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DETERMINATION OF MOTION FROM ORBIT IN THE THREE-BODY PROBLEM

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Abstract. We discuss the equal mass three-body motion in which the shape of the orbit is given. The conservation of the center of mass and a constant of motion (the total angular momentum or the total energy) leads to the uniqueness of the equal mass three-body motion in given some sorts of orbits. Although the proof was already published on an article by the present authors in 2009, here we give some complementary explanations. We show that, even in the unequal mass three-body periodic motions in which each of bodies draws its own orbit, the shape of the orbits, conservation of the center of mass and a constant of motion provide some candidates of the motion of three bodies. The reality of the motion should be tested whether the equation of motion is satisfied or not. Even if the three bodies draw unclosed orbits, we can show that similarly.

1. **Introduction.** The shape of the orbit in periodic planar motion of the equal mass N bodies attracts attention because of its beautifulness [1]. The eight shaped orbit in the equal mass three-body figure-eight choreography is so simple and beautiful that it is unforgettable, especially. The three-body figure-eight choreography is a planar motion of equal mass three bodies chasing each other with vanishing total angular momentum in an eight-shaped curve under the Newtonian gravity. It was found numerically by Moore [2] and its existence was proved by Chenciner and Montgomery [3]. Nobody yet knows the exact form of the figure-eight orbit. It is known that the orbit cannot be expressed by algebraic curves of order $4, 6, 8$ in a numerical analysis [4, 5].

The present authors felt doubt the unknown orbit to be the lemniscate and investigated that the planar three-body figure-eight choreography on the lemniscate in 2003 [6]. They showed that the three-body figure-eight choreography on the lemniscate was not realized under the Newtonian potential but realized under an inhomogeneous potential. In short, the lemniscate is not Moore, Chenciner and Montgomery's figure-eight curve. Several years later they proved that no quartic curve supports any figure-eight orbit in the planar equal mass three-body problem under the homogeneous potential [7].

When they were going to prove it, one of the authors (T.F.) asked a question: "Are the positions of bodies 1 and 2 determined uniquely if we put body 3 on the

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lemniscate?". See Figure 1 to image his question. Put body 3 somewhere on the

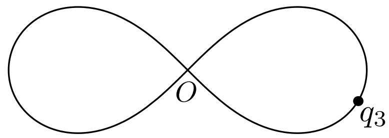

FIGURE 1. Put body 3 as you like on the lemniscate and let the position be q_3. Find the positions q_1, q_2 of bodies 1 and 2 so that the center of mass being at the origin O. There may be several configurations to satisfy $q_1 + q_2 + q_3 = 0$.

lemniscate. Then the other positions being the center of mass at the origin are determined as the solutions of the following equations:

$$F(q_1) = F(q_2) = F(q_3) = 0 \text{ such that } q_1 + q_2 + q_3 = 0,$$

where $F(q) = (x^2 + y^2)^2 - (x^2 - y^2)$ for $q = (x, y)$. One of the authors (H.O.) first tried to solve numerically for a fixed position of body 3. He got real and complex number solutions for a specific position of body 3, but his approach was not at all hopeful to answer the question properly.

In summer vacation in 2008, H.O. had a chance to exercise 'Cascade': most fundamental skill in the three-ball juggling such that the three-ball motion just looks like the three-body choreography. He struggled to master 'Cascade', but it was helpful, during exercises, to forget the problem of the uniqueness of the equal mass three-body configuration. When he faced the problem directly again, he noticed that the solutions of the equations are given by moving body 1 along the lemniscate for a fixed position of body 3:

$$F(q_3) = F(q_1) = 0 \text{ ; } q_2 \text{ is determined by } q_1 + q_2 + q_3 = 0.$$

If body 1 is located at an irrelevant position, the position of body 2 will be found either inside or outside of the lemniscate (see Figure 2 as an example). Only when

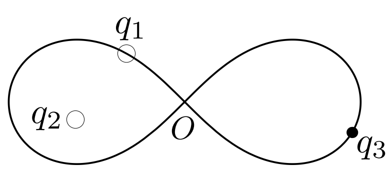

FIGURE 2. An irrelevant position of body 1 gives the position of body 2 inside of the lemniscate. The full circle is the position of body 3 and hollow circles mark positions of two bodies so that the center of mass being at the origin O.

body 1 is located at just the position, so will be body 2. It is sufficient to pass on to constructions to look over all possible positions of body 2, for the fixed position of body 3, by moving body 1 along the whole lemniscate.

On August 29, 2008, H.O. revealed a hidden lemniscate translated by $-q_3$ as a tentative orbit of body 2, and confirmed that intersection points of the original and translated lemniscates are the solutions of $F(q_1 + q_3) = F(q_1) = 0$. See Figure 3.

Thus he made sure that the configuration of three bodies was determined uniquely every instant. His approach to the configuration problem was sophisticated by H.F.

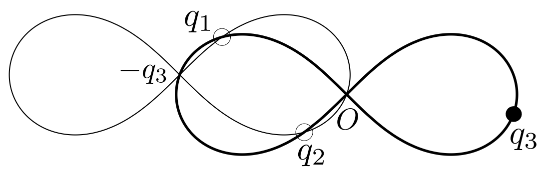

FIGURE 3. For a given position of body 3, the other positions of bodies 1 and 2 being the center of mass of three bodies at the origin O are the intersection points (hollow circles) of the original lemniscate and the lemniscate translated by $-q_3$.

and T.F. to form a theorem called 'constructions of three points' or 'the three points theorem'. It was applicable to point symmetric closed convex curves and to the figure-eight curves in which Moore, Chenciner and Montgomery's figure-eight curve is included. In this note, we will give some complementary explanations to our recent paper [8] and discuss the unequal mass three-body periodic motion and the motion in unclosed orbits.

2. **Constructions of three points.** Let us observe how the condition for the center of mass being at the origin,

$$q_1 + q_2 + q_3 = 0, \tag{1}$$

determines the mutual positions of the three bodies. Consider a circle in a plane, and take a position q_3 on the circle. We know that the set of two points $\{q_1, q_2\}$ on the circle that satisfies (1) is located with 120° separation for q_3. If equal mass three-bodies are on the q_1, q_2, and q_3 and they chase each other on the circle with the same speed, the total angular momentum determines the speed of three bodies.

Even for some curves which are invariant under the inversion $q \mapsto -q$, we can determine the position of two points for a given position of q_3 so that the center of mass being at the origin.

Theorem 2.1 (three points)**.** *If a curve γ in $\mathbb{R}^d$ with $d = 2, 3, 4, \ldots$ is invariant under the inversion $q \mapsto -q$ then the set $\{\{q_1, q_2\} | q_1, q_2 \in \gamma, q_1 + q_2 + q_3 = 0\}$ for a given $q_3 \in \gamma$ is equal to the set $\{\{q, q^*\} | q \in \gamma \cap \gamma_{\|}\}$ where $\gamma_{\|}$ is the parallel translation $q \mapsto q - q_3$ of the curve γ and $q^* = -q - q_3$.*

Although Theorem 2.1 was proved by the present authors [8], we will give another proof here. It is convenient to introduce a function Γ of $q \in \mathbb{R}^d$ that characterizes the curve γ, such that,

$$\Gamma(q) = 0 \iff q \in \gamma.$$

Then, the invariance under the inversion $q \to -q$ of the curve γ is expressed by

$$\Gamma(q) = 0 \iff \Gamma(-q) = 0. \tag{2}$$

Proof. If there is a solution $q_1, q_2 \in \gamma$ with $q_1 + q_2 + q_3 = 0$, these points satisfy

$$\Gamma(q_1) = \Gamma(q_2) = 0.$$

Using (1) and the symmetry (2), we get

$$\Gamma(q_1 + q_3) = \Gamma(-q_2) = \Gamma(q_2) = 0,$$
$$\Gamma(q_2 + q_3) = \Gamma(-q_1) = \Gamma(q_1) = 0.$$

The equations show that the points q_1 and q_2 satisfy the equations

$$\Gamma(q) = 0 \text{ and } \Gamma(q + q_3) = 0. \tag{3}$$

Namely, $q_1, q_2 \in \gamma \cap \gamma_{\|}$. Inversely, if there is an intersection point q of the curves γ and $\gamma_{\|}$, q satisfies the equation (3). Then, $q \in \gamma$ by $\Gamma(q) = 0$, and $(-q - q_3) \in \gamma$ by $\Gamma(-q - q_3) = \Gamma(q + q_3) = 0$. Moreover, $q + (-q - q_3) + q_3 = 0$ is satisfied. □

As a simple application of the three points theorem 2.1, we construct an equal mass three-body motion in a given closed convex curve γ that is invariant under the inversion $q \mapsto -q$. See [8] for details.

We can apply the three points theorem 2.1 to Moore, Chenciner and Montgomery's figure-eight curve γ, and we get Figure 4. A set of two points $\{q_1, q_2\}$ on the figure-eight curve is determined uniquely for a given point q_3. See [8] for details.

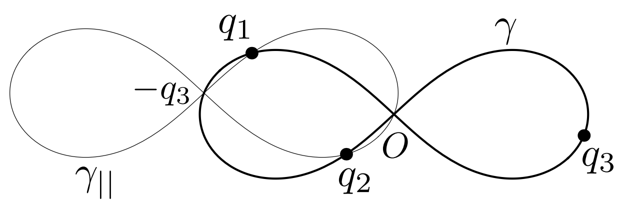

FIGURE 4. The figure-eight curve is invariant under the inversion of $q \mapsto -q$. The thin curve $\gamma_{||}$ is the parallel translation of the thick curve γ by $-q_3$. For a given point q_3, the two points $q_1$and q_2 on γ that satisfy $q_1 + q_2 + q_3 = 0$ are given by $\gamma \cap \gamma_{||}$.

The position and momentum of three bodies are determined uniquely by the total energy E (see [8] and [9]) if the figure-eight orbit is given. This corresponds to Kepler's second law in the equal mass three-body figure-eight motion.

3. **Extensions of the three points theorem.** Now we would like to find a solution of $\{q_1, q_2\}$ satisfying $q_1 + q_2 + q_3 = 0$ for a given q_3 in the case of $q_1 \in \gamma_1$ and $q_2 \in \gamma_2$, where γ_1 and γ_2 are two independent given curves. It is simply given as follows. Let us move body 1 along γ_1 for a given q_3, then q_2 draws an orbit so that $q_1 + q_2 + q_3 = 0$ holds. Such a tentative orbit of body 2, writing it as γ_1^*, is the inversion of γ_1 with respect to the point $-q_3/2$, which is the midpoint of q_1 and q_2. Since γ_1^* certainly intersects with γ_2 at least once, each of intersection points becomes a candidate for the solution q_2. Thus we can find $\{q_1, q_2\}$ satisfying $q_1 + q_2 + q_3 = 0$ for a given q_3.

Remark 1. Figure 5 shows how to obtain γ_1^* and γ_2^*. Make inversion of γ_1 around $-q_3/2$, then we obtain γ_1^*. If we make the same operation for γ_2, we obtain γ_2^*. We will notice that there are two kinds of three-body configurations in this case. We have nothing further to say which configuration is realized since the orbit of body 3 is not given yet. Here we give Lagrange's equilateral triangular motion of the equal mass three bodies as an example, see Figure 6. We find two candidates for q_1's $\in \gamma_1 \cap \gamma_2^*$ and q_2's $\in \gamma_2 \cap \gamma_1^*$. When body 3 moves around the whole γ_3, one of the q_1's collides with body 3, and so does one of the q_2's. Therefore the configuration of equal mass three bodies is unique.

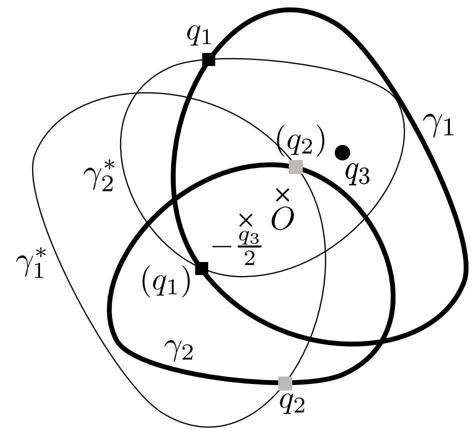

FIGURE 5. We assumed that q_3 is on a closed curve, but the orbit is not displayed. The curve γ_i^* that is the image of γ_i $(i = 1, 2)$ by the map $q_i \mapsto q_i^* = -q_i - q_3$. We find two kinds of intersection points $\{q_1, q_2\}$ and $\{(q_1), (q_2)\}$ so that the center of mass being at the origin where $\{q_1, (q_1)\} \in \gamma_1 \cap \gamma_2^*$ and $\{q_2, (q_2)\} \in \gamma_1^* \cap \gamma_2$.

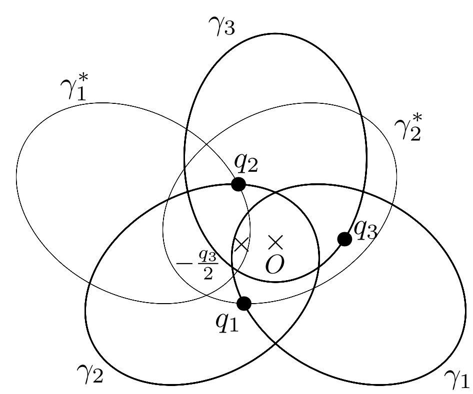

FIGURE 6. The orbits of the equal mass three bodies are the same size ellipses. For a given point $q_3 \in \gamma_3$, the other points q_1 and q_2 that satisfy $q_1 + q_2 + q_3 = 0$ are given by the intersection points of γ_i and γ_j^* $(i, j = 1, 2)$ with $i \neq j$.

We have been restricted to the equal mass three-body system. The above method can be easily extended to the unequal mass three-body systems. We will use a notation, writing the three masses as m_i $(i = 1, 2, 3)$. The center of mass of three bodies being at the origin satisfies

$$m_1 q_1 + m_2 q_2 + m_3 q_3 = 0. \tag{4}$$

Then a pair solution $\{q_1, q_2\}$ of (4) for a given q_3 can be found by the following steps: (i)Make inversion of γ_1 and γ_2 around $-m_3 q_3/(m_1 + m_2)$ and denote inverted images γ_1' and γ_2' respectively. (ii) Enlarge γ_1' by a factor of m_1/m_2 and γ_2' by a factor of m_2/m_1, then name the mapped curves γ_1^* and γ_2^*. (iii) Find $\gamma_1 \cap \gamma_2^*$ and $\gamma_2 \cap \gamma_1^*$. It should be noticed that the center of the inversion in (i) is nothing but a center of mass of bodies 1 and 2:

$$\frac{m_1q_1 + m_2q_2}{m_1 + m_2} = -\frac{m_3}{m_1 + m_2}q_3.$$

We may find several $\{q_1, q_2\}$'s, and each of them is a candidate of solutions of $q_1 + q_2 + q_3 = 0$ for a given q_3. But it is not evident that all $\{q_1, q_2\}$'s satisfy the equation of motion. The reality of motion should be tested whether the equation of motion is satisfied or not. Thus we have

Theorem 3.1. *For a given set $\gamma_1, \gamma_2 \subset \mathbb{R}^d$ and $q_3 \in \mathbb{R}^d$, we have the following equalities,*

$$\begin{aligned} & \{\{q_1, q_2\} | q_1 \in \gamma_1, q_2 \in \gamma_2, m_1q_1 + m_2q_2 + m_3q_3 = 0\} \\ = & \{\{q_1, q_1^*\} | q_1 \in \gamma_1 \cap \gamma_2^*\} \qquad (5) \\ = & \{\{q_2, q_2^*\} | q_2 \in \gamma_1^* \cap \gamma_2\}, \qquad (6) \end{aligned}$$

where * represents a map $q_i \mapsto q_i^* = \frac{1}{m_j}(-m_iq_i - m_3q_3)$ with $i, j = 1, 2$ but *$i \neq j$ and γ_i^* is the image of γ_i by this map.*

The proof of this theorem is similar to that of 'Theorem 2' in [8].

We exemplify Euler's streight-line motion and Langrange's equilateral triangular motion in which each body has its own mass. Figure 7 shows that a configuration of three bodies with $m_1 = 3, m_2 = 4, m_3 = 1$ in Euler's streight-line motion. We

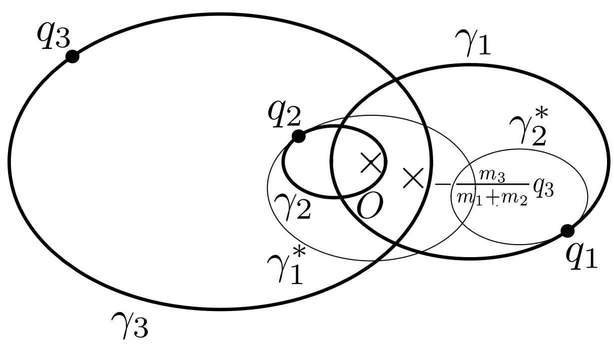

FIGURE 7. The orbits of the three bodies are ellipses. For a given point $q_3 \in \gamma_3$, the two points that satisfy $m_1q_1 + m_2q_2 + m_3q_3 = 0$ are given by the intersection points of γ_i and γ_j^* $(i, j = 1, 2)$ with $i \neq j$.

find one intersection point for $\gamma_1 \cap \gamma_2^*$ and also for $\gamma_2 \cap \gamma_1^*$ in Figure 7, so the configuration is unique. Figure 8 shows that a configuration of three bodies with $m_1 = 3, m_2 = 2, m_3 = 1$ in Langrange's equilateral triangular motion. We find two intersection points for $\gamma_1 \cap \gamma_2^*$ and also for $\gamma_2 \cap \gamma_1^*$ in Figure 8. When body 3 moves around the whole γ_3, one of the q_1's collides with body 3, and so does one of the q_2's. So the configuration of unequal mass three bodies is also unique. The

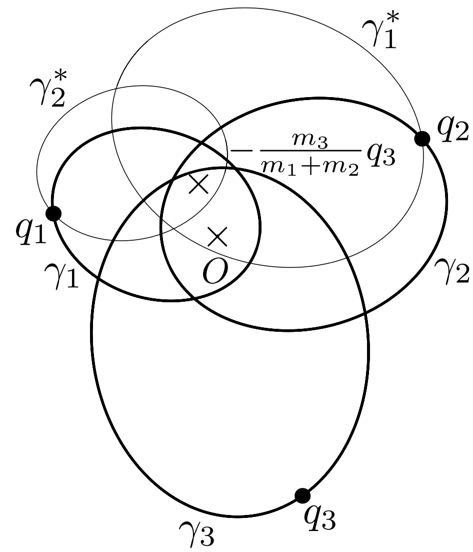

FIGURE 8. The orbits of three bodies are the ellipses. For a given point $q_3 \in \gamma_3$, the two points that satisfy $m_1q_1 + m_2q_2 + m_3q_3 = 0$ are given by the intersection points of γ_i and γ_j^* $(i, j = 1, 2)$ with $i \neq j$.

momentum of each body in these motions is obtained by

$$p_i(\sigma(t)) = m_i \frac{c}{J(\sigma)} \frac{dq_i(\sigma)}{d\sigma}, \quad \text{where } i = 1, 2, 3, \text{and}$$

$$J(\sigma) = \sum_{j=1}^{3} q_j(\sigma) \times \frac{dq_j(\sigma)}{d\sigma}.$$

for the total angular momentum c. Thus both Euler's straight-line motion and Lagrange's equilateral triangular motion are determined uniquely by their own three closed orbits and by their own total angular momentum.

Moreover, Theorem 3.1 is applicable to the unclosed orbits. For example, let us apply Theorem 3.1 to the equal mass three-body 3:4:5 free fall problem. If three orbits γ_i $(i = 1, 2, 3)$ are known and body 3 is on a point in γ_3, the positions of bodies 1 and 2 staying the center of mass at the origin can be found in a lot of intersection points $\gamma_1 \cap \gamma_2^*$ and $\gamma_2 \cap \gamma_1^*$, see Figure 9. It is not manifest at a glance whether the motion of equal mass three bodies is unique or not for a given position q_3 because of the existence of so many sets of intersection points. But it will be evident that the configuration at any time is unique if we trace back to the initial configuration such that at $t = 0$ equal mass three bodies are located at $q_1(0) = (1, 8/3), q_2(0) = (1, -4/3), q_3(0) = (-2, -4/3)$. When body 3 starts moving from $q_3(0)$, the positions of q_1 and q_2 will be pinpointed in γ_1 and γ_2. Since the total angular momentum in this motion is vanishing as well as in the figure-eight motion, the momentum of each body is determined by the total energy E:

$$p_i(\sigma(t)) = m_i \sqrt{\frac{2(E - V)}{\sum_{l=1}^{3} \left|\frac{dq_l(\sigma)}{d\sigma}\right|^2}} \frac{dq_i}{d\sigma} \quad \text{with } i = 1, 2, 3.$$

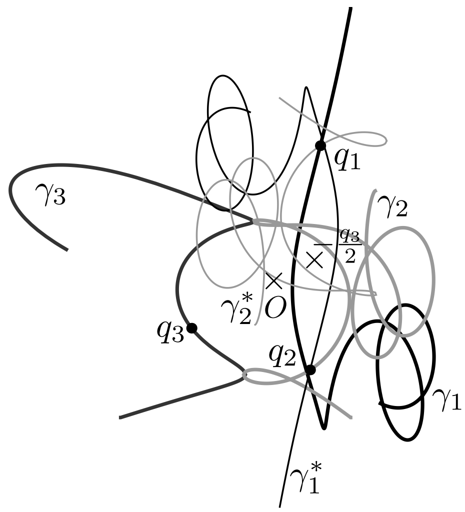

FIGURE 9. The 3:4:5 three-body free fall problem does not give closed orbits. For a given point $q_3 \in \gamma_3$, the other points q_1 and q_2 that satisfy $q_1 + q_2 + q_3 = 0$ are given by the intersection points of γ_i and γ_j^* $(i, j = 1, 2)$ with $i \neq j$.

If the configuration is unique, the motion along their own curves is determined uniquely modulo time-reparametrization, $q_i(t) \mapsto q_i(\sigma(t))$ with the function $\sigma(t)$ given by the total energy.

4. **Summary.** In this note, we showed how to find positions of bodies 1 and 2 for a given position of body 3 in the equal mass three-body motion along a closed orbit. Owing to the three points theorem (Theorem 2.1), we were able to show that the motion of equal mass three bodies in given curves (smooth point symmetric convex curves and the eight-shaped curve) is determined uniquely by the conservation of the center of mass and a constant of motion. Even in the unequal mass three-body periodic motions in which each of thee bodies draws its own orbit, we obtained some candidates of the configuration of three bodies. The reality of the motion should be tested whether the equation of motion is satisfied or not. We showed an extended three points theorem (Theorem 3.1) and it was applied to the unclosed orbits which were appeared in the 3:4:5 free fall problem.

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