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# CRITICAL SECOND ORDER OPERATORS ON TIME SCALES

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Abstract. In this paper we introduce the concept of critical operators for dynamic operators of second order. Next, we show that an arbitrarily small (in a certain sense) negative perturbation of a non-negative critical operator leads to an operator which is no longer non-negative.

1. **Introduction.** We extend to time scales a result of [9], which deals with the second order Sturm–Liouville difference equation

$$-\Delta(a_k\Delta y_{k-1}) + c_k y_k = 0, \quad a_k > 0, k \in \mathbb{Z}, \tag{1}$$

and the corresponding three-term symmetric recurrence relation

$$\tau(y) = -a_{k+1}y_{k+1} + b_k y_k - a_k y_{k-1}, \quad b_k = a_{k+1} + a_k + c_k. \tag{2}$$

Let us recall the definition of critical operators as introduced in [9] and the main result of that paper.

**Definition 1.1.** Let (1) be disconjugate on $\mathbb{Z}$, i.e., $\tau \geq 0$. We say that the operator $\tau$ is *critical* if the recessive solutions at $\infty$ and $-\infty$ are linearly dependent, in the opposite case $\tau$ is said to be *subcritical*. If $\tau \ngeq 0$, i.e., (1) is not disconjugate, $\tau$ is said to be *supercritical*.

**Theorem 1.2.** *The following statements are equivalent.*

*(i) The operator $\tau$ is critical on $\mathbb{Z}$.*
*(ii) For any $\varepsilon > 0$ and $m \in \mathbb{Z}$, the operator $\tilde{\tau}$ which we get from $\tau$ by replacing $a_m$ by $a_m - \varepsilon$, is supercritical on $\mathbb{Z}$, i.e., $\tilde{\tau} \ngeq 0$.*
*(iii) For any $\varepsilon > 0$ and $m \in \mathbb{Z}$, the operator $\hat{\tau}$ which we get from $\tau$ by replacing $b_m$ by $b_m + \varepsilon$, is supercritical on $\mathbb{Z}$, i.e., $\hat{\tau} \ngeq 0$.*

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In [9], the authors used a matrix operator associated with (2) and proved Theorem 1.2 by a construction of solutions of (1). More recently, this result was extended in [6] to $2n$-order Sturm–Liouville difference operators, where variational approach was used. We use a similar technique to prove the main result of this paper.

2. **Preliminaries.** In [11, 12], Stefan Hilger established the time scale calculus as the unification of the continuous and discrete calculus. Fundamental parts of this theory were summarized in the monographs [4, 5] and we refer to these books for proofs of properties and assertions mentioned in this section (if not specified otherwise).

A *time scale* $\mathbb{T}$ is any nonempty closed subset of the real numbers $\mathbb{R}$. In this paper we deal only with an unbounded time scale $\mathbb{T}$, i.e., $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = \infty$. (One can proceed similarly on bounded time scale under some additional conditions.)

The *forward* and *backward jump operators* $\sigma(t)$ and $\rho(t)$ and the *graininess functions* $\mu(t) := \sigma(t) - t$ and $\nu(t) := t - \rho(t)$ have the usual time scale meaning. The notations of *left/right scattered* or *dense point* are also standard.

For a function $f : \mathbb{T} \to \mathbb{C}$ it is possible to define the $\Delta$-*derivative* of $f$ at $t \in \mathbb{T}$ (denoted by $f^{\Delta}(t)$) and the $\nabla$-*derivative* of $f$ at $t \in \mathbb{T}$ (denoted by $f^{\nabla}(t)$) in the following way

$$f^{\Delta}(t) := \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t}, & \text{if } \mu(t) = 0, \\ \frac{f^{\sigma}(t)-f(t)}{\mu(t)}, & \text{if } \mu(t) > 0, \end{cases}$$

$$f^{\nabla}(t) := \begin{cases} \lim_{s\to t} \frac{f(s)-f(t)}{s-t}, & \text{if } \nu(t) = 0, \\ \frac{f(t)-f^{\rho}(t)}{\nu(t)}, & \text{if } \nu(t) > 0. \end{cases}$$

It is well known that for $\mathbb{T} = \mathbb{R}$ we have $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$ and for $\mathbb{T} = \mathbb{Z}$ the relations $f^{\Delta} = f(t+1) - f(t)$ and $f^{\nabla} = f(t) - f(t-1)$ hold. For brevity, we use the notation $f^{\sigma}(t) := f(\sigma(t))$, $f^{\rho}(t) := f(\rho(t))$, and $[f(t)]_a^b := f(b) - f(a)$.

We say that the function $f(t)$ is $\Delta$-differentiable and $\nabla$-differentiable on $\mathbb{T}$, provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$ and $f^{\nabla}(t)$ exists for all $t \in \mathbb{T}$, respectively. A function $f(t)$ is said to be *rd-continuous* ($f \in C_{\mathrm{rd}}$) if it is continuous at each right-dense point and there exists a finite left-hand limit at each left-dense point.

Now, let $c, d \in \mathbb{T}$ and $c < d$. The $\Delta$-integral and $\nabla$-integral are defined in such a way that they reduce to the usual Riemann integral, so that $\int_c^d f(t)\Delta t = \int_c^d f(t)\nabla t = \int_c^d f(t)\mathrm{d}t$ holds if $\mathbb{T} = \mathbb{R}$, and $\int_c^d f(t)\Delta t = \sum_{t=c}^{d-1} f(t)$ and $\int_c^d f(t)\nabla t = \sum_{t=c+1}^{d} f(t)$ hold if $\mathbb{T} = \mathbb{Z}$. The rule for the integration by parts takes the following form. If $f, g \in C_{\mathrm{rd}}$, then

$$\int_c^d f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_c^d - \int_c^d f^{\Delta}(t)g^{\sigma}(t)\Delta t. \tag{3}$$

Our investigation is connected with the (formally self-adjoint) second order dynamic equation

$$-(py^{\nabla})^{\Delta} + qy = 0, \tag{4}$$

where $p$ is continuous and positive, $q$ is rd-continuous, and $p, q$ are real-valued functions on $(-\infty, \infty)_{\mathbb{T}}$. Such equations were intensively studied in e.g. [3, 5, 14]. In connection with equation (4) we consider the operator

$$\mathcal{L}(y)(t) := -(py^{\nabla})^{\Delta} + qy.$$

Let us introduce a set

$$\mathbb{D} := \Big\{ y : \mathbb{T} \to \mathbb{R},\ y^{\nabla}(\cdot) \text{ is continuous and } (py^{\nabla})^{\Delta}(\cdot) \text{ is rd-continuous} \Big\}.$$

Then $y \in \mathbb{D}$ is a solution of (4) if and only if $\mathcal{L}(y)(t) = 0$ for all $t \in \mathbb{T}$.

The existence of a unique solution for boundary value problem given by (4) with boundary conditions was shown in [2, Theorem 3.6].

**Theorem 2.1** (Recessive and dominant solutions, [5, Theorem 4.55])**.** *Let $a \in \mathbb{T}$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If (4) is non-oscillatory on $[a, \omega)_{\mathbb{T}}$, i.e., there is a nontrivial solution having only finitely many generalized zeros in $[a, \omega)_{\mathbb{T}}$, then there is a solution $u(\cdot)$ called a recessive solution at $\omega$, such that $u(\cdot)$ is positive on $[t_0, \omega)_{\mathbb{T}}$ for some $t_0 \in \mathbb{T}$, and if $v(\cdot)$ is any second, linearly independent solution, called a dominant solution at $\omega$, the following hold:*

*(i)* $\lim_{t \to \omega^-} \frac{u(t)}{v(t)} = 0$,

*(ii)* $\int_{t_0}^{\omega} \frac{1}{p(t)u(t)u^{\rho}(t)} \nabla t = \infty$,

*(iii)* $\int_{b}^{\omega} \frac{1}{p(t)v(t)v^{\rho}(t)} \nabla t < \infty$ *for $b < \omega$ sufficiently close, and*

*(iv)* $\frac{p(t)v^{\nabla}(t)}{v(t)} > \frac{p(t)u^{\nabla}(t)}{u(t)}$ *for $t < \omega$, sufficiently close.*

*The recessive solution $u$ is unique up to multiplication by a nonzero constant.*

The statement for recessive and dominant solutions at $\inf \mathbb{T}$ is analogical.

3. **Critical operators.** Let $y \in \mathbb{D}$ be a solution of (4) with bounded support. Then we have

$$\begin{aligned} \langle \mathcal{L}(y), y \rangle &= -\int_{-\infty}^{\infty} \left[ y(py^{\nabla})^{\Delta} + qy^2 \right] \Delta t \\ &\overset{(3)}{=} \left[ -pyy^{\nabla} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left[ y^{\Delta} p^{\sigma} y^{\nabla}(\sigma(t)) + qy^2 \right] \Delta t \\ &= \int_{-\infty}^{\infty} \left[ p^{\sigma} (y^{\Delta})^2 + qy^2 \right] \Delta t =: \mathcal{F}(y). \end{aligned}$$

Now, we introduce the definitions of generalized zeros and disconjugacy in an analogy with [13, Definition 4.43 and Definition 4.44], see also [1, 8], and then we extend Definition 1.1 for operator $\mathcal{L}$.

**Definition 3.1.** We say that $y(\cdot)$ has a *generalized zero* in $(\rho(t_0), t_0]_{\mathbb{T}}$ if it holds $y(t_0) = 0$ or $y^{\rho}(t_0)\, y(t_0) < 0$. Equation (4) is said to be *disconjugate* on $\mathbb{T}$ provided there is no nontrivial real solution of this equation with two (or more) generalized zeros in $\mathbb{T}$.

**Definition 3.2.** Let (4) be disconjugate on $\mathbb{T}$. We say that the operator $\mathcal{L}$ is *critical* if the recessive solutions of (4) at $\infty$ and $-\infty$ are linearly dependent. In the opposite case $\mathcal{L}$ is said to be *subcritical*. If (4) is not disconjugate, $\mathcal{L}$ is said to be *supercritical*.

The main result of this paper read as follows.

**Theorem 3.3.** *Let the operator $\mathcal{L}$ be critical on $\mathbb{T}$, let $I = [a, b]_{\mathbb{T}} \subseteq \mathbb{T}$ be an arbitrary time scale interval, such that $\rho(b) > a$, and let $\varepsilon > 0$ be arbitrary. Then the operators*

$$\begin{aligned} \widetilde{\mathcal{L}}^{p} &= -(\tilde{p} y^{\nabla})^{\Delta} + qy, \\ \widetilde{\mathcal{L}}^{q} &= -(p y^{\nabla})^{\Delta} + \tilde{q} y, \end{aligned}$$

*where*

$$\tilde{p}(t) = \begin{cases} p(t) - \varepsilon, & \text{for } t \in I, \\ p(t), & \text{otherwise}, \end{cases} \qquad \tilde{q}(t) = \begin{cases} q(t) - \varepsilon, & \text{for } t \in I, \\ q(t), & \text{otherwise}, \end{cases}$$

*are supercritical on* $\mathbb{T}$.

*Proof.* Let $h$ be a recessive solution of (4) both at $\infty$ and $-\infty$. Now, let us introduce a solution $y$ of (4) in the form

$$y(t) := \begin{cases} 0, & t \in (-\infty, K)_{\mathbb{T}}, \\ f(t), & t \in [K, L)_{\mathbb{T}}, \\ h(t), & t \in [L, M)_{\mathbb{T}}, \\ g(t), & t \in [M, N)_{\mathbb{T}}, \\ 0, & t \in [N, \infty)_{\mathbb{T}}, \end{cases} \tag{5}$$

where

$$f(t) = A\, h(t) \int_K^t \frac{1}{p(s)h(s)h^\rho(s)} \nabla s \quad \text{with} \quad A := \left( \int_K^L \frac{1}{p(t)h(t)h^\rho(t)} \nabla t \right)^{-1},$$

$$g(t) = B\, h(t) \int_t^N \frac{1}{p(s)h(s)h^\rho(s)} \nabla s \quad \text{with} \quad B := \left( \int_M^N \frac{1}{p(t)h(t)h^\rho(t)} \nabla t \right)^{-1},$$

and moreover $K < L < M < N$, $K, L, M, N \in \mathbb{T}$, are chosen such that $I \subset [M, N]_{\mathbb{T}}$ in case of the operator $\widetilde{\mathcal{L}}^p$ or $I \subset [L, M]_{\mathbb{T}}$ if we deal with $\widetilde{\mathcal{L}}^q$. Because

$$f(K) = 0, \;\; f(L) = h(L), \;\; h(M) = g(M), \;\; \text{and } g(N) = 0, \tag{6}$$

$y$ is continuous. Now, we calculate the functional $\mathcal{F}$ at $y$ given by (5), i.e.,

$$\begin{aligned} \mathcal{F}(y) &= \int_{-\infty}^{\infty} \left[ p^\sigma (y^\Delta)^2 + q y^2 \right] \Delta t \\ &= \int_K^L \left[ p^\sigma (f^\Delta)^2 + q f^2 \right] \Delta t + \int_L^M \left[ p^\sigma (h^\Delta)^2 + q h^2 \right] \Delta t \\ &\qquad + \int_M^N \left[ p^\sigma (g^\Delta)^2 + q g^2 \right] \Delta t \\ &= \left[ p f f^\nabla \right]_K^L + \left[ p h h^\nabla \right]_L^M + \left[ p g g^\nabla \right]_M^N \\ &= h(L) \left[ p(L) f^\nabla(L) - p(L) h^\nabla(L) \right] \\ &\qquad + h(M) \left[ p(M) h^\nabla(M) - p(M) g^\nabla(M) \right] \\ &= A + B. \end{aligned}$$

Here we used at first the fact, that for $[\alpha, \beta]_{\mathbb{T}} \subseteq \operatorname{supp} y$ we have

$$\begin{aligned} \mathcal{F}(y) &= \int_\alpha^\beta \left[ p^\sigma (y^\Delta)^2 + q y^2 \right] \Delta t \overset{(3)}{=} \int_\alpha^\beta \left[ (p y y^\nabla)^\Delta - (p y^\nabla)^\Delta y + q y^2 \right] \Delta t \\ &= \int_\alpha^\beta \left\{ (p y y^\nabla)^\Delta + y \left[ -(p y^\nabla)^\Delta + q y \right] \right\} \Delta t = \left[ p y y^\nabla \right]_\alpha^\beta, \end{aligned}$$

then we used (6), and that

$$p(L) f^\nabla(L) = p(L)\, h^\nabla(L) + \frac{A}{h(L)}, \quad p(M) g^\nabla(M) = p(M)\, h^\nabla(M) - \frac{B}{h(M)}.$$

Next, Theorem 2.1($ii$) implies that $A$ and $B$ tend to 0 as $K \to -\infty$ and $N \to \infty$, respectively. Therefore we obtain that for any $\delta > 0$

$$\mathcal{F}(y) \le \frac{\delta}{2},$$

for $K$ sufficiently negative and $N$ sufficiently positive.

Finally, we calculate the values of the functionals $\mathcal{F}^p$ and $\mathcal{F}^q$ corresponding to the operators $\widetilde{\mathcal{L}}^p$ and $\widetilde{\mathcal{L}}^q$, respectively. We obtain (see the choice of $K, L, M, N$ above) the following inequalities

$$\mathcal{F}^p(y) = \mathcal{F}(y) - \underbrace{\varepsilon \int_I (y^\Delta(t))^2 \Delta t}_{=:\delta} \le \frac{\delta}{2} - \delta = -\frac{\delta}{2} < 0,$$

$$\mathcal{F}^q(y) = \mathcal{F}(y) - \underbrace{\varepsilon \int_I (y(t))^2 \Delta t}_{=:\delta} \le \frac{\delta}{2} - \delta = -\frac{\delta}{2} < 0.$$

Therefore $\widetilde{\mathcal{L}}^p$ and $\widetilde{\mathcal{L}}^q$ are supercritical and the theorem is proven. □

We end up this section by a statement, which is a direct consequence of Theorem 3.3.

**Corollary 1.** *Let the operator $\mathcal{L}$ be critical on $\mathbb{T}$, let $I = [a, b]_{\mathbb{T}} \subseteq \mathbb{T}$ be an arbitrary time scale interval, such that $\rho(b) > a$, and let $\varepsilon > 0$ be arbitrary. Then the operators*

$$\widehat{\mathcal{L}}^p = -(\hat{p} y^\nabla)^\Delta + qy,$$
$$\widehat{\mathcal{L}}^q = -(p y^\nabla)^\Delta + \hat{q} y,$$

*where*

$$\hat{p}(t) = \begin{cases} p(t) + \varepsilon, & \text{for } t \in I, \\ p(t), & \text{otherwise}, \end{cases} \qquad \hat{q}(t) = \begin{cases} q(t) + \varepsilon, & \text{for } t \in I, \\ q(t), & \text{otherwise}, \end{cases}$$

*are subcritical on $\mathbb{T}$.*

*Proof.* Suppose that $\widehat{\mathcal{L}}^p$ is critical. We perturb this operator in the sense of Theorem 3.3 (we change $\hat{p}$ to $\tilde{\hat{p}}$). The resulting operator should be, according to Theorem 3.3, supercritical, which is a contradiction. The part concerning $\widehat{\mathcal{L}}^q$ is proven similarly. □

4. **One term operator.** In this section we give, as an example, a criterion of criticality of one term operator

$$\ell(y)(t) := -(p y^\nabla)^\Delta. \tag{7}$$

For the discrete and continuous case see [6, 7, 10].

**Theorem 4.1.** *Operator $\ell$ is critical if and only if*

$$\int_0^\infty \frac{1}{p(t)} \nabla t = \infty = \int_{-\infty}^0 \frac{1}{p(t)} \nabla t. \tag{8}$$

*Proof.* At first we find a recessive solution of (7) at $\infty$. We use the following linearly independent solutions of (7) given by

$$y_1(t) = 1 \quad \text{and} \quad y_2(t) = \int_0^t \frac{1}{p(s)} \nabla s.$$

By a direct computation (compare Theorem 2.1($i$)) we obtain

$$\lim_{t\to\infty}\frac{y_2(t)}{y_1(t)}=\lim_{t\to\infty}\int_0^t\frac{1}{p(s)}\nabla s=\infty,$$

i.e., $y_1(t)=1$ is the recessive solution at $\infty$. At $-\infty$ we use the solutions

$$y_1(t)=1\quad\text{and}\quad y_3(t)=\int_t^0\frac{1}{p(s)}\nabla s.$$

Because

$$\lim_{t\to-\infty}\frac{y_3(t)}{y_1(t)}=\lim_{t\to-\infty}\int_t^0\frac{1}{p(s)}\nabla s=\infty$$

holds, the function $y_1(\cdot)$ is the recessive solution both at $\infty$ and $-\infty$. Hence (7) is critical.

On the other hand to verify the second implication, it suffices to show that if at least one of the integrals in (8) is convergent, the operator $\ell$ is not critical. Let us assume that $\int_0^\infty\frac{1}{p(t)}\nabla t<\infty$ and let us introduce the solutions

$$y_1(t)=1\quad\text{and}\quad y_4(t)=\int_0^\infty\frac{1}{p(s)}\nabla s-\int_0^t\frac{1}{p(s)}\nabla s=\int_t^\infty\frac{1}{p(s)}\nabla s.$$

We have

$$\lim_{t\to\infty}\frac{y_4(t)}{y_1(t)}=\lim_{t\to\infty}\int_t^\infty\frac{1}{p(s)}\nabla s=0,$$

therefore $y_4$ is the recessive solution of (7) at $\infty$. This solution is linearly independent of the recessive solution at $-\infty$, which is $y_1(t)=1$ (if the second integral is divergent) or (which we can verify analogously) it is $y_5(t)=\int_{-\infty}^t\frac{1}{p(s)}\nabla s$ (if the second integral is convergent). Altogether, the statement is proved. □

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