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THE SIMPLEST SEMILINEAR PARABOLIC EQUATION OF NORMAL TYPE

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Abstract. The notion of semilinear parabolic equation of normal type is introduced. The structure of dynamical flow corresponding to equation of this type with periodic boundary condition is investigated. Stabilization of mentioned equation with arbitrary initial condition by start control supported in prescribed subset is constructed.

1. **Introduction.** It is well-known that existence of solutions to 3D Navier-Stokes equations or to analogous PDE in a class of sufficiently smooth functions that guarantees uniqueness of these solutions is very important problem. Let us write this equation in the following abstract form:

$$\partial_t v(t,x) + Av + B(v) = 0, \quad v|_{t=0} = v_0 \tag{1}$$

where $t \in \mathbb{R}_+, x = (x_1, x_2, x_3)$, A is a linear operator, and B is nonlinear operator. Assume also that $v(t,x)$ is 2π-periodic function with respect to $x_i, i = 1, 2, 3$.

Existence and uniqueness of smooth solutions to 3D Navier-Stokes equations defined for times $t \in \mathbb{R}_+$ with small initial conditions v_0 e.g. with respect to norm of Sobolev space H^1 was known already after classical works of J.Leray [13], [14] and O.A.Ladyzhenskaya [12]. First unique solvability results with v_0 belonging to unbounded neighborhood of origin in H^1 (more precisely, to unbounded ellipsoid in H^1 that is intersection of H^1 with small ball in $H^{1/2}$) were obtained by H.Fujita and T.Kato [4]. Their result was extended onto Lebesgue spaces L_p by F.Weissler [15], onto Besov spaces by M.Cannone,Y.Meyer, and F.Planchon [2], and onto BMO^{-1} spaces by H.Koch and D.Tataru [11]. More detailed survey as well as new progress one can see in J.-Y.Chemin, I.Gallagher, and M.Paicu [3].

The case of (1) with nonzero right side independent of time was considered by A.V.Fursikov in [5] where unbounded stable invariant manifold around a steady-state solution of 3D Navier-Stokes equations has been constructed.

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All these results are based on Implicit Function Theorem that is very general but local by its nature. Hence, all aforementioned results on existence and uniqueness of smooth solutions are local as well.

The aim of this paper is to try to work out a nonlocal approach [1] to investigate equations of 3D Navier-Stokes type. Let explain the main idea. As well-known, the proof of existence of nonlocal weak solution to Navier-Stokes equation is based on energy estimate that is true because the image of nonlinear operator $B(v)$ (see (1)) is tangent to the sphere $S(L_2) = \{w \in L_2 : \|w\|_{L_2} = \|v\|_{L_2}\}$, i.e $(B(v), v)_{L_2} = 0$. Similar property for $B(v)$ with respect to sphere $S(H^1) = \{w \in H^1 : \|w\|_{H^1} = \|v\|_{H^1}\}$ is not valid in 3D case: [2] operator $B(v)$ has nonzero normal $(B_n(v))$ and tangential $(B_\tau(v))$ components, i.e. $B(v) = B_n(v) + B_\tau(v)$. The main difficulties in construction of smooth solution are connected with presence of operator $B_n(v)$ because if we would change operator $B(v)$ in (1) on $B_\tau(v)$ we get existence of strong solution by straightforward repeating the proof of weak solutions existence. That is why it seams reasonable to omit component $B_\tau(v)$ and to study at first equation (1) where operator $B(v)$ is changed on $B_n(v)$.

To minimize possible technical difficulties we begin realization of this program from the simplest example connected with Burgers equation.

In Section 2 of this paper we introduce a class of so-called normal parabolic equations (NPE). Semilinear parabolic equation is called equation of normal type if its nonlinear term B satisfies condition: vector $B(v)$ is collinear to a vector v for all v. In other words $B(v) = B_n(v)$, and equation of normal type does not satisfies energy bound " 'in the most degree"'. As was mentioned above, we study the simplest NPE that is connected with Burgers equation considered in phase space H^1.

We begin with establishing of existence and uniqueness result for solution to normal one-dimensional parabolic equation with periodic boundary conditions and with initial conditions belonging to an unbounded ellipsoid of the corresponding phase space. Although general approach used here is as in papers mentioned above in the beginning of Introduction, we are forced to give complete proof of this result because of specific of nonlinear term in considered equation.

After that, in Section 3 we derive explicit formula for solutions of NPE and establish some properties of these solutions. Note that explicit formula is the key result of this paper because it gives big opportunities for investigations. We have to emphasize that existence of explicit formula for solution is general property of NPE. To confirm this we derive explicit formula for solution of normal equations connected with 3D Navier-Stokes system: this gives us confidence that it will be possible to generalize on this, the main, case the most results obtained here for the simplest normal parabolic equation.

In Section 4 for dynamical system corresponding to normal Burgers-connected equation we describe the structure of its phase flow. Its phase space can be divided on: a) the set of stability (Solutions with initial conditions from this set tends to zero as time $t \to \infty$ with a certain prescribed rate, b) set of explosions, when solution blows up during finite time, and c) intermediate set dividing previous ones, when solution either tends to zero as $t \to \infty$, slower than solutions with initial conditions from the set a) or it increases infinitely as $t \to \infty$.

[1] that is different from approaches developed in [10] and previous papers

[2] but it is true for 2D Navier-Stokes system written in the form of Helmholtz equation i.e. when the curl of velocity is taken as unknown function

We show that all these sets are not empty, that the set of stability contains not only unbounded ellipsoid introduced in Section 1 but also some infinite-dimensional subspaces and other sets, that intermediate set contains initial conditions for solutions that tends to zero as $t \to \infty$ as well as for solutions tending to infinity as $t \to \infty$.

In Section 5 for arbitrary initial condition we construct start control supported in a given fixed subinterval such that solution of obtained boundary value problem tends to zero as $t \to \infty$. The structure of control becomes universal: only its norm and " 'sign"' depend on initial condition. Note that stabilizing control in initial condition can be used to construct stabilizing control of other types. As known (see [7, 8, 9]), this start control can be used to construct the control on the boundary (e.g. in the case of mixed Dirichlet boundary value problem) that stabilizes the solution of mentioned boundary value problem. Besides, it can be used for construction of impulse stabilizing control and, hope, of distributed stabilizing control supported in a prescribed subdomain. These kinds of control will be studied in future papers.

In the Appendix the draft of the proof for one important assertion is given. This assertion is essentially used for construction of stabilizing control from Section 4.

How to use NPE to investigate equations of Navier-Stokes type is, of course, the matter of future and not easy studies. Nevertheless, it is more less clear already now that normal equation can help. For instance, it seems quite probable that smoothness properties of solutions for equations of Navier-Stokes type became worse with increasing of time [3] when solution find itself just in the set of explosions for corresponding normal parabolic equation. There are a number other conjectures of such type. All of them should be the matter of future investigations.

2. **Semilinear parabolic equation of normal type.**

To understand better how to investigate semilinear (or quasilinear) parabolic equation that does not satisfy energy estimate we derive some semilinear parabolic equation that does not satisfy energy bound " 'in the most degree"' in the meaning that will be clear below.

2.1. **Derivation of Normal Parabolic Equation (NPE).**

We consider Burgers equation

$$\partial_t v(t,x) - \partial_{xx} v(t,x) - \partial_x v^2(t,x) = 0 \tag{1}$$

where $\partial_t v = \partial v/\partial t$, $\partial_{xx} v = \partial^2 v/\partial x^2$ with periodic boundary condition

$$v(t, x + 2\pi) = v(t,x) \tag{2}$$

and initial condition

$$v(t,x)|_{t=0} = v_0(x) \tag{3}$$

Everywhere below we consider equations only with periodic boundary condition.

Multiplying (1) on v, integrating with respect to x we obtain after integration by parts and integration on t well-known energy estimate

$$\int_0^{2\pi} v^2(t,x)dx + 2\int_0^t \int_0^{2\pi} (\partial_x v(\tau,x))^2 dx d\tau \le \int_0^{2\pi} v_0^2(x)dx \tag{4}$$

Let us derive from (1) some equation that we intend to study. Differentiating (1) on x we get

$$\partial_t v_x(t,x) - \partial_{xx} v_x(t,x) - B(v, v_x) = 0 \tag{5}$$

[3] e.g. the magnitude of their Sobolev norms begin to increase

where $v_x = \partial v/\partial x$,

$$B(v, v_x) = 2v_x^2 + 2v\partial_x v_x \tag{6}$$

Multiplying (6) on v_x scalarly in $L_2(T_1)$ where $T_1 = \mathbb{R}/2\pi\mathbb{Z}$ is circumference and integrating by parts we get

$$\int_0^{2\pi} B(v, v_x)v_x dx = \int_0^{2\pi} (2v_x^3 + 2vv_x\partial_x v_x)dx = \int_0^{2\pi} v_x^3 dx \tag{7}$$

Denote

$$L_2^0(T_1) = \{v(x) \in L_2(T_1) : \int_0^{2\pi} v(x)dx = 0\} \tag{8}$$

Let decompose operator $B(v, v_x)$ as follows:

$$B(v, v_x) = B_n(v, v_x) + B_\tau(v, v_x) \tag{9}$$

Here $B_n(v, v_x) = \Phi(v, v_x)v_x$ where $\Phi(v, v_x)$ is a functional and vector $B_\tau(v, v_x)$ is orthogonal to the vector v_x in $L_2(T_1)$:

$$\int_0^{2\pi} B_\tau(v, v_x)v_x dx = 0 \tag{10}$$

To determine functional $\Phi(v, v_x)$ we substitute (9) into (7) and use (10). As a result we get:

$$\int_0^{2\pi} v_x^3 dx = \int_0^{2\pi} \Phi(v, v_x)v_x^2 dx = \Phi(v, v_x) \int_0^{2\pi} v_x^2 dx$$

Therefore functional Φ does not depend on v depending on v_x only and for $v_x \neq 0$

$$\Phi(v_x) = \int_0^{2\pi} v_x^3 dx \Big/ \int_0^{2\pi} v_x^2 dx$$

It is clear that by continuity Φ can be defined at $v_x \equiv 0$ by the formula $\Phi(0) = 0$.

If in decomposition (9) $B_n = 0$, then $B = B_\tau$ is orthogonal to v_x and for solution v_x of (5) energy estimate can be derived. When in (9) $B_\tau = 0$ operator $B(v, v_x) = \Phi(v_x)v_x$ in the most degree does not satisfy condition that implies energy estimate. Let consider the case when in (5) $B(v, v_x) = \Phi(v_x)v_x$. Using notation $v_x = y$ we obtain from (5) the following equation:

$$\partial_t y(t, x) - \partial_{xx} y(t, x) - \Phi(y)y = 0, \tag{11}$$

where

$$\Phi(y) = \begin{cases} \int_0^{2\pi} y(x)^3 dx \Big/ \int_0^{2\pi} y(x)^2 dx, & y \neq 0 \\ 0, & y \equiv 0 \end{cases} \tag{12}$$

Equation (11) is called semilinear parabolic equation of normal type or normal parabolic equation (NPE). It is the main object of our investigation in this article.

2.2. **Unique solvability for NPE with initial conditions from unbounded ellipsoid.** We study equation (11) with periodic boundary condition supplied with initial condition

$$y(t, x)|_{t=0} = y_0(x) \tag{13}$$

It is natural to consider dynamical system (11),(13) in the phase space $L_2^0(T_1)$ defined in (8).

Let us recall definition of Sobolev spaces that will be used below. Each periodic function $z(x)$ can be decomposed in Fourier series

$$z(x) = \sum_{k=-\infty}^{\infty} z_k e^{ikx}, \qquad \text{where} \qquad z_k = \frac{1}{2\pi}\int_0^{2\pi} z(x)e^{-ikx}dx \tag{14}$$

If $z(x) \in L_2^0(T_1)$ then $z_0 = 0$. Below we always suppose that this condition on Fourier coefficient z_0 is fulfilled.

Let $s \in \mathbb{R}$ and as above $T_1 = \mathbb{R}/2\pi\mathbb{Z}$ be circumference. Sobolev space $H^s(T_1)$ is the space of periodic real-valued distributions $z(x)$ satisfying $\int_{T_1} z(x)dx = 0$ with finite norm

$$\|z\|^2_{H^s(T_1)} \equiv \|z\|^2_s = \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2s}|z_k|^2 < \infty \tag{15}$$

where z_k are Fourier coefficients of z (see (14)).

Denote $Q = \mathbb{R}_+ \times T_1$. The following spaces will be useful for us:

$$H^{1,2(-\alpha)}(Q) = \{y \in L_2(\mathbb{R}_+; H^{2-\alpha}(T_1)) : \ \partial_t y \in L_2(\mathbb{R}_+; H^{-\alpha}(T_1))\} \tag{16}$$

with some $\alpha \geq 0$. Note that it is natural to look for a solution of problem (11),(13) in the space $H^{1,2(-1)}(Q)$. In particular, in virtue of well-known embedding inequality (see e.g. [6], Ch.3 Sect.4)

$$\sup_{t\in\mathbb{R}_+} \|y(t,\cdot)\|_{L_2^0(T_1)} \leq c\|y\|_{H^{1,2(-1)(Q)}},$$

the space $L_2^0(T_1)$ should be taken as the phase space of the dynamical system generated by (11),(13).

Since for real-valued function the Fourier decomposition

$$z(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} z_k e^{ikx} = 2\sum_{k=1}^{\infty}(\mathrm{Re} z_k \cos kx - \mathrm{Im} z_k \sin kx)$$

is true, we can interpret the set

$$\begin{aligned} El_\rho &= \{z \in L_2^0(T_1) : \ \|z\|^2_{-1/2} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|z_k|^2}{|k|} \leq \rho\} \\ &= \{z \in L_2^0(T_1) : \ \sum_{k=1}^{\infty} \frac{|\mathrm{Re} z_k|^2 + |\mathrm{Im} z_k|^2}{(\rho k)/2} \leq 1\} \end{aligned} \tag{17}$$

as ellipsoid in $L_2^0(T_1)$ with length of axes directed along functions $\cos kx, \sin kx$ equal to $\sqrt{\rho k/2}$. Since $\sqrt{\rho k/2} \to \infty$ as $k \to \infty$, this ellipsoid is unbounded in $L_2^0(T_1)$. The following analog of result from [5] holds:

Theorem 2.1. *If $\rho > 0$ is sufficiently small, then for each $y_0 \in El_\rho$ there exists unique solution $y(t,x) \in H^{1,2(-1)}(Q)$ of problem (11),(13). Moreover*

$$\|y(t,\cdot)\|_0 \leq \alpha\|y_0\|_0 e^{-t} \qquad \text{as} \qquad t \to \infty \tag{18}$$

with constant $\alpha > 0$ independent of time $t > 0$ and datum $y_0 \in El_\rho$.

Remark 1. To prove part of Theorem 2.1 connected with existence of solution we need in inclusion $y_0 \in \{v \in H^{-1/2}(T_1) : \|v\|_{-1/2} < \rho\}$ only. In the second time, to prove inequality (18) we additionally need in inclusion $y_0 \in L_2^0(T_1)$. As a result we get assumption $y_0 \in El_\rho$.

2.3. **Some auxiliary lemmas.** To prove Theorem 2.1 we rewrite problem (11), (13) as follows:

$$y(t,\cdot) = e^{\Delta t} y_0 + \int_0^t e^{\Delta(t-\tau)} \Phi(y(\tau,\cdot)) y(\tau,\cdot) d\tau \tag{19}$$

where $e^{\Delta t} y_0$ is the solution of the heat equation

$$\partial_t z(t,x) - \partial_{xx} z(t,x) = 0, \qquad z|_{t=0} = y_0 \tag{20}$$

(in fact to get (19)) we invert linear part in equation (11)).

We look for solution of (19) as limit of y_k that are defined by recursion relation

$$y_k(t,\cdot) = y_1(t,\cdot) + \int_0^t e^{\Delta(t-\tau)} \Phi(y_{k-1}(\tau,\cdot)) y_{k-1}(\tau,\cdot) d\tau, \quad k = 2,3,\dots \tag{21}$$

where

$$y_1(t,\cdot) = e^{\Delta t} y_0. \tag{22}$$

To prove convergence of sequence (21) we need in some Lemmas.

Lemma 2.2. *The following bound is true:*

$$\int_0^\infty \|\Phi(u(t,\cdot))u(t,\cdot)\|^2_{-1/2} dt \le c_1 \|u\|^4_{H^{1,2(-3/2)}(Q)} \tag{23}$$

with a constant c_1 independent of $u \in H^{1,2(-3/2)}(Q)$.

Proof. Using Sobolev embedding theorem $H^s(T_1) \subset L_p(T_1)$ with $p > 2$, $(1/2 - 1/p) \le s$ and interpolation inequality

$$\|u\|_{s_2} \le \|u\|_{s_1}^{\frac{s_3-s_2}{s_3-s_1}} \|u\|_{s_3}^{\frac{s_2-s_1}{s_3-s_1}} \quad \text{with} \quad s_1 < s_2 < s_3 \tag{24}$$

we get

$$|\Phi(u)| \le \frac{\|u\|^3_{L_3}}{\|u\|^2_{L_2}} \le C \frac{\|u\|^3_{1/6}}{\|u\|^2_{L_2}} \le C \frac{\|u\|^2_0 \|u\|_{1/2}}{\|u\|^2_{L_2}} = C\|u\|_{1/2} \tag{25}$$

and therefore

$$\begin{aligned} &\int_0^\infty \|\Phi(u)u\|^2_{-1/2} dt \le \int_0^\infty |\Phi(u)|^2 \|u\|^2_{-1/2} dt \le c \int_0^\infty \|u\|^2_{1/2} \|u\|^2_{-1/2} dt \\ &\le c \sup_{t\in\mathbb{R}_+} \|u(t,\cdot)\|^2_{-1/2} \int_0^\infty \|u\|^2_{1/2} dt \le c_1 \|u\|^4_{H^{1,2(-3/2)}(Q)} \end{aligned} \tag{26}$$

□

Lemma 2.3. *The following inequality holds:*

$$\int_0^\infty \|\Phi(u)u - \Phi(v)v\|^2_{-1/2} dt \le \hat{c} \|u-v\|^2_{H^{1,2(-1)}(Q)} (\|u\|^2_{H^{1,2(-3/2)}(Q)} + \|v\|^2_{H^{1,2(-3/2)}(Q)}) \tag{27}$$

where constant $\hat{c} > 0$ does not depend on $u, v \in H^{1,2(-3/2)}(Q)$ such that $u - v \in H^{1,2(-1)}(Q)$.

Proof. Let $\mathbb{R}_+ = A_1 \cup A_2 \cup A_3$ where $A_1 = \{t \in \mathbb{R}_+ : \ u(t) = 0 \text{ or } v(t) = 0\}$, $A_2 = \{t \in \mathbb{R}_+ : \ \|u(t)\|_0 \ge \|v(t)\|_0 > 0\}$, $A_3 = \{t \in \mathbb{R}_+ : \ 0 < \|u(t)\|_0 < \|v(t)\|_0\}$. If some of these sets are empty or are of zero Lebesgue measure, the corresponding case written below should not be considered.

a) For $t \in A_1$ inequality (27) with $\int_0^\infty$ changed on $\int_{A_1}$ and with $Q = \mathbb{R}_+ \times T_1$ changed on $Q_{A_1} = A_1 \times T_1$ follows from (23) where similar changes have been made.

b) Let consider the case $t \in A_2$. We have:

$$\int_{A_2} \|\Phi(u)u - \Phi(v)v\|^2_{-1/2} dt \le 2(I_1 + I_2) \tag{28}$$

where

$$I_1 = \int_{A_2} |\Phi(u) - \Phi(v)|^2 \|u\|^2_{-1/2} dt, \quad I_2 = \int_{A_2} |\Phi(v)|^2 \|u - v\|^2_{-1/2} dt \tag{29}$$

Similarly to (23) one can get using notation $Q_2 = A_2 \times T_1$

$$I_2 \le c\|v\|^2_{H^{1,2(-3/2)}(Q_2)} \|u - v\|^2_{H^{1,2(-3/2)}(Q_2)} \le c\|v\|^2_{H^{1,2(-3/2)}(Q_2)} \|u - v\|^2_{H^{1,2(-1)}(Q_2)} \tag{30}$$

Besides, using notation $\int = \int_{T_1}$ we have

$$\begin{aligned} I_1 &= \int_{A_2} \left| \frac{\int u^3 dx \int v^2 dx - \int u^2 dx \int v^3 dx}{\int u^2 dx \int v^2 dx} \right|^2 \|u\|^2_{-1/2} dt \\ &= \int_{A_2} \left| \frac{\int (u^3 - v^3) dx \int v^2 dx + \int (v^2 - u^2) dx \int v^3 dx}{\int u^2 dx \int v^2 dx} \right|^2 \|u\|^2_{-1/2} dt \le 2(J_1 + J_2) \end{aligned} \tag{31}$$

where

$$\begin{aligned} J_1 &= \int_{A_2} \left| \frac{\int (u - v)(u^2 + uv + v^2) dx}{\int u^2 dx} \right|^2 \|u\|^2_{-1/2} dt \\ J_2 &= \int_{A_2} \left| \frac{\int (v^2 - u^2) dx \int v^3 dx}{\int u^2 dx \int v^2 dx} \right|^2 \|u\|^2_{-1/2} dt \end{aligned} \tag{32}$$

Using Lipschitz inequality, Sobolev embedding theorem, (24), (25), and definition of set A_2 we get

$$\begin{aligned} J_1 &\le c \int_{A_2} \frac{\|u - v\|^2_{L_3} (\|u\|^4_{L_3} + \|v\|^4_{L_3}) \|u\|^2_{-1/2}}{\|u\|^4_0} dt \\ &\le c \int_{A_2} \|u - v\|^{5/3}_0 \|u - v\|^{1/3}_1 \left(\left(\frac{\|u\|^3_{L_3}}{\|u\|^2_0} \right)^{4/3} + \left(\frac{\|v\|^3_{L_3}}{\|v\|^2_0} \right)^{4/3} \right) \frac{\|u\|^2_{-1/2}}{\|u\|^{4/3}_0} dt \\ &\le c \sup_t \|u - v\|^{4/3}_0 \sup_t \|u\|^{2/3}_{-1/2} \int_{A_2} \|u - v\|^{2/3}_1 \left(\|u\|^{4/3}_{1/2} + \|v\|^{4/3}_{1/2} \right) dt \\ &\le c\|u - v\|^2_{H^{1,2(-1)}(Q_2)} \left(\|u\|^2_{H^{1,2(-3/2)}(Q_2)} + \|v\|^2_{H^{1,2(-3/2)}(Q_2)} \right) dt \end{aligned} \tag{33}$$

By similar tools as in (33) we obtain

$$\begin{aligned} J_2 &\le c \int_{A_2} \|v\|^2_{1/2} \|u - v\|^2_0 \left(\frac{\|u\|^2_0}{\|u\|^2_0} + \frac{\|v\|^2_0}{\|v\|^2_0} \right) \frac{\|u\|^2_{-1/2}}{\|u\|^2_0} dt \\ &\le c \sup_t \|u - v\|^2_0 \int_{A_2} \|v\|^2_{1/2} dt \le c\|u - v\|^2_{H^{1,2(-1)}(Q_2)} \|v\|^2_{H^{1,2(-3/2)}(Q_2)} \end{aligned} \tag{34}$$

Relations (28)-(34) imply

$$\int_{A_2} \|\Phi(u)u - \Phi(v)v\|^2_{1/2} dt \le c\|u - v\|^2_{H^{1,2(-1)}(Q_2)} (\|u\|^2_{H^{1,2(-3/2)}(Q_2)} + \|v\|^2_{H^{1,2(-3/2)}(Q_2)})$$

c)The case $t \in A_3$ is considered similarly to case b) by changing $(u, v) \to (v, u)$. □

2.4. **The proof of unique solvability theorem.** Let prove now the following assertion.

Lemma 2.4. *There exists sufficiently small $\rho > 0$ such that for each initial condition $y_0 \in H^{-1/2}(T_1)$, $\|y_0\|_{-1/2} < \rho$ there exists unique solution $y \in H^{1,2(-3/2)}(Q)$ of problem (11),(13) that additionally satisfies inclusion $y - y_1 \in H^{1,2(-1)}(Q)$ where y_1 is function (22).*

Proof. i) Denote $\gamma_0 z = z|_{t=0}$. As well-known, operator

$$(\partial_t - \partial_{xx}, \gamma_0) : H^{1,2(-3/2)}(Q) \to L_2(\mathbb{R}_+; H^{-3/2}(T_1) \times H^{-1/2}(T_1)) \tag{35}$$

realizes isomorphism of the spaces (see, e.g. [6], Ch.3, Sect.4). Since in (19) the converse operator to (35) is used, we get from (21), (22) and Lemma 2.2 that

$$\begin{aligned} \|y_k - y_1\|^2_{H^{1,2(-3/2)}(Q)} &\le \alpha \int_0^\infty \|\Phi(y_{k-1})y_{k-1}\|^2_{-3/2} dt \\ &\le \alpha \int_0^\infty \|\Phi(y_{k-1})y_{k-1}\|^2_{-1/2} dt \le c^2 \|y_{k-1}\|^4_{H^{1,2(-3/2)}(Q)} \end{aligned} \tag{36}$$

Moreover, (22), (35) imply

$$\|y_1\|_{H^{1,2(-3/2)}(Q)} \le c_1 \|y_0\|_{-1/2} \le c_1 \rho. \tag{37}$$

Denote for briefness

$$c_1 \rho = \varepsilon, \qquad \|\cdot\|_{H^{1,2(-3/2)}(Q)} = \|\cdot\|. \tag{38}$$

Then (36)-(38) imply

$$\|y_k\| \le c\|y_{k-1}\|^2 + \|y_1\| \le c\|y_{k-1}\|^2 + \varepsilon \tag{39}$$

ii) We show that if constants c, ε from (39) satisfy

$$c\varepsilon < 1/4 \tag{40}$$

then

$$\|y_k\| \le 4\varepsilon \qquad \forall k \tag{41}$$

In virtue of (39)

$$\|y_2\| \le c\varepsilon^2 + \varepsilon = (c\varepsilon + 1)\varepsilon, \quad \|y_3\| \le (c\varepsilon + 1)^2 \varepsilon^2 c + \varepsilon = ((c\varepsilon + 1)^2 c\varepsilon + 1)\varepsilon. \tag{42}$$

and by induction for each $k \ge 3$

$$\|y_k\| \le (\dots (c\varepsilon + 1)^2 c\varepsilon + 1)^2 \dots)^2 c\varepsilon + 1)\varepsilon \tag{43}$$

where there are $k - 1$ opening and closing parentheses.

Let show that (43), (40) imply inequality

$$\|y_k\| \le \left(\left(\sum_{j=0}^{k-2} (3c\varepsilon)^j \right) c\varepsilon + 1 \right) \varepsilon \quad \forall k \ge 3. \tag{44}$$

Indeed, if $c\varepsilon < 1/4$ then $(c\varepsilon + 1)^2 \le 1 + 3c\varepsilon$ and therefore inequality (42) for $\|y_3\|$ implies (44) for $k = 3$. Assume that (44) is true for $k = 3, \dots, m$ and prove it for $k = m + 1$. By (40)

$$\left(\sum_{j=0}^{m-2} (3c\varepsilon)^j \right) c\varepsilon \le \left(\sum_{j=0}^{\infty} (3c\varepsilon)^j \right) c\varepsilon = \frac{c\varepsilon}{1 - 3c\varepsilon} < 1$$

and therefore in virtue of (39), (44)

$$\|y_{m+1}\| \le \left(\left(\sum_{j=0}^{m-2}(3c\varepsilon)^j\right)c\varepsilon+1\right)^2 c\varepsilon^2+\varepsilon$$

$$\le \left(\left(\left(\sum_{j=0}^{m-2}(3c\varepsilon)^j\right)^2 c^2\varepsilon^2+\left(\sum_{j=0}^{m-2}(3c\varepsilon)^j\right)2c\varepsilon+1\right)c\varepsilon+1\right)\varepsilon$$

$$\le \left(\left(\sum_{j=0}^{m-1}(3c\varepsilon)^j\right)c\varepsilon+1\right)\varepsilon$$

Now inequalities (44) and (40) imply

$$\|y_k\| \le \left(\sum_{j=0}^{k-1}(3c\varepsilon)^j\right)\varepsilon \le \frac{\varepsilon}{1-3c\varepsilon} \le 4\varepsilon.$$

iii) Using notation $|||\cdot||| = \|\cdot\|_{H^{1,2(-1)}(Q)}$ we obtain from (21), Lemma 2.3, and (41) that

$$\begin{aligned}
|||y_{k+1}-y_k||| &\le \left(\int_0^\infty \|\Phi(y_k)y_k-\Phi(y_{k-1})y_{k-1}\|_{-1}^2 dt\right)^{1/2}\\
&\le \left(\int_0^\infty \|\Phi(y_k)y_k-\Phi(y_{k-1})y_{k-1}\|_{-1/2}^2 dt\right)^{1/2}\\
&\le 8\sqrt{\hat{c}}\varepsilon|||y_k-y_{k-1}||| \le \cdots \le (8\sqrt{\hat{c}}\varepsilon)^{k-2}|||y_3-y_2|||
\end{aligned} \tag{45}$$

and therefore

$$\begin{aligned}
|||y_{k+m}-y_k||| &\le |||y_{k+m}-y_{k+m-1}||| + \cdots + |||y_{k+1}-y_k|||\\
&\le \sum_{j=k-2}^{k+m-3}(8\sqrt{\hat{c}}\varepsilon)^j|||y_3-y_2||| \le \frac{(8\sqrt{\hat{c}}\varepsilon)^{k-2}}{1-8\sqrt{\hat{c}}\varepsilon}|||y_3-y_2|||
\end{aligned}$$

This inequality proves that $y_k - y_1$ is Cauchy sequence in $H^{1,2(-1)}(Q)$ and y_k is Cauchy sequence in $H^{1,2(-3/2)}(Q)$. After passage to limit in $H^{1,2(-3/2)}(Q)$ and in $H^{1,2(-1)}(Q)$ we get

$$\lim_{k\to\infty} y_k = y \in H^{1,2(-3/2)}(Q), \quad \lim_{k\to\infty}(y_k-y_1) = y-y_1 \in H^{1,2(-1)}(Q) \tag{46}$$

where, recall, y_1 is defined in (22).

iv) Let show that y obtained in (46) satisfies (11), (13) or, what is equivalent, that y satisfies

$$y(t,\cdot)-y_1(t,\cdot)-\int_0^t e^{\Delta(t-\tau)}\Phi(y(\tau,\cdot))y(\tau,\cdot)d\tau = 0 \tag{47}$$

Subtracting from left side of (47) analogous expression obtained from (21) we get:

$$\begin{aligned}
&y(t,\cdot)-y_1(t,\cdot)-\int_0^t e^{\Delta(t-\tau)}\Phi(y(\tau,\cdot))y(\tau,\cdot)d\tau\\
=&y(t,\cdot)-y_k(t,\cdot)-\int_0^t e^{\Delta(t-\tau)}(\Phi(y(\tau,\cdot))y(\tau,\cdot)-\Phi(y_{k-1}(\tau,\cdot))y_{k-1}(\tau,\cdot))d\tau
\end{aligned} \tag{48}$$

Taking $H^{1,2(-1)}(Q)$-norm of right side in (48) and acting similarly to (45) we get by (46) the relation

$$|||y(t,\cdot)-y_k(t,\cdot)|||+|||\int_0^t e^{\Delta(t-\tau)}(\Phi(y(\tau,\cdot))y(\tau,\cdot)-\Phi(y_{k-1}(\tau,\cdot))y_{k-1}(\tau,\cdot))d\tau|||$$

$$\leq |||y(t,\cdot)-y_k(t,\cdot)|||+8\sqrt{\hat{c}}\varepsilon|||y(t,\cdot)-y_{k-1}(t,\cdot)|||\to 0 \qquad \text{as} \quad k\to\infty$$

Therefore since left side of (48) does not depend on k, it equals to zero , and (47) is proved.

v) Suppose that $y,z\in H^{1,2(-3/2)}(Q)$ such that $(y-y_1),(z-z_1)\in H^{1,2(-1)}(Q)$ satisfy (11), (13) with common initial condition y_0 and prove that $y\equiv z$. Subtracting equality (11) written for z from the similar equality for y, multiplying obtained equality on $y-z$ scalarly in $L_2(T_1)$ and integrating on t we get using (25):

$$\begin{aligned}&\frac{1}{2}\|y(t)-z(t)\|_0^2+\int_0^t\|y(\tau)-z(\tau)\|_1^2d\tau\leq|\int_0^t(\Phi(y)y-\Phi(z)z,y-z)_0d\tau|\\&\leq I+\int_0^t|\Phi(z)|\|y-z\|_0^2d\tau\leq I+c\int_0^t\|z\|_{1/2}\|y-z\|_0^2d\tau\end{aligned}\tag{49}$$

where $(\cdot,\cdot)_0$ is scalar product in $L_2(T_1)$ and

$$I=\int_0^t|\Phi(y)-\Phi(z)||(y,y-z)_0|d\tau\tag{50}$$

Acting similarly to (29),(31)-(34) we get [4]

$$\begin{aligned}&I\leq 2\int_0^t\left|\frac{\int(y-z)(y^2+yz+z^2)dx}{\int y^2dx}\right|\|y\|_0\|y-z\|_0d\tau\\&+2\int_0^t\left|\frac{\int z^3dx}{\int z^2dx}\right|\left|\frac{\int(y^2-z^2)dx}{\int y^2dx}\right|\|y\|_0\|y-z\|_0d\tau\\&\leq c\left(\int_0^t\left(\frac{\|y\|_{L_4}^2}{\|y\|_0}+\frac{\|z\|_{L_4}^2}{\|z\|_0}\right)\|y-z\|_0^2d\tau+\int_0^t\|z\|_{1/2}\left(\frac{\|y\|_0}{\|y\|_0}+\frac{\|z\|_0}{\|z\|_0}\right)\|y-z\|_0^2d\tau\right)\\&\leq c\int_0^t(\|y\|_{1/2}+\|z\|_{1/2})\|y-z\|_0^2d\tau\end{aligned}\tag{51}$$

Thus, (49), (51) imply

$$\|y(t)-z(t)\|_0^2\leq c\int_0^t\left(\|y(\tau)\|_{1/2}+\|z(\tau)\|_{1/2}\right)\|y-z\|_0^2d\tau$$

and therefore $y(t)\equiv z(t)$ in virtue of Gronwall inequality. □

3. **Explicit formula for solution of NPE.** In this section we derive explicit formula for solutions of NPE that will allow to get in next sections many important information on this equation. Explicit formula in the case of the simplest NPE is derived in Subsection 3.1. Emphasize that this key property takes place not only for the simplest NPE. To confirm this we derive in Subsection 3.2 explicit formula for solutions in the case of NPE corresponding to 3D Navier-Stokes equations.

[4]We omit here for briefness introductions of sets A_1,A_2,A_3 as in Lemma 2.3 proof and give a sketch of proof only. Complete proof can be carried on exactly as in the proof of Lemma 2.3.

3.1. **The case of the simplest NPE.** The following explicit formula for solution of problem (11),(13) holds:

Lemma 3.1. *Let $S(t,x;y_0)$ be the solution of heat equation*

$$\partial_t S - \partial_{xx} S = 0, \qquad S|_{t=0} = y_0(x) \tag{52}$$

with periodic boundary condition. The solution of problem (11),(13) has the form

$$y(t,x;y_0) = \frac{S(t,x;y_0)}{1 - \int_0^t \Phi(S(\tau,x;y_0))d\tau} \tag{53}$$

Proof. We look for the solution of (11),(13) in the form

$$y(t,x;y_0) = S(t,x;y_0)\alpha(t), \qquad \alpha(t)|_{t=0} = 1 \tag{54}$$

where S is the solution of (52) and $\alpha(t)$ is unknown function. Substitution (54) into (11) yields

$$(\partial_t \alpha)S + \alpha(\partial_t S - \partial_{xx} S - \alpha\Phi(S)S) = 0$$

This equality and (52),(54) imply that

$$(\partial_t \alpha/\alpha^2 - \Phi(S))S = 0 \qquad \Longrightarrow \qquad \alpha(t) = \frac{1}{1 - \int_0^t \Phi(S(\tau,x;y_0))d\tau}$$

□

Formula (53) implies

Lemma 3.2. *Let El_ρ be ellipsoid (17). There exists $\rho > 0$ such that for each $y_0 \in El_\rho$ solution $y(t,x;y_0)$ of problem (11),(13) satisfies the estimate*

$$\|y(t,\cdot;y_0)\|_0 \le \alpha\|y_0\|_0 e^{-t} \qquad \forall t > 0 \tag{55}$$

with a constant α independent of $y_0 \in El_\rho$.

Proof. In virtue of (25),(52) denoting by $y_{k,0}$ Fourier coefficients of y_0 we get

$$\left|\int_0^t \Phi(S(\tau,\cdot;y_0))d\tau\right| \le c\int_0^t \|S(\tau,\cdot,y_0)\|_{1/2}d\tau$$

$$= c\int_0^t e^{-\tau/2}\left(\sum_{k\neq 0} e^{-(2k^2-1)\tau}|y_{k0}|^2|k|\right)^{1/2} d\tau$$

$$\le c\left(\int_0^t e^{-\tau}d\tau\right)^{1/2}\left(\sum_{k\neq 0}\int_0^t e^{-(2k^2-1)\tau}d\tau|y_{k,0}|^2|k|\right)^{1/2}$$

$$\le c\left(\sum_{k\neq 0}|y_{k,0}|^2\frac{|k|}{2k^2-1}\right)^{1/2} \le c\|y_0\|_{-1/2}$$

where c does not depend on $y_0, t > 0$. That is why since $\|y_0\|_{-1/2} \le \rho$, inequality

$$1 - \int_0^t \Phi(S(\tau,\cdot;y_0))d\tau \ge 1 - c\rho \tag{56}$$

holds. As well-known, solution $S(t,x;y_0)$ of (52) satisfies

$$\|S(t,\cdot;y_0)\|_0^2 = \sum_{k\neq 0} e^{-2k^2 t}|y_{0,k}|^2 \le e^{-2t}\|y_0\|_0^2 \tag{57}$$

Formula (53) and bounds (56),(57) yield

$$\|y(t,\cdot;y_0)\|_0^2 \le \frac{\|S(t,\cdot;y_0)\|_0^2}{(1-c\rho)^2} \le \frac{e^{-2t}}{(1-c\rho)^2}\|y_0\|_0^2 \tag{58}$$

Inequality (58) implies (55). □

Theorem 2.1 follows from Lemmas 2.4, 3.2.

3.2. **The case of the NPE corresponding to 3D Navier-Stokes equations.** Let consider 3D Navier-Stokes equations defined on torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$:

$$\partial_t v(t,x) - \Delta v + (v,\nabla)v + \nabla p(t,x) = 0, \quad \operatorname{div} v = 0 \tag{59}$$

$$v(t,x)|_{t=0} = v_0(x) \tag{60}$$

where $t \in \mathbb{R}_+, x = (x_1,x_2,x_3) \in \mathbb{T}^3$ (i.e. periodic boundary conditions are imposed), $v(t,x) = (v_1,v_2,v_3)$ is the velocity vector field of fluid flow, ∇p is the gradient of pressure, Δ is Laplace operator, $(v,\nabla)v = \sum_{j=1}^3 v_j \partial_{x_j} v$. Let transform problem (59),(60) for velocity to the problem for the curl of velocity as unknown function:

$$\omega(t,x) = \operatorname{curl} v(t,x) = (\partial_{x_2}v_3 - \partial_{x_3}v_2,\ \partial_{x_3}v_1 - \partial_{x_1}v_3,\ \partial_{x_1}v_2 - \partial_{x_2}v_1) \tag{61}$$

Recall the following well-known formulas of vectorial analysis:

$$(v,\nabla)v = \omega \times v + \nabla\frac{|v|^2}{2}, \tag{62}$$

$$\operatorname{curl}\,(\omega \times v) = (v,\nabla)\omega - (\omega,\nabla)v, \quad \text{if} \quad \operatorname{div} v = 0, \quad \operatorname{div}\omega = 0 \tag{63}$$

where $\omega \times v = (\omega_2 v_3 - \omega_3 v_2,\ \omega_3 v_1 - \omega_1 v_3,\ \omega_1 v_2 - \omega_2 v_3)$ is vector product and $|v|^2 = v_1^2 + v_2^2 + v_3^2$. Let substitute (62) into the first equality of (59) and apply to both parts of obtained equality operator curl. Then in virtue of (61),(63), and formula $\operatorname{curl}\nabla F = 0$ we obtain so called Helmholtz equations

$$\partial_t \omega(t,x) - \Delta\omega + (v,\nabla)\omega - (\omega,\nabla)v = 0, \quad \operatorname{div}\omega = 0 \tag{64}$$

where the second equation in (64) is evident consequence of (61). We add these equations with initial condition

$$\omega(t,x)|_{t=0} = \omega_0(x) \tag{65}$$

where $\omega_0 = \operatorname{curl} v_0$.

For each $k \in \mathbb{Z}_+ = \{j \in \mathbb{Z} : j \ge 0\}$ we define the space

$$V^k = V^k(\mathbb{T}^3) = \{v(x) \in (H^k(\mathbb{T}^3))^3 : \operatorname{div} v = 0,\ \int_{\mathbb{T}^3} v(x)dx = 0\} \tag{66}$$

where $H^k(\mathbb{T}^3)$ is Sobolev space. Using decomposition in Fourier series

$$v(x) = \sum_{\xi\in\mathbb{Z}^3} \widehat{v}(\xi)e^{i(x,\xi)}, \quad \text{where} \quad \widehat{v}(\xi) = (2\pi)^{-3}\int_{\mathbb{T}^3} v(x)e^{-i(x,\xi)}dx$$

and well-known formula curl curl $v = -\Delta v$ if $\operatorname{div} v = 0$ we see that inverse operator to curl is well-defined on spaces V^k and is determined by the formula

$$\operatorname{curl}^{-1}\omega(x) = i\sum_{\xi\in\mathbb{Z}^3} \frac{\xi \times \widehat{\omega}(\xi)}{|\xi|^2} e^{i(x,\xi)} \tag{67}$$

That is why operator

$$\operatorname{curl}\ :\ V^1 \longrightarrow V^0$$

realized isomorphism of the spaces. Therefore sphere in V^1 for problem (59), (60) is equivalent to sphere in V^0 for problem (64), (65). Thus, we have to construct

normal parabolic equations with respect to sphere in V^0, corresponding to problem (64), (65).

Since summand $(v,\nabla)\omega$ from (64) is tangential operator:

$$\int_{\mathbb{T}^3}(v,\nabla)\omega\cdot\omega dx=0,$$

normal part of nonlinear operator from (64) is defined by nonlinear term $(\omega,\nabla)v$. We look it for in the form $\Phi(\omega)\omega$ where Φ is unknown functional that is found by equation

$$\int_{\mathbb{T}^3}\Phi(\omega)\omega(x)\cdot\omega(x)dx=\int_{\mathbb{T}^3}(\omega(x),\nabla)v(x)\cdot\omega(x)dx \tag{68}$$

Relation (68) implies desired formula for Φ:

$$\Phi(\omega)=\int_{\mathbb{T}^3}(\omega(x),\nabla)\mathrm{curl}\ ^{-1}\omega(x)\cdot\omega(x)dx\Big/\int_{\mathbb{T}^3}|\omega(x)|^2dx,\ \omega\neq 0, \tag{69}$$

and $\Phi(0)=0$ where curl $^{-1}\omega(x)$ is defined in (67).

So, normal parabolic equations corresponding to system (64) is defined as follows:

$$\partial_t\omega(t,x)-\Delta\omega-\Phi(\omega)\omega=0,\quad \mathrm{div}\,\omega=0 \tag{70}$$

where functional Φ is defined in (69).

The following analog of Lemma 3.1 is true:

Lemma 3.3. *Let $S(t,x,y_0)$ be resolving operator of the following Stokes system with periodic boundary conditions:*

$$\partial_t y(t,x)-\Delta y(t,x)=0,\quad div\,y=0,\quad y(t,x)|_{t=0}=y_0(x), \tag{71}$$

i.e. $S(t,x,y_0)=y(t,x)$ (we assume, of course, that $div\,y_0=0$). The solution of problem (70),(65) has the form

$$\omega(t,x;\omega_0)=\frac{S(t,x;\omega_0)}{1-\int_0^t\Phi(S(\tau,x;\omega_0))d\tau} \tag{72}$$

The proof of this lemma is reduced to substitution (72) into (70) and direct checking of obtained equality.

4. **The structure of NPE dynamics.**

Everywhere below we have deals only with the simplest NPE corresponding to Burgers equation. The aim of this section is to find out the main feature of dynamical flow corresponding to NPE. We decompose the phase space of the dynamical system on three sets with different behavior of dynamical flow inside each of them.

4.1. **Distinctive sets of phase space.**

Let define three subsets of phase space for NPE. Recall that we take $L_2^0(T_1)\equiv H^0(T_1)$ as the phase space for problem (11), (13).

Definition 4.1. The set $M_-\equiv M_-(\alpha)\subset H^0(T_1)$ of y_0 such that the solution $y(t,x;y_0)\in H^{1,2(-1)}(Q)$ of problem (11), (13) exists and satisfies inequality

$$\|y(t,\cdot;y_0)\|_0\leq\alpha\|y_0\|_0e^{-t}\qquad\forall t>0 \tag{73}$$

is called the set of stability. Here $\alpha>1$ is a certain fixed number.

The following simple sufficient condition for belonging to $M_-(\alpha)$ is true:

Lemma 4.2. *Let $\alpha > 1$. If $y_0 \in H^0(T_1)$ satisfies the bound*

$$\sup_{t\in\mathbb{R}_+} \int_0^t \Phi(S(\tau,\cdot;y_0))d\tau \le \frac{\alpha-1}{\alpha} \tag{74}$$

then $y_0 \in M_-(\alpha)$

Proof. In virtue of (53),(57),(74)

$$\|y(t,\cdot;y_0)\|_0 \le \frac{\|S(t,\cdot;y_0)\|_0}{1-\sup_{t\in\mathbb{R}_+}\int_0^t \Phi(S(\tau,\cdot;y_0))d\tau} \le \alpha e^{-t}\|y_0\|_0$$

□

Definition 4.3. The set $M_+ \subset H^0(T_1)$ of initial conditions y_0 from (11), (13) such that corresponding solution $y(t,x;y_0)$ exists only on a finite interval $t \in (0,t_0)$ with $t_0 > 0$ depending on y_0, and blows up at $t = t_0$ is called the set of explosions.

In virtue of formula (53) for solution $y(t,x;y_0)$

$$M_+ = \{y_0 \in H^0(T_1) : \ \exists t_0 > 0 \ \int_0^{t_0} \Phi(S(\tau,\cdot;y_0))d\tau = 1\} \tag{75}$$

The minimal magnitude from the set of numbers $\{t_0\}$ satisfying (75) is called the time of explosion.

Definition 4.4. The collection

$$M_I(\alpha) = H^0(T_1) \setminus \{M_-(\alpha) \cup M_+\} \tag{76}$$

is called intermediate set.

Remark 2. Definitions of stability and intermediate sets include parameter $\alpha > 1$ and from this point of view they are not absolute. Nevertheless they are convenient for using.

We study below the properties of these sets and, in particular, we show that all these sets are not empty. We begin from the set of stability. This set is the most important for us.

4.2. **Subspaces belonging to the set of stability.** Show first that $M_-(\alpha) \ne \emptyset$. This fact follows from

Lemma 4.5. *Let $\rho \le (\alpha-1)/(\alpha c)$ where c is the constant from (56). Then*

$$El_\rho \subset M_-(\alpha) \tag{77}$$

where the set El_ρ is defined in (17).

Proof. Inequality (58) implies that (77) holds if $\alpha \ge 1/(1-c\rho)$. But the last inequality is equivalent to $\rho \le (\alpha-1)/(\alpha c)$. □

We show in this subsection that, actually, $M_-(\alpha)$ is essentially wider than El_ρ. For this aim we consider different one-dimensional subspaces (that we call " 'lines" ') and show that there are many lines belonging to $M_-(\alpha)$, and the set of all such lines form infinite-dimensional subspace. Besides, there are lines such that the whole ray along one direction of this line belongs to $M_-(\alpha)$, but along opposite direction only an finite segment belongs to $M_-(\alpha)$.

Actually, we begin from considering not lines but two-dimensional subspaces

$$l_k = \{ze^{ikx} + \overline{z}e^{-ikx},\ z \in \mathbb{C}\} \equiv \{a\cos kx + b\sin kx,\ a,b \in \mathbb{R}\} \tag{78}$$

with $z = (a - ib)/2$, where $k \in \mathbb{Z} \setminus \{0\}$ is fixed.

Lemma 4.6. *$l_k \in M_-(\alpha)$ for each $k \in \mathbb{Z} \setminus \{0\}$.*

Proof. If $y_0 = ze^{ikx} + \overline{z}e^{-ikx}$, $k \neq 0$, then $S(t, x; y_0) = e^{-k^2 t} y_0$ and

$$\int_0^{2\pi} S^3(t, x; y_0)dx = e^{-3k^2 t} \int_0^{2\pi} (z^3 e^{3ikx} + 3z|z|^2 e^{ikx} + 3\overline{z}|z|^2 e^{-ikx} + \overline{z}^3 e^{-3ikx})dx = 0.$$

Hence $\Phi(S(t, x; y_0)) = 0$ and by (53) solution $y(t, x; y_0)$ of (11),(13) satisfies

$$\|y(t, x; y_0)\|_0^2 = e^{-2k^2 t} \|ze^{ikx} + \overline{z}e^{-ikx}\|_0^2 \leq e^{-2t} \|y_0\|_0^2.$$

□

Let U_L be a certain subset of natural numbers that satisfy the property: for each $k_1, k_2, k_3 \in U_L$ $k_1 + k_2 + k_3 \neq 0$, $k_1 + k_2 - k_3 \neq 0$, i.e.

$$U_L = \{k \in \mathbb{N} : \quad k_1 + k_2 \pm k_3 \neq 0 \quad \forall k_1, k_2, k_3 \in U_L\} \tag{79}$$

where, recall, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers. An example of U_L is the set of all odd natural numbers.

The following generalization of Lemma 4.6 is true:

Lemma 4.7. *For each $\alpha > 1$ the subspace*

$$L = \{y_0 = \sum_{k \in U_L} (z_k e^{ikx} + \overline{z}_k e^{-ikx}), \quad z_k \in \mathbb{C}\} \subset H^0(T_1) \tag{80}$$

belongs to $M_-(\alpha)$ if the set U_L satisfies (79).

Proof. If $y_0 \in L$ then

$$\int_0^{2\pi} S^3(t, x; y_0)dx = A(t, x) + \overline{A}(t, x)$$

where

$$A(t, x) = \sum_{k_1, k_2, k_3 \in U_L} e^{-(k_1^2 + k_2^2 + k_3^2)t} \int_0^{2\pi} \Big(z_{k_1} z_{k_2} z_{k_3} e^{i(k_1 + k_2 + k_3)x}$$

$$+ z_{k_1} z_{k_2} \overline{z}_{k_3} e^{i(k_1 + k_2 - k_3)x} + z_{k_1} \overline{z}_{k_2} z_{k_3} \; e^{i(k_1 - k_2 + k_3)x} + \overline{z}_{k_1} z_{k_2} z_{k_3} e^{i(k_2 - k_1 + k_3)x} \Big) dx = 0$$

because by definition of U_L $k_1 + k_2 \pm k_3 \neq 0 \quad \forall k_1, k_2, k_3 \in U_L$. One can get desired assertion from the last equality as well as it was made in the proof of Lemma 4.6. □

4.3. **Rays belonging to the set of stability.** Consider now examples of rays belonging to the set of stability. Let $k \in \mathbb{N}$ be fixed. We take

$$y_0(x) = z_1 e^{ikx} + \overline{z}_1 e^{-ikx} + z_2 e^{i2kx} + \overline{z}_2 e^{-i2kx} \tag{81}$$

We will use the following notation for complex numbers z_j, $j = 1, 2$:

$$z_1 = |z_1| e^{i\varphi_1}, \quad z_2 = |z_2| e^{i\varphi_2}, \quad \text{i.e.} \quad \varphi_j = \operatorname{Arg} z_j \quad j = 1, 2 \tag{82}$$

Then

$$\int_0^{2\pi} S^3(t,x;y_0)dx$$
$$=\int_0^{2\pi}\left(2|z_1|\cos(kx+\varphi_1)e^{-k^2t}+2|z_2|\cos(2kx+\varphi_2)e^{-4k^2t}\right)^3dx$$
$$=8\int_0^{2\pi}\Big[|z_1|^3\cos^3(kx+\varphi_1)e^{-3k^2t}+3|z_1|^2|z_2|\cos^2(kx+\varphi_1)\cos(2kx+\varphi_2)e^{-6k^2t}$$
$$+3|z_1||z_2|^2\cos(kx+\varphi_1)\cos^2(2kx+\varphi_2)e^{-9k^2t}+|z_2|^3\cos^3(2kx+\varphi_2)e^{-12k^2t}\Big]dx \tag{83}$$

Evidently, for each $k\in\mathbb{N}$

$$\int_0^{2\pi}\cos^3(kx+\varphi_1)dx=0,\quad \int_0^{2\pi}\cos^3(2kx+\varphi_2)dx=0,$$
$$\int_0^{2\pi}\cos(kx+\varphi_1)\cos^2(2kx+\varphi_2)dx=\int_0^{2\pi}\cos kx\cos^2(2kx+\varphi_2-2\varphi_1)dx=0 \tag{84}$$

From the other side

$$8\int_0^{2\pi}\cos^2(kx+\varphi_1)\cos(2kx+\varphi_2)dx=\int_0^{2\pi}\Big[\left(e^{i(2kx+2\varphi_1)}+e^{-i(2kx+2\varphi_1)}\right)$$
$$\times\left(e^{i(2kx+\varphi_2)}+e^{-i(2kx+\varphi_2)}\right)+2\cdot 2\cos(2kx+\varphi_2)\Big]dx$$
$$=\int_0^{2\pi}\Big[\left(e^{i(4kx+2\varphi_1+\varphi_2)}+e^{-i(4kx+2\varphi_1+\varphi_2)}\right)+\left(e^{i(2\varphi_1-\varphi_2)}+e^{-i(2\varphi_1-\varphi_2)}\right)\Big]dx$$
$$=4\pi\cos(2\varphi_1-\varphi_2) \tag{85}$$

Formulas (83)-(85) imply that

$$\int_0^{2\pi}S^3(t,x;y_0)dx=12\pi|z_1|^2|z_2|\cos(2\varphi_1-\varphi_2)e^{-6k^2t} \tag{86}$$

where initial condition y_0 is defined in (81). Similarly to (83) we obtain

$$\int_0^{2\pi}S^2(t,x;y_0)dx=4\pi e^{-2k^2t}\left(|z_1|^2+|z_2|^2e^{-6k^2t}\right), \tag{87}$$

and by definition (12) of Φ we get

$$\Phi(S(t,x;y_0))=\frac{3|z_1|^2|z_2|e^{-4k^2t}}{|z_1|^2+|z_2|^2e^{-6k^2t}}\cos(2\varphi_1-\varphi_2). \tag{88}$$

This formula provides us with new sets belonging to $M_-(\alpha)$

Lemma 4.8. *Let in initial condition y_0 from (81)*

$$|z_1|=|z_2|=\rho\in\mathbb{R}_+,\qquad \text{and}\quad \frac{\pi}{2}+2\pi j\le 2\varphi_1-\varphi_2\le\frac{3\pi}{2}+2\pi j \tag{89}$$

with a fixed $\varphi_1,\varphi_2\in(0,2\pi)$ and $j=0$ or $j=1$. Then the ray

$$\{\rho\in\overline{\mathbb{R}_+},\ \varphi_1,\varphi_2\} \tag{90}$$

of initial conditions (81) belongs to $M_-(\alpha)$

Proof. Insertion $2\varphi_1-\varphi_2 \in [\frac{\pi}{2}, \frac{3\pi}{2}]+2\pi j,\ j=0$ or $j=1$ implies $\cos(2\varphi_1-\varphi_2)\le 0$, and by (88)

$$\max_{t\in\mathbb{R}_+}\int_0^t \Phi(S(\tau,\cdot;y_0))d\tau \le 0 < \frac{\alpha-1}{\alpha} \qquad \text{for} \quad \alpha>1.$$

Hence desired assertion follows from Lemma 4.2. □

Lemma 4.9. *Let parameters φ_1,φ_2 of y_0 from (81) satisfy*

$$2\pi j \le 2\varphi_1-\varphi_2 < 2\pi j+\frac{\pi}{2} \quad \text{or} \quad 2\pi j+\frac{3\pi}{2} \le 2\varphi_1-\varphi_2 < 2\pi j+2\pi \tag{91}$$

with fixed $\varphi_1,\varphi_2\in[0,2\pi),\ j=0$ or $j=1$ and $|z_1|=|z_2|=\rho>0$. Then the segment

$$\{\rho\in[0,\rho_0],\ \varphi_1,\varphi_2\} \tag{92}$$

of initial conditions (81) belongs to $M_-(\alpha)$ if ρ_0 from (92) is defined by the equation

$$\frac{3\rho_0}{k^2}\int_0^\infty \frac{e^{-4t}dt}{1+e^{-6t}}\cos(2\varphi_1-\varphi_2)=\frac{\alpha-1}{\alpha} \tag{93}$$

Proof. By (91) we have $\cos(2\varphi_1-\varphi_2)>0$. Nevertheless, using definition (93) of ρ_0 and equality $|z_1|=|z_2|=\rho$ we get from (88) that for $\rho\in(0,\rho_0]$

$$\max_{t\in\mathbb{R}_+}\int_0^t \Phi(S(\tau,\cdot;y_0))d\tau \le \frac{3\rho_0}{k^2}\int_0^\infty \frac{e^{-4t}dt}{1+e^{-6t}}\cos(2\varphi_1-\varphi_2)=\frac{\alpha-1}{\alpha}$$

Therefore Lemma 4.9 follows from Lemma 4.2 □

It is clear that extension the segments $\{\rho\in[0,\rho_0],\ \varphi_1,\varphi_2\}$ across ρ_0 will lead to passage corresponding initial condition y_0 from $M_-(\alpha)$ to $M_I(\alpha)$ and even to M_+ if this extension is big enough.

Lemmas 4.8, 4.9 imply

Theorem 4.10. *Let initial condition y_0 from (81), (82) satisfies (89). Then the ray*

$$\begin{aligned} R(\varphi_1,\varphi_2) &= \{|z_1|=|z_2|=\rho\in\mathbb{R}_+,\ \varphi_1,\varphi_2\} \\ &\cup\{|z_1|=|z_2|=\rho\in(0,\rho_0),\ \varphi_1+\pi,\varphi_2+\pi\} \end{aligned} \tag{94}$$

where ρ_0 is defined by equation (93) belongs to $M_-(\alpha)$.

Proof. We have to note only that if φ_1,φ_2 satisfy (89) then $\varphi_1+\pi,\varphi_2+\pi$ satisfy (91) and to take into account Lemmas 4.8, 4.9. □

4.4. **Set of explosions and intermediate set.** First of all we show that $M_+\ne\emptyset$ and $M_I(\alpha)\ne\emptyset$ for each $\alpha>1$.

Lemma 4.11. *The set M_+ is not empty.*

Proof. We take initial condition (81),(82) with $|z_1|=|z_2|=\rho>0$ and with $\cos(2\varphi_1-\varphi_2)>0$. Then by (88)

$$\int_0^t \Phi(S(\tau,\cdot;y_0))d\tau = 3\rho\int_0^t \frac{e^{-4k^2\tau}d\tau}{1+e^{-6k^2\tau}}\cos(2\varphi_1-\varphi_2).$$

We take ρ such that

$$3\rho\int_0^\infty \frac{e^{-4k^2\tau}d\tau}{1+e^{-6k^2\tau}}\cos(2\varphi_1-\varphi_2)=2$$

Then there exists t_0 such that denominator in (53) will be equal to zero at $t = t_0$. Hence, solution $y(t, x; y_0)$ of (11),(13) defined by (53) blows up. □

Lemma 4.12. *For each $\alpha > 1$ the set $M_I(\alpha)$ is not empty.*

Proof. We take y_0 from (81),(82) with $k = 1$, $|z_1| = |z_2| = \rho > 0$ and with $\cos(2\varphi_1 - \varphi_2) = 1/2$. Then by (87),(88)

$$\|y_0\|_0 = 2\rho\sqrt{2\pi}, \qquad \|S(t, \cdot; y_0)\|_0 = 2\sqrt{\pi}\rho(1 + e^{-6t})^{1/2}e^{-t}, \tag{95}$$

$$\int_0^t \Phi(S(\tau, \cdot; y_0))d\tau = \frac{3\rho}{2}\int_0^t \frac{e^{-4\tau}d\tau}{1 + e^{-6\tau}} \tag{96}$$

Formula (53) for solution $y(t, x; y_0)$ of (11),(13) and (95),(96) imply that

$$\|y(t, \cdot; y_0)\|_0 = g(t)\|y_0\|_0 e^{-t} \tag{97}$$

where

$$g(t) = \frac{(1 + e^{-6t})^{1/2}}{\sqrt{2}\left(1 - \frac{3}{2}\rho\int_0^t \frac{e^{-4\tau}d\tau}{1+e^{-6\tau}}\right)} \tag{98}$$

Suppose that ρ in (98) satisfies:

$$\rho \in (0, \widehat{\rho}), \qquad \widehat{\rho} = \frac{2}{3}\left(\int_0^\infty \frac{e^{-4\tau}d\tau}{1 + e^{-6\tau}}\right)^{-1} \tag{99}$$

Function $g(t)$ from (98) is defined for $t \in \mathbb{R}_+$ and it increases monotonically for growing $t \geq t_0 > 0$ [5] from $g(t_0)$ up to

$$g(t)|_{t=\infty} = \left[\sqrt{2}\left(1 - \frac{3}{2}\rho\int_0^\infty \frac{e^{-4\tau}d\tau}{1 + e^{-6\tau}}\right)\right]^{-1}$$

For each $1 < \alpha < \beta$ we can choose $\rho \in (0, \widehat{\rho})$ satisfying

$$\left[\sqrt{2}\left(1 - \frac{3}{2}\rho\int_0^\infty \frac{e^{-4\tau}d\tau}{1 + e^{-6\tau}}\right)\right]^{-1} = \beta$$

Then by (97),(98)

$$\|y(t, x; y_0)\|_0 \leq \beta\|y_0\|_0 e^{-t} \qquad \text{for each} \quad t > 0, \tag{100}$$

but there exists $\hat{t} > t_0$ such that

$$\|y(t, x; y_0)\|_0 \geq \alpha\|y_0\|_0 e^{-t} \qquad \text{for} \quad t \in (\hat{t}, \infty).$$

Hence, for such ρ $y_0 \in M_I(\alpha)$. □

It follows from Lemma 4.12 proof that the set $M_I(\alpha)$ contains initial conditions y_0 such that corresponding solution of (11),(13) satisfies (100) with some $\beta > \alpha$ but does not satisfies (100) with $\beta = \alpha$. Let show that $M_I(\alpha)$ contains also initial conditions of other type.

Lemma 4.13. *For each $\alpha > 1$ there exists initial condition $y_0 \in M_I(\alpha)$ such that*

$$\|y(t, \cdot; y_0)\|_0 \longrightarrow \infty \qquad \text{as} \quad t \to \infty \tag{101}$$

where $y(t, x; y_0)$ is the solution of problem (11),(13).

[5] Simple equation for determination of $t_0 > 0$ can be obtained by direct differentiation of $g^2(t)$

Proof. Let take $y_0(x)$ as in the proof of Lemma 4.12. More exactly, we take $\rho = \widehat{\rho}$ where $\widehat{\rho}$ is defined by (99). Then $g(t)$ from (98) with $\rho = \widehat{\rho}$ is defined by the formula

$$g(t) = \frac{(1+e^{-6t})^{1/2} \int_0^\infty \frac{e^{-4\tau} d\tau}{1+e^{-6\tau}}}{\sqrt{2} \int_t^\infty \frac{e^{-4\tau} d\tau}{1+e^{-6\tau}}} \tag{102}$$

Since

$$\int_t^\infty \frac{e^{-4\tau} d\tau}{1+e^{-6\tau}} \le \int_t^\infty e^{-4\tau} d\tau = \frac{1}{4} e^{-4t}$$

we get from (102) that

$$g(t) \ge \frac{4e^{4t}(1+e^{-6t})^{1/2} \int_0^\infty \frac{e^{-4\tau} d\tau}{1+e^{-6\tau}}}{\sqrt{2}} \ge c e^{4t} \tag{103}$$

with a positive c. Relations (103),(97) imply (101). □

5. **Stabilization of solution to NPE by start control.** Our purpose now is to stabilize solution of normal parabolic equation by control in initial condition when this control is supported on a given subinterval of $(0, 2\pi)$.

5.1. **Formulation of the main result on stabilization.** We consider semilinear parabolic equation (11):

$$\partial_t y(t,x) - \partial_{xx} y(t,x) - \Phi(y) y = 0 \tag{104}$$

where

$$\Phi(y) = \begin{cases} \int_0^{2\pi} y^3(x) dx \Big/ \int_0^{2\pi} y^2(x) dx, & y \ne 0 \\ 0, & y \equiv 0 \end{cases} \tag{105}$$

with periodic boundary condition

$$y(t, x+2\pi) = y(t,x) \tag{106}$$

and initial condition

$$y(t,x)|_{t=0} = y_0(x) + u(x). \tag{107}$$

Here $y_0(x) \in H^0(T_1) \equiv L_2^0(T_1)$ is an arbitrary given initial datum and $u(x) \in H^0(T_1)$ is a control. We assume that on a circumference $T_1 = \mathbb{R}/2\pi\mathbb{Z}$ a segment $[a,b]$ is given, and control $u(x)$ is supported in $[a,b]$:

$$\operatorname{supp} u \subset [a,b] \tag{108}$$

The setting of stabilization problem is as follows:

Given $y_0(x) \in H^0(T_1)$, find a control $u \in H^0(T_1)$ satisfying (108) such that there exists unique solution $y(t,x;y_0+u) \in H^{1,2(-1)}(Q)$, and this solution satisfies the estimate

$$\|y(t,\cdot;y_0+u)\|_0 \le \alpha \|y_0+u\|_0 e^{-t} \quad \forall t > 0 \tag{109}$$

with a certain $\alpha > 1$.

Note that if initial condition $y_0 \in M_-(\alpha)$ then by Definition 4.1 of the stability set $M_-(\alpha)$ the control $u \equiv 0$ is a solution of stabilization problem. In other words in this case stabilization problem is trivial. If $y_0 \in M_+$ or $y_0 \in M_I(\alpha)$ and corresponding solution $y(t,x;y_0)$ satisfies (101), then stabilization problem is reach of content.

The following main theorem holds:

Theorem 5.1. *Let $y_0 \in H^0(T_1)$ be given. Then there exists a control $u \in H^0(T_1)$ satisfying (108) such that there exists unique solution $y(t,x;y_0+u) \in H^{1,2(-1)}(Q)$ of (104)-(107) and this solution satisfies bound (109) with a certain $\alpha > 1$.*

5.2. **Formulation of the main preliminary result.** We make in (104)-(107) change of variable $\tilde{x} = x - \frac{a+b}{2}$ and denote

$$\tilde{y}(t,\tilde{x}) = y(t,\tilde{x} + \frac{a+b}{2}), \quad \tilde{y}_0(\tilde{x}) = y_0(\tilde{x} + \frac{a+b}{2}), \quad \tilde{u}(\tilde{x}) = u(\tilde{x} + \frac{a+b}{2}). \tag{110}$$

Then substitution (110) into (104)-(107), (109) yields relations (104)-(107), (109) with y, y_0, u, x changed on $\tilde{y}, \tilde{y}_0, \tilde{u}, \tilde{x}$, segment $[a,b]$ transforms to segment $[-\rho,\rho]$ with $\rho = (b-a)/2$. If now we make renaming by omitting tilde everywhere, we get again relations (104)-(107), (109) but inclusion (108) transforms to

$$\operatorname{supp} u(x) \subset [-\rho,\rho], \qquad 0 < \rho < \pi \tag{111}$$

Below we write circumference T_1 as a segment $[-\pi,\pi]$ with identified endpoints, and we consider stabilization problem (104)-(107), (109) with condition (111) instead of (108).

In fact we propose universal stabilizing start control that up to constant multiplier can be defined as follows. For a given $\rho \in (0,\pi)$ we choose $p \in \mathbb{N}$ such that

$$\frac{\pi}{2p} < \rho \tag{112}$$

and denote by $\chi(x)$ the characteristic function of interval $(-\frac{\pi}{2p}, \frac{\pi}{2p})$:

$$\chi(x) = \begin{cases} 1, & |x| \le \frac{\pi}{2p} \\ 0, & \frac{\pi}{2p} < |x| \le \pi \end{cases} \tag{113}$$

The desired control function is defined as follows:

$$u(x) = \chi(x)(\cos 2px + \cos 4px) \tag{114}$$

Let consider the boundary value problem for heat equation

$$\partial_t S(t,x) - \partial_{xx} S(t,x) = 0, \qquad S(t,x)|_{t=0} = u(x) \tag{115}$$

with periodic boundary condition.

The following theorem is true:

Theorem 5.2. *For each $\rho \in (0,\pi)$ the function $u(x)$ defined in (114) with help of natural number p satisfying (112) and characteristic function (113), satisfies conditions:*

$$u(x) \in H^0(T_1), \quad \textit{supp}\, u \subset [-\rho,\rho], \quad \text{and} \quad \widehat{u}_0 = 0,\ \widehat{u}_1 > 0 \tag{116}$$

where $\widehat{u}_k$ are Fourier coefficients of $u(x)$, and

$$\int_{-\pi}^{\pi} S^3(t,x;u)dx \ge \beta e^{-6t} \qquad \forall\, t \ge 0 \tag{117}$$

with a positive constant β

Remark 3. The proof of (116) is evident, of course, what we can not tell about (117). As well-known the kernel of resolving operator for heat equation is positive. Therefore if initial condition $u(x)$ would be nonnegative on $[-\rho,\rho]$ and positive on a set of positive Lebesgue measure then $S(t,x;u) > 0$ for each t and (117) is evidently true. But the point is that inclusion $u \in H^0(T_1) = L_2^0(T_1)$ implies $\int_{-\pi}^{\pi} u(x)dx = 0$ (see (8)). In virtue of derivation of NPE fulfilled in subsection 2.1 this condition is very essential for normal parabolic equation. Just because of

condition $\int_{-\pi}^{\pi} u(x)dx = 0$ the proof of inequality (117) is complicated and cumbersome. In Appendix below a draft of Theorem 5.2 proof is given. Complete proof of Theorem 5.2 will be published in some other place.

5.3. **Auxiliary assertions.** We begin with the following particular case of Sobolev embedding theorem.

Lemma 5.3. *Let $w^j \in H^{1/6}(T_1)$, $j = 1, 2, 3$ and $w_k^j = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-ikx} w^j(x)dx$, $k \in \mathbb{Z}$ be Fourier coefficients of w^j. Then there exists $c > 0$ such that*

$$\left| \int_{-\pi}^{\pi} \Pi_{j=1}^3 w^j(x)dx \right| \le c \sum_{m=1}^{3} \left(\sum_{k=2}^{\infty} |w_k^m|^2 k^{1/3} \right)^{1/2} \|w^{[m+1]}\|_{1/6} \|w^{[m+2]}\|_{1/2} \quad (118)$$

where for $m = 1, 2, 3$, $n = 1, 2$ $[m+n] = m+n$ if $m+n \le 3$ and $[m+n] = m+n-3$ if $m+n > 3$.

Remark 4. Additional information in comparison with Sobolev embedding theorem that we get here is presence in right side of (118) multiplier $\sum_{k=2}^{\infty} |w_k^m|^2 k^{1/3}$ where summation begins not from $k = 1$ but from $k = 2$. We get this from condition $w_0^m = 0$ for $w^m \in H^{1/6}(T_1)$.

Proof. Using decomposition of $w^j(x)$ in Fourier series we get

$$\int_{\pi}^{\pi} \Pi_{j=1}^3 w^j(x)dx = 2\pi \sum_{\substack{k_1+k_2+k_3=0 \\ k_j \ne 0}} w_{k_1}^1 w_{k_2}^2 w_{k_3}^3 = 2\pi \sum_{\substack{k_1,k_2 \in \mathbb{Z} \\ k_j \ne 0 \\ k_1+k_2 \ne 0}} w_{k_1}^1 w_{k_2}^2 \overline{w}_{k_1+k_2}^3 \quad (119)$$

where we used that $w_{-k}^j = \overline{w}_k^j$ because $w^j(x)$ is a real-valued function. Using the last equality again we obtain from (119)

$$\begin{aligned} \left| \int_{\pi}^{\pi} \Pi_{j=1}^3 w^j(x)dx \right| &\le 4\pi \left(\sum_{k_1,k_2=1}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_1+k_2}^3| \right. \\ &+ \left. \sum_{\substack{k_2,k_2=1 \\ k_1-k_2 \ne 0}}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_1-k_2}^3| \right) = 4\pi \left(\sum_{k_1,k_2=1}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_1+k_2}^3| \right. \\ &+ \left. \sum_{\substack{k_1,k_2=1 \\ k_1>k_2}}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_1-k_2}^3| + \sum_{\substack{k_1,k_2=1 \\ k_2>k_1}}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_1-k_2}^3| \right) \end{aligned} \quad (120)$$

In sum with $k_1 > k_2$ we make change of variables $(k_1, k_2) \to (m = k_1 - k_2, k_2)$, and in sum with $k_2 > k_1$ we change (k_1, k_2) on $m = k_2 - k_1, k_1$:

$$\sum_{\substack{k_1,k_2=1 \\ k_1>k_2}}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_1-k_2}^3| = \sum_{m=1}^{\infty} \sum_{k_2=1}^{\infty} |w_{m+k_2}^1||w_{k_2}^2||w_m^3|$$

$$\sum_{\substack{k_1,k_2=1 \\ k_2>k_1}}^{\infty} |w_{k_1}^1||w_{k_2}^2||w_{k_2-k_1}^3| = \sum_{m=1}^{\infty} \sum_{k_1=1}^{\infty} |w_{k_1}^1||w_{m+k_1}^2||w_m^3|$$

Hence, (120) can be rewritten as follows:

$$\begin{aligned}&\left|\int_{\pi}^{\pi}\Pi_{j=1}^{3}w^{j}(x)dx\right|\\&\leq 4\pi\sum_{k,m=1}^{\infty}\left(|w_k^1||w_m^2||w_{k+m}^3|+|w_k^1||w_{k+m}^2||w_m^3|+|w_{k+m}^1||w_k^2||w_m^3|\right)\end{aligned}\tag{121}$$

Cauchy-Bunyakovskii inequality implies

$$\begin{aligned}&\sum_{k,m=1}^{\infty}|w_k^1||w_m^2||w_{k+m}^3|\leq\left(\sum_{k=1}^{\infty}|w_k^1|^2k^{1/3}\right)^{1/2}\left(\sum_{k=1}^{\infty}k^{-1/3}\left(\sum_{m=1}^{\infty}|w_m^2||w_{k+m}^3|\right)^2\right)^{1/2}\\&\leq\|w^1\|_{1/6}\left(\sum_{k=1}^{\infty}k^{-1/3}\left(\sum_{m=1}^{\infty}m^{1/3}|w_m^2|^2\right)\left(\sum_{m=1}^{\infty}m^{-1/3}|w_{k+m}^3|^2\right)\right)^{1/2}\\&=\|w^1\|_{1/6}\|w^2\|_{1/6}\left(\sum_{k,m=1}^{\infty}k^{-1/3}m^{-1/3}|w_{k+m}^3|^2\right)^{1/2}\end{aligned}\tag{122}$$

Using change of variables $(k,m)\to(k,n=k+m)$ we get

$$\sum_{k,m=1}^{\infty}k^{-1/3}m^{-1/3}|w_{k+m}^3|^2=\sum_{n=2}^{\infty}|w_n^3|^2\sum_{k=1}^{n-1}k^{-1/3}(n-k)^{-1/3},\tag{123}$$

and by simple calculation we obtain

$$\begin{aligned}&\sum_{k=1}^{n-1}k^{-1/3}(n-k)^{-1/3}\leq\int_0^n\frac{dx}{(x(n-x))^{1/3}}=\int_{-n/2}^{n/2}\frac{dy}{((n/2)^2-y^2)^{1/3}}\\&=\left(\frac{2}{n}\right)^{2/3}\frac{n}{2}\int_{-n/2}^{n/2}\frac{d(2y/n)}{(1-(2y/n)^2)^{1/3}}=cn^{1/3}\end{aligned}\tag{124}$$

where

$$c=2^{-1/3}\int_{-1}^{1}\frac{dz}{(1-z^2)^{1/3}}$$

In virtue of estimates (122)-(124)

$$\sum_{k,m=1}^{\infty}|w_k^1||w_m^2||w_{k+m}^3|\leq c\|w^1\|_{1/6}\|w^2\|_{1/6}\left(\sum_{k=2}^{\infty}k^{1/3}|w_k^3|^2\right)^{1/2}\tag{125}$$

The second and the third summands from(121) can be estimated similarly to (125). As a result we get (118). □

Lemma 5.4. *Let $S(t,x;u)$ be the solution of problem (115) with $u\in L_2^0(T_1)$, and u_k, $k\in\mathbb{Z}$ be Fourier coefficients of u. Then*

$$\|S(t,\cdot,;u)\|_\alpha^2=2\sum_{k=1}^{\infty}|u_k|^2e^{-2k^2t}|k|^{2\alpha}\leq 2e^{-2t}\gamma_1(\alpha,t)\|u\|_0^2\tag{126}$$

$$\sum_{k=2}^{\infty}|u_k|^2e^{-2k^2t}|k|^{2\alpha}\leq e^{-8t}\gamma_2(\alpha,t)\|u\|_0^2\tag{127}$$

where

$$\gamma_1(\alpha,t)=\begin{cases} e^{2(t-\frac{\alpha}{2})}\left(\frac{\alpha}{2}\right)^{\alpha}t^{-\alpha}, & t\le \alpha/2\\ 1, & t>\alpha/2\end{cases};\ \gamma_2(\alpha,t)=\begin{cases} e^{8(t-\frac{\alpha}{8})}\left(\frac{\alpha}{2}\right)^{\alpha}t^{-\alpha}, & t\le \alpha/8\\ 4^{\alpha}, & t>\alpha/8\end{cases} \tag{128}$$

Proof. Denote $f_1(k,t)=e^{-2(k^2-1)t}|k|^{2\alpha}$, $f_2(k,t)=e^{-2(k^2-4)t}|k|^{2\alpha}$ with $k\in\mathbb{R}$. Then

$$\max_{k\ge 1} f_1(k,t)=\gamma_1(\alpha,t) \qquad \text{and} \qquad \max_{k\ge 2} f_2(k,t)=\gamma_2(\alpha,t)$$

where γ_1,γ_2 are defined in (128). This imply (126),(127). □

5.4. **Proof of the stabilization result.** In this subsection we prove Theorem 5.1. We take $\lambda u(x)$ as a desired control where $u(x)\in L_2^0(T_1)$ is the function constructed in Theorem 5.2 and $\lambda<0$, $|\lambda|\gg 1$ is a parameter. In virtue of explicit formula (53)[6] for solution of stabilization problem (104)-(107),(111),(109), in order to prove desired result it is enough to choose parameter λ such that the function

$$1-\int_0^t \Phi(S(\tau,\cdot;y_0+\lambda u))d\tau \tag{129}$$

for each $t>0$ would be bounded from below with a positive constant independent of t. For this aim we estimate function $\Phi(S(t,\cdot;y_0+\lambda u))$.

By definition (105) of Φ

$$\Phi(S(t,\cdot;y_0+\lambda u))=\lambda(\Phi(S(t,\cdot;u))+\Psi(t,\lambda,y_0,u)) \tag{130}$$

where

$$\Psi(t,\lambda,y_0,u))=\Phi(S(t,\cdot;u+y_0/\lambda))-\Phi(S(t,\cdot;u)) \tag{131}$$

In virtue of Lagrange Theorem on finite increment and (105) there exists $\theta=\theta(\lambda,t)\in(0,1)$ such that

$$\begin{aligned} &|\Psi(S(t,\lambda,y_0,u))|\le|\Phi'(S(t,\cdot;u+\theta y_0/\lambda))[S(t,\cdot,y_0/\lambda)]|\\ &=\frac{1}{|\lambda|}\left|\frac{3\int_{-\pi}^{\pi}S^2(t,x,u+\theta y_0/\lambda)\,S(t,x,y_0)dx}{\int_{-\pi}^{\pi}S^2(t,x,u+\theta y_0/\lambda)dx}\right.\\ &\left.-\frac{2\int_{-\pi}^{\pi}S^3(t,x,u+\theta y_0/\lambda)dx\int_{-\pi}^{\pi}S(t,x,u+\theta y_0/\lambda)S(t,x,y_0)dx}{\left(\int_{-\pi}^{\pi}S^2(t,x,u+\theta y_0/\lambda)dx\right)^2}\right| \end{aligned} \tag{132}$$

Denominator from right side of (132) can be estimated from below as follows:

$$\begin{aligned} &\left|\int_{-\pi}^{\pi}S^2(t,x,u+\theta y_0/\lambda)dx\right|=\left|\int_{-\pi}^{\pi}(S(t,x,u)+\frac{\theta}{\lambda}S(t,x;y_0))^2dx\right|\\ &\ge\int_{-\pi}^{\pi}\frac{1}{2}S^2(t,x,u)-\frac{1}{|\lambda|^2}S^2(t,x;y_0)dx\\ &\ge 2\pi e^{-2t}\left(u_1^2-\frac{\|y_0\|_0^2}{|\lambda|^2}\right)\ge\pi e^{-2t}u_1^2 \end{aligned} \tag{133}$$

for

$$|\lambda|>\lambda_0=\frac{\sqrt{2}\|y_0\|_0}{u_1} \tag{134}$$

[6] with initial condition y_0 replaced on y_0+u

Here u_1 is the first Fourier coefficient of function u defined in (114). To estimate numerators from right side of (132) we use Lemmas 5.3,5.4:

$$\begin{aligned}
&\left|\int_{-\pi}^{\pi} S^2(t,x,u+\theta y_0/\lambda)S(t,x;y_0)dx\right| \\
&\le c\left(\sum_{k=2}^{\infty} e^{-2k^2t}|u_k+\theta y_{0k}/\lambda|^2k^{1/3}\right)^{1/2}\|S(t,\cdot,u+\theta y_0/\lambda)\|_{1/6}\|S(t,\cdot,y_0)\|_{1/6} \\
&+\left(\sum_{k=2}^{\infty} e^{-2k^2t}|y_{0k}|^2k^{1/3}\right)^{1/2}\|S(t,\cdot,u+\theta y_0/\lambda)\|_{1/6}^2 \\
&\le ce^{-6t}\gamma_2(1/6,t)^{1/2}\gamma_1(1/6,t)\|y_0\|_0(\|u\|_0^2+\|y_0\|_0^2/|\lambda|^2)
\end{aligned} \tag{135}$$

where γ_1,γ_2 are defined in (128), and analogously

$$\left|\int_{-\pi}^{\pi} S^3(t,x,u+\theta y_0/\lambda)dx\right| \le ce^{-6t}\gamma_2(1/6,t)^{1/2}\gamma_1(1/6,t)(\|u\|_0^3+\|y_0\|_0^3/|\lambda|^3) \tag{136}$$

Besides, to bound the numerator from (132) we need in the estimate:

$$\begin{aligned}
&\int_{-\pi}^{\pi} S(t,x,u+\theta y_0/\lambda)S(t,x;y_0)dx \\
&\le \left(\int_{-\pi}^{\pi} S^2(t,x,u+\theta y_0/\lambda)dx\right)^{1/2}\left(\int_{-\pi}^{\pi} S^2(t,x;y_0)dx\right)^{1/2} \\
&\le c\left(\int_{-\pi}^{\pi} S^2(t,x,u+\theta y_0/\lambda)dx\right)^{1/2} e^{-t}\|y_0\|_0
\end{aligned} \tag{137}$$

Getting together estimates (132)-(137) we obtain that if $|\lambda|$ satisfies (134) then

$$\begin{aligned}
&|\lambda||\Psi(t,\lambda,y_0,u)| \le c\left(\frac{e^{-6t}\gamma_2(1/6,t)^{1/2}\gamma_1(1/6,t)\|y_0\|_0(\|u\|_0^2+\|y_0\|_0^2/|\lambda|^2)}{e^{-2t}u_1^2}\right. \\
&\left.+\frac{e^{-6t}\gamma_2(1/6,t)^{1/2}\gamma_1(1/6,t)e^{-t}\|y_0\|_0(\|u\|_0^3+\|y_0\|_0^3/|\lambda|^3)}{e^{-3t}u_1^3}\right) \\
&\le ce^{-4t}\|y_0\|_0\gamma_2(1/6,t)^{1/2}\gamma_1(1/6,t)\left(\frac{\|u\|_0^3}{u_1^3}+\frac{\|u\|_0^2}{u_1^2}+1\right)
\end{aligned} \tag{138}$$

where c does not depend on t, y_0, and $|\lambda|>\lambda_0$. Recall that u here is fixed function (114).

Estimate now function (129). In virtue of (130),(138), and (117)

$$\begin{aligned}
&1-\int_0^t \Phi(S(\tau,\cdot;y_0+\lambda u))d\tau \\
&\ge 1+\int_0^t \frac{\beta|\lambda|e^{-6\tau}}{e^{-2\tau}\|u\|_0^2}d\tau - C(\|y_0\|_0,\|u\|_0/u_1)\int_0^t e^{-4\tau}\gamma_2(1/6,\tau)^{1/2}\gamma_1(1/6,\tau)d\tau \\
&=\left(1-C(\|y_0\|_0,\|u\|_0/u_1)\int_0^t e^{-4\tau}\gamma_2(1/6,\tau)^{1/2}\gamma_1(1/6,\tau)d\tau\right)+\frac{|\lambda|\beta}{4\|u\|_0^2}\left(1-e^{-4t}\right)
\end{aligned} \tag{139}$$

The first term in right side of (139) is a function bounded from below with a constant (possibly, negative), and on a certain time interval $(0,t_0)$ with $t_0>0$ this function

monotonically decreases from 1 to 0. Hence, for sufficiently large $|\lambda|$ right side of (139) will be bounded from below with positive constant. Therefore (53) with y_0 replaced on $y_0 + u$ imply (109) with a certain $\alpha > 1$. This complete the proof of Theorem 5.1.

6. **Appendix. Draft of Theorem 5.2 proof.** In this section we outline very briefly the way that leads to proof of Theorem 5.2

6.1. **Fourier coefficients of control function (114).** Recall that function (114) is as follows:

$$u(x) = \chi(x)(\cos 2px + \cos 4px) \tag{140}$$

where

$$\chi(x) = \begin{cases} 1, & |x| \le \frac{\pi}{2p} \\ 0, & \frac{\pi}{2p} < |x| \le \pi \end{cases} \tag{141}$$

is the characteristic function of interval $(-\frac{\pi}{2p}, \frac{\pi}{2p})$ with arbitrary fixed $p \in \mathbb{N}$.

Our first aim is to calculate Fourier coefficients $\widehat{u}_k$ of function (140). Fourier coefficients of function (141) are as follows:

$$\widehat{\chi}_k = \frac{1}{2\pi} \int_{-\frac{\pi}{2p}}^{\frac{\pi}{2p}} e^{-ikx} dx = \frac{1}{k\pi} \sin \frac{k\pi}{2p}, \text{ for } k \neq 0; \quad \widehat{\chi}_0 = \frac{1}{2p} \tag{142}$$

Denote

$$\psi(x) = \frac{1}{2} \sum_{m=1}^{2} (e^{2pmix} + e^{-2pmix}) \equiv \sum_{n \in \mathbb{Z}} \widehat{\psi}_n e^{inx}$$

Then

$$\widehat{u}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} \psi(x) \chi(x) dx = \sum_n \widehat{\psi}_n \widehat{\chi}_{k-n} = \frac{1}{2} \sum_{m=1}^{2} (\widehat{\chi}_{k-2pm} + \widehat{\chi}_{k+2pm})$$

This relation and (142) imply that

$$\widehat{u}_{2p} = \widehat{u}_{-2p} = \widehat{u}_{4p} = \widehat{u}_{-4p} = 1/(4p) \tag{143}$$

and for $k \neq \pm 2p \pm 4p$

$$\widehat{u}_k = \frac{k}{\pi} \sin \frac{k\pi}{2p} \sum_{m=1}^{2} \frac{(-1)^m}{k^2 - (2pm)^2} = \frac{12p^2 k \sin \frac{k\pi}{2p}}{\pi(k^2 - 16p^2)(k^2 - 4p^2)} \tag{144}$$

Now condition (116) from Theorem 5.2 can be derived easily from (144).

Let consider the boundary value problem for heat equation

$$\partial_t S(t,x,u) - \partial_{xx} S(t,x,u) = 0, \qquad S(t,x,u)|_{t=0} = u(x) \tag{145}$$

with periodic boundary condition and with function (140) as initial condition. In next subsection we study this function.

6.2. **Properties of solution to problem (145).** Since (116) has been proved already, to prove Theorem 5.2 we have to verify inequality (117). In virtue of (143),(144) solution $S(t,x,u)$ of problem (145) can be rewritten as follows:

$$S(t,x,u) = S_1(t,x,u) + S_2(t,x,u) \tag{146}$$

where

$$S_1(t,x,u) = \frac{1}{4p}\sum_{m=1}^{2}(e^{2pmix} + e^{-2pmix})e^{-(2pm)^2 t} \tag{147}$$

$$S_2(t,x,u) = \frac{12p^2}{\pi}\sum_{\substack{k\in\mathbb{Z}\\ k\neq\pm 2p\pm 4p}}\frac{k\sin\frac{k\pi}{2p}}{(k^2-16p^2)(k^2-4p^2)}e^{ikx-k^2 t} \tag{148}$$

By (146) we get

$$\begin{aligned}&\int_{-\pi}^{\pi} S^3(t,x,u)dx\\ &= \int_{-\pi}^{\pi}(S_1^3(t,x,u) + 3S_1^2(t,x,u)S_2(t,x,u) + 3S_1(t,x,u)S_2^2(t,x,u) + S_2^3(t,x,u))dx\end{aligned} \tag{149}$$

and we have to prove positiveness of integrals from (149). We begin from the integral connected with the first term in right side of (149).Denoting by $c_{\pm 2pm}$ coefficient before exponent $e^{\pm 2pmix}$ in (147) we obtain:

$$\int_{-\pi}^{\pi} S_1^3(t,x,u)dx = 2\pi(c_{2p}^2 c_{-4p} + c_{-2p}^2 c_{4p}) = \frac{\pi e^{-24p^2 t}}{16p^3} \geq 0 \tag{150}$$

Let prove that

$$\int_{-\pi}^{\pi} S_1^2(t,x,u)S_2(t,x,u)dx = 0 \tag{151}$$

Indeed, by (147) only Fourier coefficients of $S_1^2(t,x,u)$ with numbers multiple to $2p$ are not equal to zero, and by (148) Fourier coefficients of $S_2(t,x,u)$ with numbers multiple to $2p$ are equal to zero. Therefore zero Fourier coefficient of integrand $S_1^2(t,x,u)S_2(t,x,u)$ equal to zero, and we get (151).

Investigation of third and forth terms from right side of (149) is much more complicate, and below we give only its short draft.

6.3. **Positiveness of the third term from right side of (149).** Our aim now is to prove

Theorem 6.1. *The following inequality is true:*

$$3\int_{-\pi}^{\pi} S_1(t,x,u)S_2^2(t,x,u)dx > 0$$

where S_1, S_2 are defined in (147), (148).

Rename for briefness Fourier coefficients from (147), (148) as follows:

$$S_1(t,x,u) = \sum_{m=1}^{2}(a_{2pm}e^{2pmix} + a_{-2pm}e^{-2pmix}), \quad S_2(t,x,u) = \sum_{k\neq\pm 2p\pm 4p} b_k e^{ikx} \tag{152}$$

Since by (147), (148) $a_{2pm} = a_{-2pm}, b_k = b_{-k}$, we get

$$\int_{-\pi}^{\pi} S_1(t,x,u)S_2^2(t,x,u)dx = 2\pi a_{2p}\left(\sum_{\substack{k\neq\pm 2p\pm 4p\\ k-2p\neq\pm 2p\pm 4p}} b_k b_{k-2p} + \sum_{\substack{k\neq\pm 2p\pm 4p\\ -k-2p\neq\pm 2p\pm 4p}} b_k b_{k+2p}\right)$$
$$+ 2\pi a_{4p}\left(\sum_{\substack{k\neq\pm 2p\pm 4p\\ k-4p\neq\pm 2p\pm 4p}} b_k b_{k-4p} + \sum_{\substack{k\neq\pm 2p\pm 4p\\ -k-4p\neq\pm 2p\pm 4p}} b_k b_{k+4p}\right)$$
$$= 4\pi a_{2p}\sum_{\substack{k\neq\pm 2p\pm 4p\\ k-2p\neq\pm 2p\pm 4p}} b_k b_{k-2p} + 4\pi a_{4p}\sum_{\substack{k\neq\pm 2p\pm 4p\\ k-4p\neq\pm 2p\pm 4p}} b_k b_{k-4p} \equiv I_1 + I_2 \tag{153}$$

where to get the second equality we change variables $k \to k-2p$ in the second term and $k \to k-4p$ in the forth term. Besides, by definition I_1 from right side of (153) is the first sum in the left part of the last equality and I_2 is the second sum. Returning from (152) to notations of (147), (148) we rewrite the last equality from (153) as follows:

$$I_1 = \frac{3^2 4^2 p^3}{\pi}\sum_{\substack{k\in\mathbb{Z}\\ k\neq\pm 2p\pm 4p\\ k-2p\neq\pm 2p\pm 4p}} \frac{-k(k-2p)\sin^2\frac{\pi k}{2p}e^{-(k^2+(k-2p)^2+4p^2)t}}{(k^2-16p^2)(k^2-4p^2)((k-2p)^2-16p^2)(((k-2p)^2-4p^2)} \tag{154}$$

$$I_2 = \frac{3^2 4^2 p^3}{\pi}\sum_{\substack{k\in\mathbb{Z}\\ k\neq\pm 2p\pm 4p\\ k-4p\neq\pm 2p\pm 4p}} \frac{k(k-4p)\sin^2\frac{\pi k}{2p}e^{-(k^2+(k-4p)^2+16p^2)t}}{(k^2-16p^2)(k^2-4p^2)((k-4p)^2-16p^2)((k-4p)^2-4p^2)} \tag{155}$$

If we make in (154) change $k = n+p$, the function under sign of summation will be even with respect to n and therefore we will be able to pass to summation on $n \in \mathbb{N}$. As a result we get

$$I_1 = \frac{2^4 3^2 p^3}{\pi}\left(\frac{e^{-10p^2 t}}{3^4 5^2 p^6} + 2\sum_{n=1}^{\infty}\frac{(p^2-n^2)\cos^2\frac{\pi n}{2p}e^{-(2n^2+12p^2)t}}{((n^2-15p^2)^2-4p^2n^2)((n^2-3p^2)^2-4p^2n^2)}\right) \tag{156}$$

Similarly, doing in (155) change $k = n+2p$ we obtain

$$I_2 = \frac{2^5 3^2 p^3}{\pi}\sum_{n=1}^{\infty}\frac{(n^2-4p^2)\sin^2\frac{\pi n}{2p}e^{-(2n^2+24p^2)t}}{((n^2-12p^2)^2-16p^2n^2)(n^4-16p^2n^2)} \tag{157}$$

The following assertions hold:

Proposition 1. *i)Summands in sum (156) are positive when $n \in [1,p)\cup(p,3p)\cup(3p,5p)$ and are negative for $n > 5p$.*

ii)Summands in sum (157) are positive when $n \in [1,2p)\cup(2p,4p)\cup(6p,\infty)$ and are negative for $n \in (4p,6p)$.

This proposition can be verified by direct calculations.

For natural a, b or for $a \in \mathbb{N}, b = \infty$ we denote

$$I(a,b) = \sum_{k=a}^{b} c_k, \quad \text{where} \quad I = \sum_{k=0}^{\infty} c_k$$

Lemma 6.2. *The following inequality holds:*

$$I_2(6p+1,\infty)+I_1(7p+1,\infty)>0, \quad I_1(3p+1,7p-1)>0$$
$$I_2(1,2p-1)+I_2(4p+1,6p-1)>0$$

Lemma 6.2 can be proved by some algebraic transformations and calculations. Now Theorem 6.1 follows from Proposition 1 and Lemma 6.2.

6.4. **Positiveness of the forth term from right side of (149).** In this subsection we show briefly the way to study the most difficult forth term from (149). Our aim is

Theorem 6.3. *The following inequality is true:*

$$\int_{-\pi}^{\pi} S_2^3(t,x,u)dx > \beta e^{-6t} > 0 \tag{158}$$

where $S_2(t,x,u)$ is defined in (148), and $\beta>0$ is a constant.

First of all we find expression for left part of (158). Denote

$$B(k)=\frac{k\sin\frac{\pi k}{2p}}{(k^2-16p^2)(k^2-4p^2)} \tag{159}$$

Then substitution (148) into the left part of (158) yields:

$$\int_{-\pi}^{\pi} S_2^3(t,x,u)dx = \frac{2^7 3^3 p^3}{\pi^2}\sum_{\substack{k,m\in\mathbb{Z}\setminus\{\pm 2p,\pm 4p\}\\ k+m\neq\pm 2p,\pm 4p}} B(k)B(m)B(k+m)e^{-(k^2+m^2+(k+m)^2)t} \tag{160}$$

Since $B(k)$ is even function under k one can prove that (160) is equivalent to

$$\int_{-\pi}^{\pi} S_2^3(t,x,u)dx = \frac{2^8 3^4 p^3}{\pi^2}\sum_{\substack{k,m\in\mathbb{N}\setminus\{2p,4p\}\\ k+m\neq 2p,4p}} B(k)B(m)B(k+m)e^{-(k^2+m^2+(k+m)^2)t} \tag{161}$$

Find now distribution of signs for summands in the sum from right side of equality (161). For this we need in the following direct consequence of formula (159)

Lemma 6.4. *Let $B(k), k\in\mathbb{N}\setminus\{2p,4p\}$ be function (159). Then*

$$\begin{aligned} &i)\ B(k)=0 \quad \text{for} \quad k=2pl \quad \text{where} \quad l\in\mathbb{N}, l\geq 3\\ &ii)\ B(k)>0 \quad \text{for} \quad k\in(0,2p)\cup(2p,4p)\cup\{\cup_{l\in\mathbb{N}}(4lp,(4l+2)p)\\ &iii)\ B(k)<0 \quad \text{for} \quad \cup_{l\in\mathbb{N}}((4l+2)p,4(l+1)p) \end{aligned} \tag{162}$$

One can deduce from Lemma 6.4 the following assertion.

Lemma 6.5. *The sign of function $B(k)B(m)B(k+m)$ from (161) is distributed as follows:*

i) In each square $\{(k,m)\in(2pa,2p(a+1))\times(2pb,2p(b+1))\}$ with $a,b\in\mathbb{N}\setminus\{1\}$

$$\operatorname{sign}(B(k)B(m)B(k+m))=\begin{cases}+, & \text{if } k+m<2p(a+b+1)\\ -, & \text{if } k+m>2p(a+b+1)\end{cases} \tag{163}$$

ii) In each set $\{(k,m)\in(2pa,2p(a+1))\times(0,2p)\cup(0,2p)\times(2pa,2p(a+1))\}$ with $a\in\mathbb{N}\setminus\{1\}$

$$\operatorname{sign}(B(k)B(m)B(k+m))=\begin{cases}+, & \text{if } k+m<2p(a+1)\\ -, & \text{if } k+m>2p(a+1)\end{cases} \tag{164}$$

iii) In each square $\{(k,m) \in (2pa, 2p(a+1)) \times (2p,4p) \cup (2p,4p) \times (2pa, 2p(a+1))\}$ *with* $a \in \mathbb{N} \setminus \{1\}$

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \begin{cases} -, & \text{if } k+m < 2p(a+2) \\ +, & \text{if } k+m > 2p(a+2) \end{cases} \tag{165}$$

iv) In the square $\{(k,m) \in (0,4p) \times (0,4p)\}$

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \begin{cases} +, & \text{if } k+m < 6 \\ -, & \text{if } k+m > 6p \end{cases} \tag{166}$$

Denote by J the sum from right side of (161). Our last aim is to prove the positiveness of J. We denote the parts of sum J with summands indicated in points i),ii),iii),iv) by $J(i), J(ii), J(iii), J(iv)$ correspondingly. It is possible to prove positiveness all of these sums with help of detailed investigation of function (159) properties, straightforward but complicate algebraic transformations, and cumbersome arithmetical calculations. Complete proof of this fact will be published in some other place. Theorem 5.2 follows from (149),(150),(151), and Theorems 6.1,6.3.

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