wealth classes, the resulting analytical expression of $\gamma[\mathbf{f}]$ turns out to be

$$\gamma[\mathbf{f}] = \left\lfloor \frac{2\gamma_0(S[\mathbf{f}]^2 - 1) - n(S_0 + 1)(S[\mathbf{f}]^2 - S_0)}{2(S_0^2 - 1)} + \frac{n}{2} S[\mathbf{f}] \right\rfloor, \tag{12}$$

where $\lfloor \cdot \rfloor$ denotes integer part. In particular, if the reference social gap is taken to be $S_0 = 0$ (i.e., when $N^- = N^+$) then the expression of $\gamma[\mathbf{f}]$ specializes as (see Fig. 2)

$$\gamma[\mathbf{f}] = \left\lfloor \frac{n - 2\gamma_0}{2} S[\mathbf{f}]^2 + \frac{n}{2} S[\mathbf{f}] + \gamma_0 \right\rfloor. \tag{13}$$

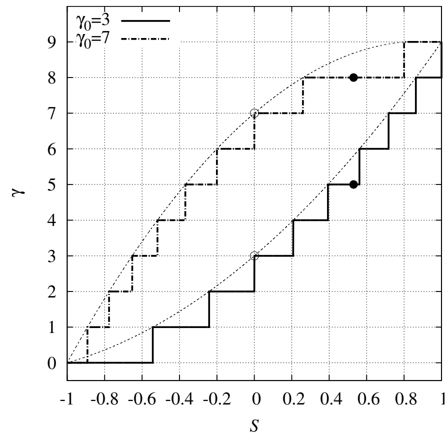

FIGURE 2. The critical distance $\gamma[\mathbf{f}]$ vs. the macroscopic social gap $S[\mathbf{f}]$ as in Eq. (13) for the two cases $\gamma_0 = 3, 7$ and $n = 9$ social classes. Empty bullets indicate the reference value γ_0 corresponding to the reference social gap $S_0 = 0$. Filled bullets indicate instead the actual initial critical distance $\gamma[\mathbf{f}](t = 0)$ corresponding to the actual initial social gap $S[\mathbf{f}](t = 0)$ for the case study addressed in Fig. 3. Dotted lines, drawing the parabolic profile of function (13) without integer part, are plotted for visual reference.

The evolution of the system predicted by the model depends essentially on the four parameters n (the number of wealth classes), μ (the relative encounter rate for cooperation, cf. Eq. (5)), U_0 (the average wealth of the population), and γ_0 (the reference critical distance). The next simulations aim at exploring some aspects of the role that they play on the asymptotic configurations of the system. In more detail:

- $n = 9$ and $\mu = 0.3$ are selected;
- two case studies for U_0 are addressed, namely $U_0 = -0.4 < 0$ and $U_0 = 0$, in order to compare, respectively, the economic dynamics of a society in which poor classes dominate with those of a society in which the initial distribution of active particles encompasses uniformly poor and wealthy classes;
- in addition, in each of the case studies above the asymptotic configurations for both constant and variable $\gamma[\mathbf{f}]$ are investigated, assuming, for duly comparison, that in the former the critical distance coincides with γ_0. Particularly, $\gamma_0 = 3$, corresponding to a mainly cooperative attitude, and $\gamma_0 = 7$, corresponding instead to a strongly competitive attitude, are chosen.

Figure 3 illustrates the asymptotic configurations in case of negative average wealth status U_0 (poor society). The model predicts, in general, a consolidation of the poorest classes. Nevertheless, in a basically cooperative framework ($\gamma_0 = 3$) a certain redistribution of part of the wealth is observed, which for constant γ involves moderately poor and moderately rich classes whereas for variable $\gamma[\mathbf{f}]$ further stresses the difference between the poorest and the wealthiest classes. In fact, imposing a constant critical distance coinciding with the reference value γ_0 corresponds to forcing the society to behave as if the social gap were $S[\mathbf{f}] \equiv S_0 = 0$. On the other hand, the spontaneous attitude of the modeled society, in which

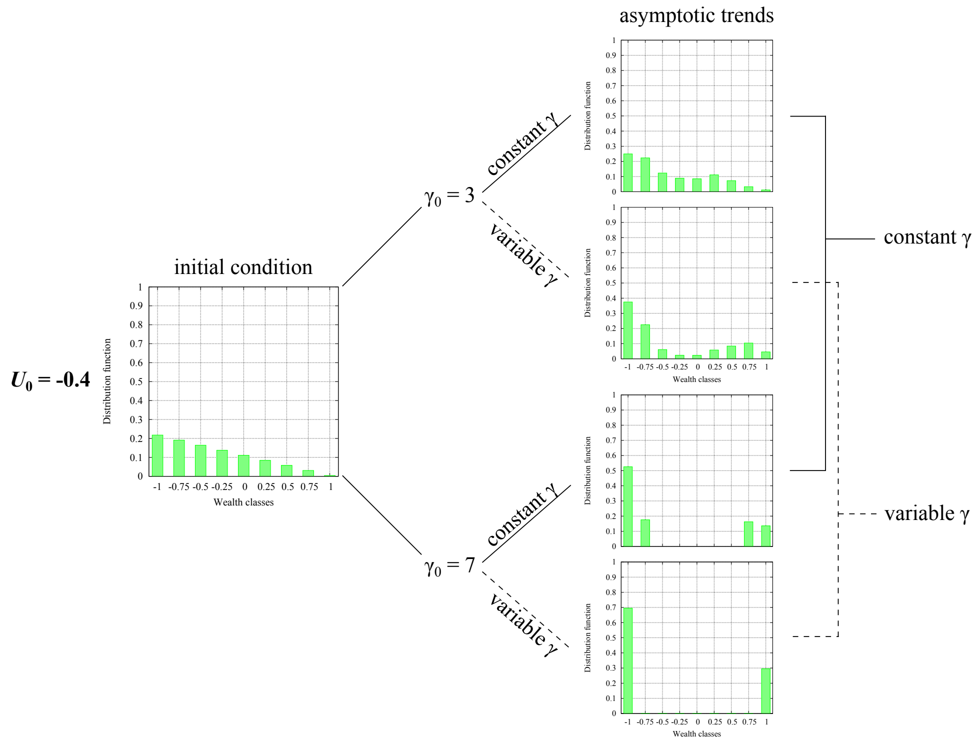

FIGURE 3. Asymptotic distributions of active particles over wealth classes for $U_0 = -0.4$.

the actual initial social gap computed from the given initial condition is $S[\mathbf{f}](t = 0) = \frac{8}{15} \approx 0.53 > 0$, is much more competitive than that implied by $\gamma_0 = 3$, as Fig. 2 demonstrates. Analogous considerations can be repeated in a competitive framework ($\gamma_0 = 7$). Now the tendency is a strong concentration in the extreme classes, which in particular results in the consolidation of oligarchic wealthy classes which were nearly absent at the beginning.

Figure 4 illustrates instead the asymptotic trend in case of null average wealth status (economically "neutral" society). In this case there is no difference between the asymptotic configurations reached under constant and variable $\gamma[\mathbf{f}]$. Indeed, the initial symmetry of the distribution about the intermediate class $u_5 = 0$, which ensures $U_0 = 0$ and is preserved during the subsequent evolution, forces $S[\mathbf{f}] \equiv 0$, hence $\gamma[\mathbf{f}] \equiv \gamma_0$, at all later times. The stationary configurations show a quite intuitive progressive clustering of the population in the extreme classes as the level of competition increases from $\gamma_0 = 3$ to $\gamma_0 = 7$.

As a general concluding remark, we notice that, even with a variable critical distance, none of the asymptotic configurations of the system seems to be properly identifiable as a Black Swan. On the other hand, the simple case studies addressed in this subsection are preliminary to the contents of the next subsection, which will focus on the joint effect of socio-economic and political opinion dynamics. It is from the complex interplay between these two social aspects that Black Swans are mostly expected to arise.

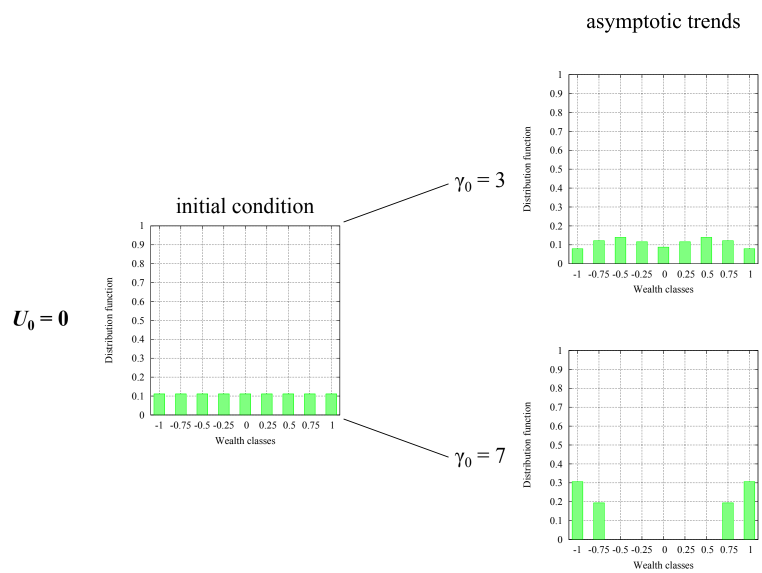

FIGURE 4. Asymptotic distributions of active particles over wealth classes for $U_0 = 0$.

3.2. **Modeling support/opposition to a Government.** In this section we investigate how the welfare dynamics considered in Section 3.1 can induce changes of personal opinions in terms of support/opposition to a certain political regime. In doing so, we will keep in mind recent results in the literature of Social Sciences; see for instance [2, 3, 6].

The mathematical structures to be used are those presented in Section 2.1, in particular the additional variable v represents here the attitude of the individuals to the Government. It is customary to use also for v the uniformly spaced lattice

$$I_v = \{v_1 = -1, \ldots, v_{\frac{m+1}{2}} = 0, \ldots, v_m = 1\},$$

$$v_r = \frac{2}{m-1} r - \frac{m+1}{m-1}, \quad r = 1, \ldots, m,$$

agreeing that $v_1 = -1$ corresponds to the strongest opposition whereas $v_m = 1$ to the maximum support.

Mathematical models based on Eq. (3) are obtained by prescribing the interaction rate η_{hk}^{pq} and the transition probabilities $\mathcal{B}_{hk}^{pq}(i, r)$. A very simple approach is proposed here, deferring to the next section a discussion on possible improvements.

For the **interaction rate** the same model given by Eq. (5) is used, according to the assumption that interactions among active particle are mainly driven by the wealth status rather than by the difference of political opinion. Thus η_{hk}^{pq} is independent of the opinion classes that candidate and field particles belong to, $\eta_{hk}^{pq} = \eta_{hk}$.

Notice that this amounts, in a way, to disregarding political persuasion dynamics. The model could be made more precise, for instance, by allowing the interaction rate to depend on the proximity of political point of view of the interacting particles.

For the **transition probabilities** the following factorization on the output test state $(u_i,\, v_r)$ is proposed, relying simply on intuition:

$$\mathcal{B}^{pq}_{hk}(i,\, r) = \bar{\mathcal{B}}^{pq}_{hk}(i)\hat{\mathcal{B}}^{pq}_{hk}(r),$$

where:

- $\bar{\mathcal{B}}^{pq}_{hk}(i)$ encodes the transitions of wealth class, which are further supposed to be independent of the political feelings of the interacting pairs: $\bar{\mathcal{B}}^{pq}_{hk}(i) = \bar{\mathcal{B}}_{hk}(i)$. For this term the structure given by Eqs. (8)–(10) is used;
- $\hat{\mathcal{B}}^{pq}_{hk}(r)$ encodes the changes of political opinion resulting from interactions. Coherently with the observation made above that political persuasion is neglected, so that political feelings originate in the individuals in consequence of their own wealth condition, this term is assumed to depend on the wealth status and political opinion of the candidate particle only: $\hat{\mathcal{B}}^{pq}_{hk}(r) = \hat{\mathcal{B}}^{p}_{h}(r)$.

In view of the special structure

$$\mathcal{B}^{pq}_{hk}(i,\, r) = \bar{\mathcal{B}}_{hk}(i)\hat{\mathcal{B}}^{p}_{h}(r),$$

it turns out that sufficient conditions ensuring the conservation in time of both the total number of active particles and the average wealth status of the system are:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \bar{\mathcal{B}}_{hk}(i) = 1, & \forall\, h,\, k = 1,\, \dots,\, n \\ \displaystyle\sum_{i=1}^{n} u_i\bar{\mathcal{B}}_{hk}(i) = u_h + \sigma_{hk}, & \forall\, h,\, k = 1,\, \dots,\, n,\ \sigma_{hk} \text{ antisymmetric} \\ \displaystyle\sum_{r=1}^{m} \hat{\mathcal{B}}^{p}_{h}(r) = 1, & \forall\, h = 1,\, \dots,\, n,\ \forall\, p = 1,\, \dots,\, m; \end{cases}$$

in particular, the first two statements are directly borrowed from Eqs. (8)–(10).

As far as the modeling of $\hat{\mathcal{B}}^{p}_{h}(r)$ is concerned, the following set of transition probabilities is proposed:

1. Poor individual ($u_h < 0$) in a poor society ($U_0 < 0$):

$$\begin{aligned} p = 1 &\begin{cases} \hat{\mathcal{B}}^{1}_{h}(1) = 1 \\ \hat{\mathcal{B}}^{1}_{h}(r) = 0,\ \forall\, r \neq 1 \end{cases} \\ p > 1 &\begin{cases} \hat{\mathcal{B}}^{p}_{h}(p-1) = 2\beta \\ \hat{\mathcal{B}}^{p}_{h}(p) = 1 - 2\beta \\ \hat{\mathcal{B}}^{p}_{h}(r) = 0,\ \forall\, r \neq p-1,\, p \end{cases} \end{aligned} \tag{14}$$

2. Wealthy individual ($u_h \geq 0$) in a poor society ($U_0 < 0$) or poor individual ($u_h < 0$) in a wealthy society ($U_0 \geq 0$):

$$
\begin{aligned}
p = 1 \quad & \begin{cases} \hat{\mathcal{B}}_h^1(1) = 1 - \beta \\ \hat{\mathcal{B}}_h^1(2) = \beta \\ \hat{\mathcal{B}}_h^1(r) = 0, \ \forall\, r \neq 1,\, 2 \end{cases} \\
p \neq 1,\, m & \begin{cases} \hat{\mathcal{B}}_h^p(p-1) = \beta \\ \hat{\mathcal{B}}_h^p(p) = 1 - 2\beta \\ \hat{\mathcal{B}}_h^p(p+1) = \beta \\ \hat{\mathcal{B}}_h^p(r) = 0, \ \forall\, r \neq p-1,\, p,\, p+1 \end{cases} \\
p = m \quad & \begin{cases} \hat{\mathcal{B}}_h^m(m-1) = \beta \\ \hat{\mathcal{B}}_h^m(m) = 1 - \beta \\ \hat{\mathcal{B}}_h^m(r) = 0, \ \forall\, r \neq m-1,\, m \end{cases}
\end{aligned}
\tag{15}
$$

3. Wealthy individual ($u_h \geq 0$) in a wealthy society ($U_0 \geq 0$):

$$
\begin{aligned}
p < m & \begin{cases} \hat{\mathcal{B}}_h^p(p) = 1 - 2\beta \\ \hat{\mathcal{B}}_h^p(p+1) = 2\beta \\ \hat{\mathcal{B}}_h^p(r) = 0, \ \forall\, r \neq p,\, p+1 \end{cases} \\
p = m & \begin{cases} \hat{\mathcal{B}}_h^m(m) = 1 \\ \hat{\mathcal{B}}_h^m(r) = 0 \ \forall\, r \neq m, \end{cases}
\end{aligned}
\tag{16}
$$

where $\beta \in [0,\, \frac{1}{2}]$ is a parameter expressing the basic probability of changing political opinion. According to Eqs. (14)–(16), changes of political opinion are triggered jointly by the individual wealth status of the candidate particle and the average collective one of the population, in such a way that:

- poor individuals in a poor society tend to distrust markedly the Government policy, sticking in the limit at the strongest opposition;
- wealthy individuals in a poor society and poor individuals in a wealthy society exhibit, in general, the most random behavior. In fact, they may trust the Government policy either because of their own wealthiness, regardless of the possibly poor general condition, or because of the collective affluence, in spite of their own poor economic status. On the other hand, they may also distrust the Government policy either because of the poor general condition, in spite of their individual wealthiness, or because of their own poor economic status, regardless of the collective affluence;
- wealthy individuals in a wealthy society tend instead to trust earnestly the Government policy, sticking in the limit at the maximum support.

Leaving asidea number of possible refinements of the model, some preliminary numerical simulations can be developed toward the main target of this paper. Specifically, we consider again the two cases corresponding to an economically neutral ($U_0 = 0$) and a poor ($U_0 = -0.4 < 0$) society, assuming that the political feelings are initially uniformly distributed within the various wealth classes. The relevant parameters related to welfare dynamics are set like in Section 3.1. Additionally, the basic probability of changing political orientation is set to $\beta = 0.4$, and $m = 9$ opinion classes are selected corresponding to as many levels of political support/opposition.

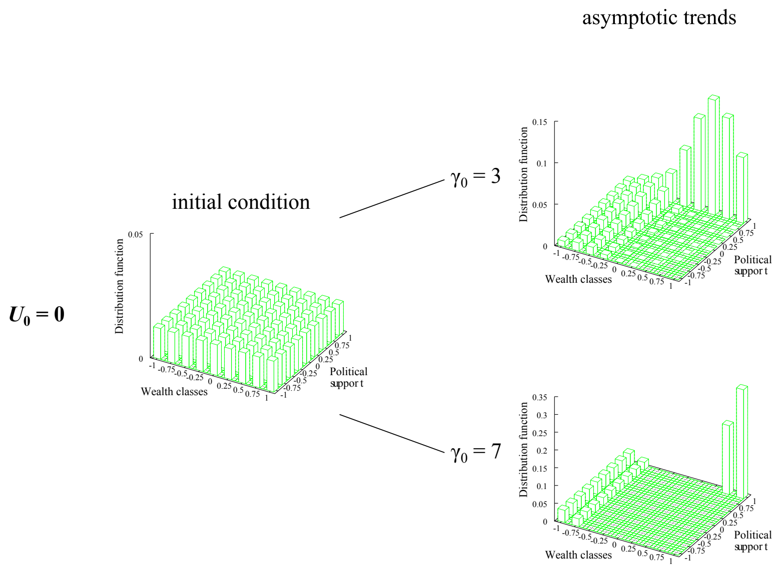

FIGURE 5. Asymptotic distributions of active particles over wealth classes and political orientation for $U_0 = 0$.

Figure 5, referring to the case $U_0 = 0$, shows that in an economically neutral society with uniform wealth distribution not only do wealthy classes stick at an earnest support to the Government policy, but also poor ones do not completely distrust them, especially in a context of prevalent cooperation among the classes ($\gamma_0 = 3$). Therefore, this example does not suggest the development of significant polarization in that society. On the other hand, Figure 6, corresponding to the case $U_0 = -0.4$, clearly shows a strong radicalization of the opposition. The model predicts indeed that, in such a poor society, poor classes stick asymptotically at the strongest opposition, whereas wealthy classes spread over the whole range of political orientations, however with a mild tendency toward opposition for the moderately rich ones (say, $u_5 = 0$, $u_6 = 0.25$, and $u_7 = 0.5$). The growth of political aversion is especially emphasized under variable critical distance $\gamma[\mathbf{f}]$, when the marked clustering of the population in the lowest wealth classes, due to a more competitive spontaneous attitude, entails in turn a clustering in the highest distrust of the government.

Remark 3. The case studies discussed above indicate that an effective interpretation of the social phenomena under consideration requires a careful examination of the probability distribution over the microscopic states. Indeed, Figs. 5, 6 show entirely different scenarios, that might not be completely caught simply by average macroscopic quantities.

3.3. **Looking for early signals of the Black Swan.** The simulations presented in the preceding sections have put in evidence that an unfair welfare distribution

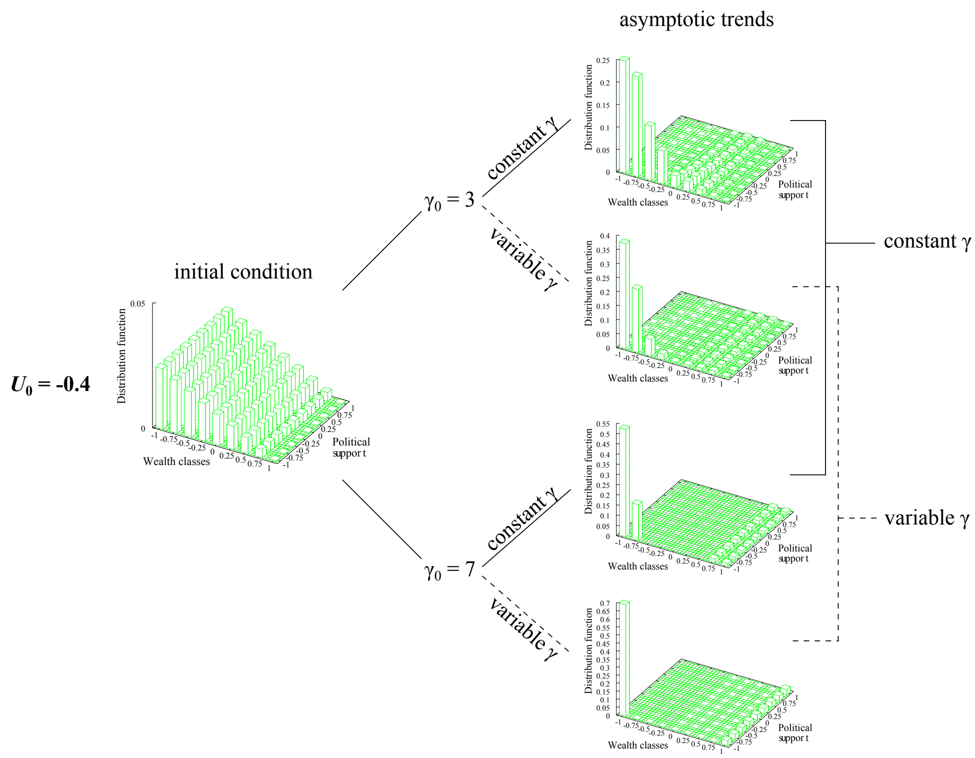

FIGURE 6. Asymptotic distributions of active particles over wealth classes and political orientation for $U_0 = -0.4$.

can cause a radical surge of opposition to the government. Therefore, it is of some practical interest to look for early signals that may precede the occurrence of such a situation.

To begin with, it is worth detailing a little more the expression *Black Swan*, introduced in the specialized literature to describe unpredictable events, which are far away from those generally observed by repeated empirical evidence. In [29], a Black Swan is specifically characterized as follows:

> *"A Black Swan is a highly improbable event with three principal characteristics: It is unpredictable; it carries a massive impact; and, after the fact, we concoct an explanation that makes it appear less random, and more predictable, than it was."*

and a critical analysis is developed about the failure of the existing mathematical approaches to address such situations. In the author's opinion, this is due to the fact that mathematical models usually rely on what is already known, thus failing to predict what is instead unknown. It is worth observing that [29] is a rare example of research moving against the main stream of the traditional approaches, generally focused on well-predictable events. The book [29] had an important impact on the search for new research perspectives: for instance, it motivated applied mathematicians and other scholars to propose formal approaches to study the Black Swan, in an attempt to forecast conditions for its onset. Concerning this, it is worth stressing that individual behavioral rules and strategies are not, in most cases, constant in time due to the evolutionary characteristics of living complex system. Particularly, some parameters of the models, related to the interactions among individuals, can change in time depending on the global state of the system. Such a variability can actually generate unpredictable events. In addition, it is plain that the mathematical search for the Black Swan can hardly rely on a purely macroscopic viewpoint. On the other hand, early signals of upcoming extreme events can be profitably sought at a macroscopic level, in order for them to be observable, hence recognizable, in practice.

Bearing in mind the previous remarks, we now provide some suggestions for the possible detection of a Black Swan within our current mathematical framework.

Let us assume that a specific model, derived from the mathematical structures presented in Section 2, is expected to exhibit a stationary trend described by some phenomenologically guessed distribution $\tilde{\mathbf{f}} = \{\tilde{f}_i^r\}_{\substack{i=1,\dots,n \\ r=1,\dots,m}}$ (which, in principle, has to be determined heuristically for each specific case study as we will see in the following). Then we define the following time-evolving distance d_{BS} (the subscript "BS" standing for Black Swan):

$$d_{\mathrm{BS}}(t) := \max_{r=1,\dots,m} \|\tilde{f}_\cdot^r - f_\cdot^r(t)\|, \tag{17}$$

where $\|\cdot\|$ is a suitable norm in $\mathbb{R}^n$ over the activity $u \in I_u$, for instance the ℓ^1-norm:

$$\|f_\cdot^r(t)\|_1 = \sum_{i=1}^{n} |f_i^r(t)|. \tag{18}$$

The quantity d_{BS} will generally not approach zero as $t \to +\infty$, because the heuristic asymptotic distribution $\tilde{\mathbf{f}}$ does not translate the actual trend of the system for large times. Using the terminology introduced in [25], the mapping $t \mapsto d_{\mathrm{BS}}(t)$ can be possibly regarded as one of the *early-warning signals* for the emergence of critical transitions to rare events, because it may highlight the onset of strong deviations from expectations.

Remark 4. Alternative metrics to (18) can also be introduced depending on the phenomenology of the system at hand, which may need, for instance, either uniform or averaged ways of measuring the distance between different configurations. In addition, specific applications may suggest other early-warning mappings different from (17). For instance, linear or quadratic moments might be taken into account.

It is interesting to examine the time evolution of the distance d_{BS} with reference to the case study $U_0 = -0.4$ with variable critical distance addressed in Section 3.2. A meaningful choice of the expected asymptotic distribution is, for both $\gamma_0 = 3$ and $\gamma_0 = 7$, the one resulting from the corresponding dynamics with constant critical distance. Figure 7 shows the (common) qualitative trends of the mapping $t \mapsto d_{\mathrm{BS}}(t)$: an initial decrease of the distance, which may suggest a convergence to the guessed distribution, is then followed by a sudden increase (notice the singular point in the graph of d_{BS}) toward a nonzero steady value, which ultimately indicates a deviation from the expected outcome. Such a turnround is possibly a macroscopic signal that a Black Swan is about to appear. However, in order to get a complete picture, the average gross information delivered by $d_{\mathrm{BS}}(t)$ has to be supplemented by the detailed knowledge of the probability distribution over the microscopic states, which is the only one able to properly distinguish between lower ($\gamma_0 = 3$) and higher ($\gamma_0 = 7$) radicalization of political feelings when welfare dynamics are ruled by a variable critical threshold $\gamma[\mathbf{f}]$.

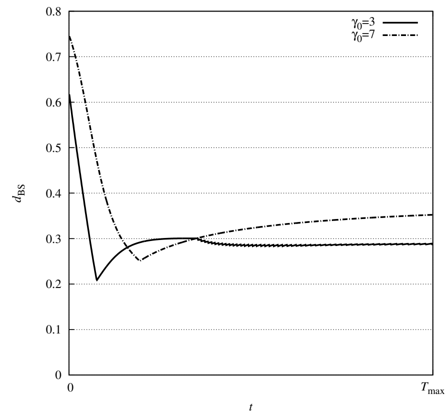

FIGURE 7. The mapping $t \mapsto d_{BS}(t)$ computed in the case studies with variable γ illustrated in Fig. 6, taking as phenomenological guess the corresponding asymptotic distributions obtained with constant γ.

4. **Critical analysis.** In this paper we have considered the problem of modeling complex systems of interacting individuals, focusing in particular on the ability of the models to predict the onset of rare events that cannot be generally foreseen on the basis of past empirical evidences. The results presented in the preceding sections are encouraging, yet a critical analysis is necessary in order to understand how far we still are from the challenging goal of devising suitable mathematical tools for studying the emergence of highly improbable events. We feel confident that a first step in such a direction has been made in this paper. On the other hand, we do not naively pretend that the ultimate target has been met. With the aim of contributing to further improvements, we propose in the following some considerations about specific problems selected according to our scientific bias. Hopefully, this selection addresses key issues of the theory.

Mathematical tools for complex systems. The leading idea of the present paper is that the previous modeling approach to interconnected socio-economic and political systems, where individual behaviors can play a relevant role on the collective dynamics, needs to consider the latter as living complex systems. This implies characterizing them in terms of qualitative complexity issues proper of Social Sciences, that have then to be translated in the mathematical language. The mathematical tools presented in the preceding sections are potentially able to capture such issues, taking advantage of a procedure of complexity reduction in modeling heterogeneous behaviors and expression of strategies. Yet, no matter how promising this approach may appear, we are not pretending that it is sufficient "as-is" for chasing the Black Swan. Instead, models such as those described in Section 3 can provide a detailed analysis of events whose broad dynamics are rather well understood. Furthermore, simulations contribute to put in evidence the role of some key parameters and can indicate how to devise external actions in order to eventually obtain a specific behavior of the society under consideration.

Modeling interplays toward the Black Swan. Based on the preliminary results that we have obtained, we believe that rare events can only be generated by several concomitant causes. In this paper we have addressed the interplay between welfare dynamics and the level of consent/dissent to the policies of a Government, with the aim of detecting the onset of the opposition to a certain government. The latter can indeed be regarded as a possible prototype of a Black Swan.

Further generalizations of the model. The modeling approach can be generalized for instance by considering the case of open systems, in which external actions can significantly modify both individual and collective system dynamics. Other interesting applications concern the case of several interacting societies, for example the study of how and when a domino effect can arise. Even more challenging appears to be the generalization of the model to large social networks [11, 31]. Recent studies, among others [12, 24], indicate that the role and structure of the networks can act as additional inputs for determining the predominance of either cooperation or competition. However, exploring this issue requires a substantial development of the mathematical structures presented in this paper.

Analytical problems. The qualitative analysis of models herein presented rises interesting analytical problems. As a matter of fact, showing the existence and uniqueness of solutions to the initial value problem is not a difficult task, because one can exploit the conservation of the total number of individuals and of their average wealth status. For linearly additive interactions the proof can be obtained by a simple application of fixed point theorems in a suitable Banach space, see [7]; the generalization to nonlinearly additive interactions has been recently proposed in the already cited paper [9] for nonlinearities involving moments of the distribution function. Similar techniques apply also to the class of models presented in this paper. Furthermore, simulations suggest that, for a given initial condition, the system reaches a unique asymptotic configuration in quite a broad range of parameters, but the existing literature still lacks precise analytical results able to confirm or reject such a conjecture.

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REFERENCES

[1] W. B. Arthur, S. N. Durlauf and D. A. Lane, eds., "The Economy as an Evolving Complex System II," Studies in the Sciences of Complexity, Vol. XXVII, Addison-Wesley, 1997.

[2] D. Acemoglu and J. A. Robinson, "Economic Origins of Dictatorship and Democracy," Cambridge University Press, 2005.

[3] D. Acemoglu, D. Ticchi and A. Vindigni, *Emergence and persistence of inefficient states*, J. Eur. Econ. Assoc., **9** (2011), 177–208.

[4] G. Ajmone Marsan, N. Bellomo and M. Egidi, *Towards a mathematical theory of complex socio-economical systems by functional subsystems representation*, Kinet. Relat. Models, **1** (2008), 249–278.

[5] G. Ajmone Marsan, *On the modelling and simulation of the competition for a secession under media influence by active particles methods and functional subsystems decomposition*, Comput. Math. Appl., **57** (2009), 710–728.

[6] A. Alesina and E. Spolaore, *War, peace, and the size of countries*, J. Public Econ., **89** (2005), 1333–1354.

[7] L. Arlotti and N. Bellomo, *Solution of a new class of nonlinear kinetic models of population dynamics*, Appl. Math. Lett., **9** (1996), 65–70.

[8] L. Arlotti, N. Bellomo and E. De Angelis, *Generalized kinetic (Boltzmann) models: Mathematical structures and applications*, Math. Models Methods Appl. Sci., **12** (2002), 567–591.

[9] L. Arlotti, E. De Angelis, L. Fermo, M. Lachowicz and N. Bellomo, *On a class of integro-differential equations modeling complex systems with nonlinear interactions*, Appl. Math. Lett., **25** (2012), 490–495.

[10] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, *Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study*, Proc. Natl. Acad. Sci. USA, **105** (2008), 1232–1237.

[11] A.-L. Barabási, R. Albert and H. Jeong, *Mean-field theory for scale-free random networks*, Physica A, **272** (1999), 173–187.

[12] U. Bastolla, M. A. Fortuna, A. Pascual-García, A. Ferrera, B. Luque and J. Bascompte, *The architecture of mutualistic networks minimizes competition and increases biodiversity*, Nature, **458** (2009), 1018–1020.

[13] N. Bellomo and J. Soler, *On the mathematical theory of the dynamics of swarms viewed as complex systems*, Math. Models Methods Appl. Sci., **22** (2012), 1140006, 29 pp.

[14] A. Bellouquid, E. De Angelis and L. Fermo, *Towards the modeling of vehicular traffic as a complex system: A kinetic theory approach*, Math. Models Methods Appl. Sci., **22** (2012), 1140003, 35 pp.

[15] M. L. Bertotti and M. Delitala, *Conservation laws and asymptotic behavior of a model of social dynamics*, Nonlinear Anal. Real World Appl., **9** (2008), 183–196.

[16] M. L. Bertotti and M. Delitala, *On a discrete generalized kinetic approach for modelling persuader's influence in opinion formation processes*, Math. Comput. Modelling, **48** (2008), 1107–1121.

[17] C. F. Camerer, "Behavioral Game Theory: Experiments in Strategic Interaction," Princeton University Press, 2003.

[18] V. Coscia, L. Fermo and N. Bellomo, *On the mathematical theory of living systems II: The interplay between mathematics and system biology*, Comput. Math. Appl., **62** (2011), 3902–3911.

[19] E. Cristiani, P. Frasca and B. Piccoli, *Effects of anisotropic interactions on the structure of animal groups*, J. Math. Biol., **62** (2011), 569–588.

[20] M. Delitala, P. Pucci and M. C. Salvatori, *From methods of the mathematical kinetic theory for active particles to modeling virus mutations*, Math. Models Methods Appl. Sci., **21** (2011), 843–870.

[21] B. Düring, P. Markowich, J.-F. Pietschmann and M.-T. Wolfram, *Boltzmann and Fokker-Planck equations modelling opinion formation in the presence of strong leaders*, P. R. Soc. London Ser. A Math. Phys. Eng. Sci., **465** (2009), 3687–3708.

[22] D. Helbing, "Quantitative Sociodynamics. Stochastic Methods and Models of Social Interaction Processes," Second edition, Springer, Heidelberg, 2010.

[23] J. C. Nuño, M. A. Herrero and M. Primicerio, *A mathematical model of criminal-prone society*, Discrete Contin. Dyn. Syst. Ser. S, **4** (2011), 193–207.

[24] D. G. Rand, S. Arbesman and N. A. Christakis, *Dynamic social networks promote cooperation in experiments with humans*, Proc. Natl. Acad. Sci. USA, **108** (2011), 19193–19198.

[25] M. Scheffer, J. Bascompte, W. A. Brock, V. Brovkin, S. R. Carpenter, V. Dakos, H. Held, E. H. van Nes, M. Rietkerk and G. Sugihara, *Early-warning signals for critical transitions*, Nature, **461** (2009), 53–59.

[26] H. A. Simon, *Theories of decision-making in economics and behavioral science*, Am. Econ. Rev., **49** (1959), 253–283.

[27] H. A. Simon, "Models of Bounded Rationality: Economic Analysis and Public Policy," Vol. 1, MIT Press, Cambridge, MA, 1982.

[28] H. A. Simon, "Models of Bounded Rationality: Empirically Grounded Economic Reason," Vol. 3, MIT Press, Cambridge, MA, 1997.

[29] N. N. Taleb, "The Black Swan: The Impact of the Highly Improbable," Random House, New York City, 2007.

[30] G. Toscani, *Kinetic models of opinion formation*, Commun. Math. Sci., **4** (2006), 481–496.

[31] F. Vega-Redondo, "Complex Social Networks," Econometric Society Monographs, **44**, Cambridge University Press, Cambridge, 2007.

[32] G. F. Webb, "Theory of Nonlinear Age-dependent Population Dynamics," Monographs and Textbooks in Pure and Applied Mathematics, **89**, Marcel Dekker, Inc., New York, 1985.

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