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SOME NEW BOUNDS FOR TWO MAPPINGS RELATED TO THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. Some new results concerning two mappings associated to the celebrated Hermite-Hadamard integral inequality for convex function with applications for special means are given.

1. Introduction.

The Hermite-Hadamard integral inequality for convex functions $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}\tag{HH}$$

is well known in the literature and has many applications for special means.

In order to provide various refinements of this result, the first author introduced in 1991, see [2], the following associated mapping $H:[0,1]\to\mathbb{R}$ defined by

$$H(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

for a given convex function $f:[a,b]\to\mathbb{R}$.

Some of the main properties of H are as follows (see also [2], [3], [4] and [8]):

1. H is convex on $[0,1]$;
2. One has the bounds:

$$\inf_{t\in[0,1]}H(t)=H(0)=f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]}H(t)=H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx;$$

3. H increases monotonically on $[0,1]$;

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4. The following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \le \int_0^1 H(t)\,dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right]. \tag{1}$$

The corresponding double integral mapping in connection with the Hermite-Hadamard inequalities was considered first in [3] and is defined as

$$F:[0,1]\to\mathbb{R},\, F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy.$$

Some of the main results concerning this mapping [3] (see also [4]) are as follows:

1. $F\left(\tau+\frac{1}{2}\right)=F\left(\frac{1}{2}-\tau\right)$ for all $\tau\in\left[0,\frac{1}{2}\right]$ and $F(t)=F(1-t)$ for all $t\in[0,1]$;
2. F is convex on $[0,1]$;
3. We have the bounds:

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy;$$

4. The following inequality holds:

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right);$$

5. F decreases monotonically on $\left[0,\frac{1}{2}\right]$ and increases monotonically on $\left[\frac{1}{2},1\right]$;
6. We have the inequality:

$$H(t) \le F(t) \text{ for all } t\in[0,1].$$

For other related results, see for instance the research papers [1], [10], [11], [12], [14], [13], [15], [16], [17], the monograph online [9] and the references therein.

In the recent paper [7] we proved the following result where upper and lower bounds for the associated functions

$$\frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)\,f\left(\frac{a+b}{2}\right) - H(t)$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,dx - F(t)$$

with $t\in[0,1]$, have been given.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned} 0 &\le 2\min\{t,1-t\} \\ &\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right] \\ &\le \frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)\,f\left(\frac{a+b}{2}\right)-H(t) \\ &\le 2\max\{t,1-t\} \\ &\times\left[\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+f\left(\frac{a+b}{2}\right)\right]-\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\right] \end{aligned} \tag{2}$$

and

$$\begin{aligned} 0 &\le 2\min\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)\,dx-F(t) \\ &\le 2\max\{t,1-t\}\left[\frac{1}{b-a}\int_a^b f(x)\,dx-F\left(\frac{1}{2}\right)\right], \end{aligned} \tag{3}$$

for any $t\in[0,1]$.

Motivated by the above results we establish in this paper some new bounds involving these two mappings. Applications for special means are also provided.

2. **The Results.**

The following result holds:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned} &\frac{t}{b-a}\int_a^b f(x)\,dx+(1-t)\,f\left(\frac{a+b}{2}\right)-H(t) \\ &\le t(1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\,dx-F(t) \\ &\le 2t(1-t)\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned} \tag{5}$$

for any $t\in[0,1]$.

Proof. Since the class of convex differentiable functions is dense in the uniform topology in the class of all convex functions defined on the interval $[a,b]$, we can assume that f is differentiable on (a,b).

Utilising the convexity of the function we can write the gradient inequality

$$f(u)-f(v)\ge f'(v)(u-v) \tag{6}$$

for any $u, v \in (a, b)$.

On making use of (6) we have

$$f(tx + (1-t)y) - f(x) \geq (1-t) f'(x)(y-x) \tag{7}$$

and

$$f(tx + (1-t)y) - f(y) \geq -t f'(y)(y-x) \tag{8}$$

for any $x, y \in (a, b)$ and $t \in (0, 1)$.

Now, if we multiply (7) by t and (8) by $1-t$, with $t \in (0, 1)$, and add the obtained inequalities, we get

$$\begin{aligned} & f(tx + (1-t)y) - t f(x) - (1-t) f(y) \\ & \geq t(1-t)[f'(x) - f'(y)](y-x), \end{aligned}$$

which is equivalent with

$$\begin{aligned} & t f(x) + (1-t) f(y) - f(tx + (1-t)y) \\ & \leq t(1-t)[f'(y) - f'(x)](y-x) \end{aligned} \tag{9}$$

for any $x, y \in (a, b)$ and $t \in (0, 1)$.

If we choose $y = \frac{a+b}{2}$ in (9) and integrate over x on $[a, b]$ then we get

$$\begin{aligned} & t \int_a^b f(x)\,dx + (1-t) f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx \\ & \leq t(1-t) \int_a^b \left[f'(x) - f'\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right) dx, \end{aligned} \tag{10}$$

which holds for any $t \in [0, 1]$ (we notice that for either $t = 0$ or $t = 1$, the inequality reduces to the equality $0 = 0$).

Now, observe that

$$\begin{aligned} & \int_a^b \left[f'(x) - f'\left(\frac{a+b}{2}\right)\right]\left(x - \frac{a+b}{2}\right) dx \\ & = \int_a^b f'(x)\left(x - \frac{a+b}{2}\right) dx \\ & = f(x)\left(x - \frac{a+b}{2}\right)\Big|_a^b - \int_a^b f(x)\,dx \\ & = \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(x)\,dx, \end{aligned}$$

which by (10) produces the desired result (4).

Further, if we integrate the inequality (9) over x and y on $[a, b]$ we have

$$\begin{aligned} & t(b-a) \int_a^b f(x)\,dx + (1-t)(b-a) \int_a^b f(y)\,dy \\ & - \int_a^b \int_a^b f(tx + (1-t)y)\,dxdy \\ & \leq t(1-t) \int_a^b \int_a^b [f'(y) - f'(x)](y-x)\,dxdy \end{aligned} \tag{11}$$

for any $t \in [0, 1]$.

Observe that

$$\begin{aligned}
&\int_a^b \int_a^b \left[f'(y) - f'(x)\right](y-x)\,dxdy \\
&= (b-a)\int_a^b f'(x)\,xdx + (b-a)\int_a^b f'(y)\,ydy \\
&- \int_a^b f'(x)\,dx \int_a^b ydy - \int_a^b xdx \int_a^b f'(y)\,dy \\
&= 2\left[(b-a)\int_a^b f'(x)\,xdx - \frac{b^2-a^2}{2}\left(f(b)-f(a)\right)\right] \\
&= 2(b-a)\int_a^b f'(x)\left(x - \frac{a+b}{2}\right)dx \\
&= 2(b-a)\left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)\,dx\right],
\end{aligned}$$

and by (11) we get the desired result (5). □

Remark 1. By replacing t with $1-t$ in (4), adding the obtained results and dividing by 2 we get the symmetric inequality

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\right] - \frac{H(t)+H(1-t)}{2} \qquad (12)\\
&\leq t(1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]
\end{aligned}$$

for any $t \in [0,1]$.

Since it is known that for convex functions whose lateral derivatives $f'_+(a)$ and $f'_-(b)$ are finite we have the inequality

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{8}\left[f'_-(b) - f'_+(a)\right](b-a)$$

with the sharp constant $\frac{1}{8}$ (see [5]), hence we can obtain the simpler, however coarser upper bounds as follows:

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on the interval $[a,b]$. If the lateral derivatives $f'_+(a)$ and $f'_-(b)$ are finite, then we have*

$$\begin{aligned}
&\frac{t}{b-a}\int_a^b f(x)\,dx + (1-t)\,f\left(\frac{a+b}{2}\right) - H(t) \qquad (13)\\
&\leq \frac{1}{8}t(1-t)\left[f'_-(b) - f'_+(a)\right](b-a)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx - F(t) \qquad (14)\\
&\leq \frac{1}{8}t(1-t)\left[f'_-(b) - f'_+(a)\right](b-a)
\end{aligned}$$

for any $t \in [0,1]$.

Corollary 2. *Under the assumptions of Corollary 1, we have*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + f\left(\frac{a+b}{2}\right)\right] - H\left(\frac{1}{2}\right) \qquad (15)\\
&\leq \frac{1}{4}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]\\
&\leq \frac{1}{32}\left[f'_-(b) - f'_+(a)\right](b-a)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)\,dx - F\left(\frac{1}{2}\right) \qquad (16)\\
&\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right]\\
&\leq \frac{1}{16}\left[f'_-(b) - f'_+(a)\right](b-a).
\end{aligned}$$

Remark 2. We observe that the first inequality in (15) is equivalent with

$$\frac{3}{4}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{2}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{8} \leq H\left(\frac{1}{2}\right),$$

while the first inequality in (16) is equivalent with

$$\frac{3}{2}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{4} \leq F\left(\frac{1}{2}\right).$$

3. **Applications for L_p-means.** Let us consider the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^p$, $p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $0<a<b$. Define the mapping

$$H_p(t) := \frac{1}{b-a}\int_a^b (tx+(1-t)A(a,b))^p\,dx,\ t\in[0,1].$$

It is obvious that $H_p(0)=A^p(a,b)$, $H_p(1)=L_p^p(a,b)$ where, we recall that $A(a,b)=\frac{a+b}{2}$,

$$L_p^p(a,b) := \frac{1}{p+1}\frac{b^{p+1}-a^{p+1}}{b-a},\ p\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$$

and for $t\in(0,1)$ we have

$$\begin{aligned}
H_p(t) &= \frac{1}{[tb+(1-t)A(a,b)]-[ta+(1-t)A(a,b)]}\int_{ta+(1-t)A(a,b)}^{tb+(1-t)A(a,b)} y^p\,dy \qquad (17)\\
&= L_p^p(ta+(1-t)A(a,b), tb+(1-t)A(a,b)).
\end{aligned}$$

Now, consider the function

$$F_p(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b (tx+(1-t)y)^p\,dxdy.$$

We observe that $F_p(1) = F_p(0) = L_p^p(a,b)$ and for $t \in (0,1)$ we have

$$F_p(t) = \frac{1}{b-a}\int_a^b \left(\frac{1}{b-a}\int_a^b (tx+(1-t)y)^p\,dx\right)dy \tag{18}$$
$$= \frac{1}{b-a}\int_a^b \left(\frac{1}{[tb+(1-t)y]-[ta+(1-t)y]}\int_{ta+(1-t)y}^{tb+(1-t)y} s^p ds\right)dy$$
$$= \frac{1}{b-a}\int_a^b L_p^p(ta+(1-t)y, tb+(1-t)y)\,dy.$$

We can calculate the double integral

$$F_p\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\frac{x+y}{2}\right)^p dxdy$$
$$= \begin{cases} \frac{4}{(b-a)^2(p+1)(p+2)}\left[b^{p+2} - 2\left(\frac{b+a}{2}\right)^{p+2} + a^{p+2}\right] & p \neq -2, \\ \frac{8}{(b-a)^2}\ln\left(\frac{A(a,b)}{G(a,b)}\right) & p = -2 \end{cases}$$

for $p \neq -1$, where $G(a,b)$ denotes the geometric mean of a, b (see [7]).

We can state the following result:

Proposition 1. *We have the following inequalities:*

$$tL_p^p(a,b) + (1-t)A^p(a,b) - H_p(t) \tag{19}$$
$$\leq t(1-t)\left[A(a^p,b^p) - L_p^p(a,b)\right]$$

and

$$L_p^p(a,b) - F_p(t) \leq 2t(1-t)\left[A(a^p,b^p) - L_p^p(a,b)\right] \tag{20}$$

for any $t \in [0,1]$.

In particular, for $t = \frac{1}{2}$ we get

$$A\left(L_p^p(a,b), A^p(a,b)\right) - L_p^p(A(a,A(a,b)), A(A(a,b),b)) \tag{21}$$
$$\leq \frac{1}{4}\left[A(a^p,b^p) - L_p^p(a,b)\right]$$

$$L_p^p(a,b) - F_p\left(\frac{1}{2}\right) \leq \frac{1}{2}\left[A(a^p,b^p) - L_p^p(a,b)\right]. \tag{22}$$

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