ELECTRONIC RESEARCH ANNOUNCEMENTS
IN MATHEMATICAL SCIENCES
Volume 20, Pages 51–54 (April 15, 2013)
S 1935-9179 AIMS (2013)
doi:10.3934/era.2013.20.51

NEW RESULTS ON FAT POINTS SCHEMES IN $\mathbb{P}^2$

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(Communicated by Igor Dolgachev)

Abstract. The purpose of this note is to announce two results, Theorem A and Theorem B below, concerning geometric and algebraic properties of fat points in the complex projective plane. Their somewhat technical proofs are available in [10] and will be published elsewhere. Here we present only main ideas which are fairly transparent.

1. Introduction

Fat points have been studied in algebraic geometry and commutative algebra for a long time, see, for example, [11], [4], [5], [14], [12]. Recently, they have come up naturally as a source of interesting examples in various aspects the comparison problem for symbolic and usual powers of homogeneous ideal, see for example [1], [2], [13], [9].

Recall that a *fat points scheme* $Z \subset \mathbb{P}^n$ is a scheme defined by an ideal of the form

$$I_Z = I(P_1)^{m_1} \cap \cdots \cap I(P_s)^{m_s}, \tag{1}$$

where $I(P)$ denotes the maximal ideal of a point $P \in \mathbb{P}^n$ and $s, m_1, \ldots, m_s$ are positive integers. To a fat point scheme Z as above one associates a linear series

$$\mathcal{L}_n(t; m_1, \ldots, m_s) \tag{2}$$

consisting of homogeneous polynomials of degree t having in points $P_1, \ldots, P_s$ multiplicities $m_1, \ldots, m_s$ respectively. We omit in the notation (2) the dimension n of the ambient space if it is clear from the context. Also if the points are general, then we write down only multiplicities. One of the classical problems in algebraic geometry is to determine the numbers

$$\dim \mathcal{L}(t; m_1, \ldots, m_s).$$

Working somewhat naively, one would expect that a linear series of the form (2) is empty whenever its *virtual dimension*

$$\operatorname{vdim} \mathcal{L}_n(t; m_1, \ldots, m_s) = \binom{n+t}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n} - 1$$

is negative. This is not quite true. In case of the projective plane this problem is governed by the Segre-Harbourne-Gimigliano-Hirschowitz conjecture, SHGH for short, see, e.g., [3, Section 4] for diverse variants and historical background.

Received by the editors February 6, 2013.
1991 *Mathematics Subject Classification.* Primary: 14C20; Secondary: 14F17.
Key words and phrases. vanishing, symbolic powers, fat points, postulation problems.
The second named author was partially supported by NCN grant UMO-2011/01/B/ST1/04875.

Conjecture A. [SHGH] *Let I be the ideal of a fat points scheme $Z \subset \mathbb{P}^2$ supported on general points. Assume that there exists a homogeneous polynomial of degree t in I. Then*

(i) *either*
$$h^1(\mathbb{P}^2, \mathcal{O}_{P^2}(t) \otimes I) = 0, \tag{3}$$
i.e., the linear system $\mathcal{L}$ of homogeneous polynomials of degree t vanishing along Z is non-special;

(ii) *or the base locus of $\mathcal{L}$ contains a multiple (-1)-curve.*

The next conjecture concerns algebraic properties of fat points ideals as defined in (1). Note that in this Conjecture the points need not to be general. However our partial solutions of the Conjecture deals with general points only.

Conjecture B. [Harbourne, Huneke] *Let I be a fat points scheme ideal in $\mathbb{P}^n$. Then for all $r \geqslant 1$ there is the containment*
$$I^{(nr)} \subset M^{r(n-1)} I^r, \tag{4}$$
where M denotes the irrelevant ideal in the ring $\mathbb{C}[\mathbb{P}^n]$.

These conjectures are closely related. Indeed, for an ideal I_Z as in (1), the elements in $I_Z^{(m)}$ are exactly those homogeneous polynomials which vanish to order $(m \cdot m_i)$ at each point P_i. Conjecture B provides immediate bounds on the degree of such polynomials and this gives in turn useful information towards the statements of Conjecture A. Similarly, in the opposite direction, vanishing postulated in (3) is useful towards proving (or disproving) containment claimed in (4).

2. Results

Conjecture A has attracted a lot of attention recently, and there are a number of results verifying it under additional assumptions. For example, the first author showed in [6] that SHGH Conjecture is true for fat points schemes with all multiplicities m_i equal and bounded from above by 42. Our vanishing stated in Theorem A below concerns an infinite family of fat points scheme supported in general points.

Theorem A. *Let $s \geqslant 4$ and $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_s \geqslant 1$ be fixed integers. If*
$$t \geqslant m_1 + m_2 \quad \text{and} \quad \operatorname{vdim}(\mathcal{L}(t; m_1, \ldots, m_s)) \geqslant \frac{1}{2}(3m_4^2 - 7m_4 + 2),$$
then
$$h^1(\mathcal{L}(t; m_1, \ldots, m_s)) = 0$$
i.e., the system $\mathcal{L}(t; m_1, \ldots, m_s)$ is non-special.

Conjecture B has been proved recently for *general* points with all multiplicities equal to 1 by Harbourne and Huneke in $\mathbb{P}^2$, [13, Proposition 3.10] and by the first author in $\mathbb{P}^3$, [7, Theorem 3]. Here we announce the following result, which expands considerably the range of cases in which Conjecture B holds true.

Theorem B. *Let $I = I(P_1)^{m_1} \cap \cdots \cap I(P_s)^{m_s}$ be a fat points ideal supported on $s \geqslant 9$ general points in $\mathbb{P}^2$. If one of the following conditions holds*

a) *at least $s-1$ among m_i's are equal (*almost homogeneous *case);*
b) *$m_1 \geqslant \ldots \geqslant m_s \geqslant \frac{m_1}{2}$ (*uniformly fat *case),*

then Conjecture B holds, i.e., there is the containment
$$I^{(2r)} \subset M^r \cdot I^r$$
for all $r \geqslant 1$.

3. Ideas of proofs

Cremona transformation is one of classical methods in investigating linear series of type $\mathcal{L}(t; m_1, \ldots, m_s)$. For example, Cremona transformation centered in the first three points P_1, P_2, P_3 results in a new linear series

$$\mathcal{L}(2t - m_1 - m_2 - m_3; t - m_2 - m_3, t - m_1 - m_3, t - m_1 - m_2, m_4, \ldots, m_s),$$

which under favorable circumstances might be easier to handle than the original one. Jarnicki and the first author developed in [8] another approach based on the following reduction algorithm.

Let $S = (a_1, \ldots, a_s)$ be sequence of positive integers. Let m be a fixed integer and let $Z = \{1, 2, \ldots, m\}$ be an initial set of reducers. In the first step of the algorithm we subtract from a_s the maximal number $r \in Z$ satisfying the condition $r \leqslant a_s$. Since by assumption $a_s \geqslant 1$, this first step is always possible. Then we replace a_s by $a_s - r$ and we throw away the number r from the set Z of reducers. Thus in the second step we have $S = (a_1, \ldots, a_{s-1}, a_s - r)$ and we apply the reduction to its $s - 1$ first elements with the set $Z \setminus \{r\}$. We “reduce” the number a_{s-1} with the largest possible reducer from the set of reducers available at this stage and so on until there are no more reducers (in which case we say that the reduction was successful or that the set S was m–reducible) or until all left reducers are larger than the element in S which we try to reduce (in this case the algorithm stops and we say that S was not m–reducible).

Example 3.1. The sequence $(5, 5, 5, 4, 2, 1)$ reduced with $Z = \{1, 2, 3\}$ gives in each reduction step:

$$\begin{array}{ll} r = 1, & (5,5,5,4,2,0) \\ r = 2, & (5,5,5,4,0,0) \\ r = 3, & (5,5,5,1,0,0) \end{array}$$

The same sequence reduced with $Z = \{1, 2, 3, 4\}$ gives

$$\begin{array}{ll} r = 1, & (5,5,5,4,2,0) \\ r = 2, & (5,5,5,4,0,0) \\ r = 4, & (5,5,5,0,0,0) \\ r = 3, & (5,5,2,0,0,0) \end{array}$$

For the set $Z = \{1, 2, 3, 4, 5\}$ we obtain

$$\begin{array}{ll} r = 1, & (5,5,5,4,2,0) \\ r = 2, & (5,5,5,4,0,0) \\ r = 4, & (5,5,5,0,0,0) \\ r = 5, & (5,5,0,0,0,0) \\ r = 3, & (5,2,0,0,0,0) \end{array}$$

This set is not 6-reducible because each element is less than 6. It is also not $m \geqslant 7$ reducible because it is too long.

The interest in the above algorithm stems from the following fact following from [8, Corollary 19].

Theorem 3.2. *Fix positive integers $t, m_1, \ldots, m_s$ and let $S_1 = (1, 2, \ldots, t, t+1)$. Assume inductively that S_j is m_j-reducible with the reduction equal to S_{j+1} for $j = 1, \ldots, s$. If not all elements of S_{s+1} are zero, then the linear system $\mathcal{L}(t; m_1, \ldots, m_s)$ is effective and non-special.*

The proof of Theorem A boils down to checking that under its assumptions it is always possible to perform a favorable sequence of reductions leading to the vanishing conclusion. The precise proof is elaborated in [10]. Note that one obtains in particular information on the Castelnuovo-Mumford regularity of $\mathcal{L}$.

The proof of Theorem B relies on the following observation due to Harbourne and Huneke which follows from [13, Lemma 2.3].

Proposition 3.3. *Let $\mathcal{I}$ be a fat points ideal in $\mathbb{P}^2$. Assume that*

$$\alpha(\mathcal{I}^{(2r)}) \geqslant r \cdot (\mathrm{reg}(\mathcal{I}) + 1).$$

Then $\mathcal{I}^{(2r)} \subset \mathcal{M}^r \mathcal{I}^r$.

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