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# ORDER ISOMORPHISMS IN WINDOWS

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Abstract. We characterize order preserving transforms on the class of lower-semi-continuous convex functions that are defined on a convex subset of $\mathbb{R}^n$ (a "window") and some of its variants. To this end, we investigate convexity preserving maps on subsets of $\mathbb{R}^n$. We prove that, in general, an order isomorphism is induced by a special convexity preserving point map on the epi-graph of the function. In the case of non-negative convex functions on $K$, where $0 \in K$ and $f(0) = 0$, one may naturally partition the set of order isomorphisms into two classes; we explain the main ideas behind these results.

In this note, we announce results regarding order isomorphisms on classes of convex functions that are defined on a convex region $K$ in $\mathbb{R}^n$. The case $K = \mathbb{R}^n$ was studied in [2], [3] and [4]. In this paper, we consider closed, convex proper subsets $K$ of $\mathbb{R}^n$ having non-empty interior; such a $K$ is called a "window" in $\mathbb{R}^n$. Denote by $Cvx(K)$ the collection of all lower-semi-continuous convex functions defined on $K$ that assume values in $\mathbb{R} \cup \{+\infty\}$. Note that we may extend $f$ to $\mathbb{R}^n$ by assigning to $x \notin K$ the value $f(x) = +\infty$. This extension preserves convexity and lower-semi-continuity since $K$ is closed, and consequently, we may consider $Cvx(K)$ as a subclass of $Cvx(\mathbb{R}^n)$; in this setting, some new and remarkable properties appear. The central new notion employed in this context is that of fractional linear maps, coming from projective geometry. In general, the intuition of projective geometry plays an important role, and developing the corresponding theory in the linear case (i.e. in $\mathbb{R}^n$), where it seems to have been rarely considered (from the convexity point of view), leads to an interesting interplay between the two approaches. The full details, results and proof of theorems from this note will appear in [1].

The original goal of this investigation was to study duality in $Cvx(K)$, where $K$ is a window. It was shown in [2], [3] that the standard duality on the whole class $Cvx(\mathbb{R}^n)$, the Legendre transform, is essentially, up to linear terms, the only order reversing isomorphism. Recall that, for a fixed Euclidean structure with inner product given by $\langle \cdot, \cdot \rangle$, the Legendre transform is given by

$$(\mathcal{L}f)(y) = \sup_{x \in \mathbb{R}^n} \left( \langle x, y \rangle - f(x) \right).$$

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With one order reversing transform in hand, studying order reversing transforms is equivalent to studying order preserving transforms (simply compose one with the known order reversing transform to obtain the other). We mainly work with order preserving transforms in this note, and when applicable (this may involve specific cases for the set $K$), we use this device to characterize order reversing transforms as well.

Let us begin by quoting the corresponding theorem for the class $Cvx(\mathbb{R}^n)$:

**Theorem 1** ([2])**.** *Let $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ be a bijection satisfying, for all $\phi, \psi \in Cvx(\mathbb{R}^n)$,*

$$\phi \leq \psi \qquad \text{if and only if} \qquad \mathcal{T}\phi \leq \mathcal{T}\psi.$$

*Then there exist $C_0 \in \mathbb{R}, C_1 \in \mathbb{R}^+, v_0, v_1 \in \mathbb{R}^n$ and $B \in GL_n$ such that*

$$(\mathcal{T}\phi)(x) = C_1\phi(Bx + v_0) + \langle v_1, x\rangle + C_0.$$

Such bijective transforms, which preserve order in both directions ($\mathcal{T}$, $\mathcal{T}^{-1}$), are called *order preserving isomorphisms*. The last theorem states that, up to linear terms, the only order isomorphism on $Cvx(\mathbb{R}^n)$ is the identity transform composed with an affine change of variables, $x \mapsto Bx + v_0$. By linear terms, we mean multiplication by a positive scalar and an addition of an affine linear function. When the class is taken to be the set of functions defined on a convex set $K \subsetneq \mathbb{R}^n$, the situation is different, and it involves the notion of fractional linear maps.

Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a linear map, $b, c \in \mathbb{R}^n$ two vectors and $d \in \mathbb{R}$ some constant such that the matrix

$$\hat{A} = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$$

is invertible. The map

$$x \mapsto \frac{1}{\langle c, x\rangle + d}(Ax + b),$$

defined on $\{x : \langle c, x\rangle \neq -d\}$, is what we call "fractional linear". A remark about this name is needed. In the literature, the name "fractional linear maps" sometimes refers to Möbius transformations, which is not the case here: a Möbius transformation on a subset of $\mathbb{C} = \mathbb{R}^2$ does not preserve intervals, but these notions are indeed connected. One sometimes uses the name "permissible projective transformations", see e.g. [5]. Another option is "convexity preserving maps", but this describes their action rather than their functional form, thus hiding the fact that they are of a simple and easily described form.

It is known ([8], [1]), and stated as Fact 3 below, that any injective map on a convex set $K \subset \mathbb{R}^n$, where $n \geq 2$ and $int(K) \neq \emptyset$, that preserves intervals must be fractional linear – the opposite is also true: every fractional linear map is injective and preserves intervals wherever it is defined. We refer to the epi-graph of a function $f$ by $epi(f) \subset \mathbb{R}^{n+1}$, i.e. $epi(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathbb{R}^n, f(x) < y\}$. In the next theorem, we see that the "change of variables" for order isomorphisms on $Cvx(K)$ is fractional linear and may be non-affine.

**Theorem 2.** *Let $n \geq 1$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non-empty interior. If $\mathcal{T} : Cvx(K_1) \to Cvx(K_2)$ is an order preserving isomorphism, then there exists a bijective fractional linear map $F : K_1 \times \mathbb{R} \to K_2 \times \mathbb{R}$ such that $\mathcal{T}$ is given by*

$$epi(\mathcal{T}f) = F(epi(f)).$$

*It follows, in particular, that $K_2$ is a fractional linear image of $K_1$.*

Let us elaborate on the meaning of the equation $epi(\mathcal{T}f) = F(epi(f))$. It is shown in [1] that when $F$ induces a transform on $Cvx(K)$, up to some affine linear functional $L_1$, $F$ is of the form

$$F(x,y) = \left(\frac{Ax+u}{\langle v,x\rangle + d}, \frac{y}{\langle v,x\rangle + d}\right).$$

Denote by $L_0 = \langle v,\cdot\rangle + d$ the affine linear functional present in the denominator and by $F_b(x) = \frac{Ax+u}{\langle v,x\rangle+d}$ the base-map; that is, the projection of $F$ to the first $n$ coordinates, which is itself a bijective fractional linear map, from $K_1$ to $K_2$. We conclude that

$$\mathcal{T}f = \left(\frac{f}{L_0}\right) \circ F_b^{-1} + L_1, \tag{1}$$

where $L_1$ is some affine linear functional (and $F_b^{-1} : K_2 \to K_1$ is a bijective fractional linear map as an inverse of such). Note that $L_0$ and $F_b$ are not independent, since $L_0$ must vanish on the defining hyperplane of $F_b$ (where $F_b$ is not defined). Moreover, note that for a general $f$, the function $\frac{f}{L_0}$ may not be convex, but the composition with $F_b^{-1}$ exactly compensates this problem. The result of this composition is again a convex function. In the special case of $A = I, u = 0, L_0(x) = x_1+1$ and $L_1(x) \equiv 0$, we obtain $F(x,y) = (\frac{x}{x_1+1}, \frac{y}{x_1+1})$ and $(\mathcal{T}f)(x) = (1-x_1)f(\frac{x}{1-x_1})$. This simpler form of the transform is not general, but if one allows linear actions on the epi-graphs, before and after $F$ acts on them, it suffices to consider this form, as is explained in Proposition 4.

Similarly to Theorem 1, Theorem 2 states that up to linear terms, the only transforms in this setting are induced by a *fractional linear* "change of variables". The change of variables, which is not linear, usually includes a compensation term outside the function to keep it convex, as in formula (1).

There is another important, *different,* instance of the equation $epi(\mathcal{T}f) = F(epi(f))$, which may occur when the transform is defined on the subset of $Cvx(K)$ consisting of non-negative functions vanishing at the origin (assuming $0 \in K$). We state it now for comparison and elaborate below (Theorem 13). A transform of this second, and essentially different, type (a-la-$\mathcal{J}$, see [4]), corresponds to the inducing fractional linear map:

$$F_{\mathcal{J}}(x,y) = (\frac{x}{y}, \frac{1}{y}),$$

and to the explicit formula:

$$(\mathcal{J}f)(x) = \inf\{r > 0 : rf(\frac{x}{r}) \le 1\}.$$

An immediate corollary of Theorem 2 is that a transform $\mathcal{T} : Cvx(K_1) \to Cvx(K_2)$ can always be extended to a half space, meaning there exist two open half spaces $H_1, H_2$, $K_i \subseteq H_i$ and $\tilde{\mathcal{T}} : Cvx(H_1) \to Cvx(H_2)$ with $\tilde{\mathcal{T}}|_{Cvx(K_1)} = \mathcal{T}$. Extension of these transforms to the whole of $Cvx(\mathbb{R}^n)$ is not always possible; if there is no singularity (i.e. the denominator $L_0$ is a constant function), then $\mathcal{T}$ may indeed be extended to the whole space, and the inducing map $F$ is affine linear. Otherwise, $F$ is a non-affine fractional linear map.

The classical uniqueness theorem of fractional linear maps comes from projective geometry, and is equivalent to the fundamental theorem of projective geometry.

The corresponding uniqueness theorem for *subsets* of $\mathbb{R}P^n$ exists, for example see [8]. The following fact is similar to the result in [8] (there, continuity is assumed, whereas we get it automatically from injectivity and interval preservation). We reprove it in a different and more geometric form in [1].

**Fact 3.** *Let $n \geq 2$ and $K \subseteq \mathbb{R}^n$ a convex set with non empty interior. If $F : K \to \mathbb{R}^n$ is an injective interval preserving map, then $F$ is a fractional linear map.*

It is useful to note that the group of fractional linear maps is generated by its subgroup of affine linear maps and another single non-affine map.

**Proposition 4.** *Denote by $H^+$ the half space $\{x_1 > 1\}$ (where $x_1$ is the first coordinate of $x$) and let the map $F_0 : H^+ \to H^+$ (called the* canonical form *of a fractional linear map) be given by*

$$F_0(x) = \frac{x}{x_1 - 1}.$$

*For any $x_0, y_0 \in \mathbb{R}^n$ and a non-affine fractional linear map $F$ with $F(x_0) = y_0$, there exist $B, C \in GL_n$ such that for every $x \in \mathbb{R}^n$,*

$$B(F(Cx + x_0) - y_0) = F_0(x).$$

Fractional linear maps turn up naturally in convexity. For example, they are strongly connected to the action of polarity, as can be seen in the following, easily verified proposition.

**Proposition 5.** *Let $K \subseteq \{x_1 < 1\} \subset \mathbb{R}^n$ be a closed convex set containing* $0$. *Then for the canonical form $F_0(x) = \frac{x}{x_1 - 1}$ the following holds:*

$$F_0(K) = (e_1 - K^\circ)^\circ,$$

*where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.*

Thus, a general fractional linear image of a convex set (affine or not) may be constructed by two applications of polarity, composed with some affine maps.

The next fact can actually be viewed as a transitivity result on the Klein model of hyperbolic geometry. Let $B_2^n$ denote the open Euclidean unit ball in $\mathbb{R}^n$.

**Fact 6.** *Let $x, y$ be two points in $B_2^n$. There exists a bijective fractional linear map $F : B_2^n \to B_2^n$ such that $F(x) = y$.*

For example, for $n = 2$, $x = (0, 0)$ and $y = (-\frac{1}{2}, 0)$, we have

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2x-1}{2-x} \\ \frac{\sqrt{3}y}{2-x} \end{pmatrix}.$$

A similar transitivity result exists for a simplex in $\mathbb{R}^n$:

**Fact 7.** *Let $\Delta \subset \mathbb{R}^n$ be a non degenerate open simplex, and let $x, y \in \Delta$. There exists a bijective fractional linear map $F : \Delta \to \Delta$ such that $F(x) = y$.*

It is worthwhile to mention that such questions regarding transitivity, and moreover, characterizing the sets $K$ such that on their interior the action of fractional linear maps is transitive, have been considered in the past; for example, see the discussion in [6].

Whereas the structures of the simplex and $B_2^n$ allow for *non-affine* fractional linear maps to preserve these bodies, this is not the case for the cross-polytope $B_1^n = conv\{\pm e_i\}_{i=1}^n$ or the cube $B_\infty^n = [-1, 1]^n$.

**Theorem 8.** *Let $K$ stand for either $B_1^n$ or $B_\infty^n$. Any bijective fractional linear map $F : K \to K$ is linear.*

Next, we briefly present our main theorems regarding characterization of order isomorphisms in different classes of convex functions, generalizing and extending Theorem 2. We introduced earlier the classes $Cvx(K)$ and $Cvx_0(K)$. An equivalent definition of $Cvx_0(K)$ is

$$Cvx_0(K) = \{f \in Cvx(\mathbb{R}^n) : 1_K^\infty \le f \le 1_{\{0\}}^\infty\},$$

where $1_K^\infty$ is the convex characteristic function of $K$ (we assume $K$ includes the origin), equaling zero on $K$ and $+\infty$ elsewhere (likewise $1_{\{0\}}^\infty$ is zero at 0 and $+\infty$ elsewhere). The elements of $Cvx_0(K)$ are called *geometric convex functions* on $K$.

The point-wise supremum of a family of convex functions is convex, thus both $Cvx(K)$ and $Cvx_0(K)$ are closed under this operation. We define the regularized infimum of a family of convex functions to be:

$$\hat{\inf_{\alpha \in A}} \{f_\alpha\} := \sup\{g \in Cvx(K) : \forall \alpha \in A \quad g \le f_\alpha\}.$$

First, we state the generalizations of Theorem 2:

**Theorem 9.** *Let $n \ge 1$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior. If $\mathcal{T} : Cvx(K_1) \to Cvx(K_2)$ is an injective transform satisfying*

1. *$\mathcal{T}(\sup_\alpha f_\alpha) = \sup_\alpha \mathcal{T} f_\alpha$, and*
2. *$\mathcal{T}(\hat{\inf}_\alpha f_\alpha) = \hat{\inf}_\alpha \mathcal{T} f_\alpha$*

*for any family $\{f_\alpha\} \subseteq Cvx(K_1)$, then there exist $K_2' \subseteq K_2$, and a bijective fractional linear map $F : K_1 \times \mathbb{R} \to K_2' \times \mathbb{R}$, such that $\mathcal{T}$ is given by*

$$epi(\mathcal{T} f) = F(epi(f)).$$

*Note that for $x \notin K_2'$ we have $(\mathcal{T} f)(x) = +\infty$, and that $K_2'$ is a fractional linear image of $K_1$.*

Defining $S_{f_0} = Cvx(\mathbb{R}^n) \cap \{f : f_0 \le f\}$ for some fixed $f_0 \in Cvx(\mathbb{R}^n)$, we can show:

**Theorem 10.** *Let $n \ge 1$, and let $f_1, f_2 \in Cvx(\mathbb{R}^n)$ be convex functions with support of full dimension. If $\mathcal{T} : S_{f_1} \to S_{f_2}$ is an order isomorphism, then there exists a bijective fractional linear map $F : epi(f_1) \to epi(f_2)$ such that $\mathcal{T}$ is given by*

$$epi(\mathcal{T} f) = F(epi(f)).$$

*In particular, if there exists such a $\mathcal{T}$, then the epi-graph of $f_2$ is a fractional linear image of the epi-graph of $f_1$.*

**Theorem 11.** *Let $n \ge 1$, and let $f_1, f_2 \in Cvx(\mathbb{R}^n)$ be convex functions with support of full dimension. If $\mathcal{T} : S_{f_1} \to S_{f_2}$ is an injective transform satisfying*

1. *$\mathcal{T}(\sup_\alpha f_\alpha) = \sup_\alpha \mathcal{T} f_\alpha$, and*
2. *$\mathcal{T}(\hat{\inf}_\alpha f_\alpha) = \hat{\inf}_\alpha \mathcal{T} f_\alpha$*

*for any family $\{f_\alpha\} \subseteq S_{f_1}$, then there exists $\tilde{f}_2 \in S_{f_2}$ and a bijective fractional linear map $F : epi(f_1) \to epi(\tilde{f}_2)$ such that $\mathcal{T}$ is given by*

$$epi(\mathcal{T} f) = F(epi(f)).$$

*In particular, $\mathcal{T}$ is an order isomorphism from $S_{f_1}$ to $S_{\tilde{f}_2}$.*

Regarding order reversing isomorphisms, let us remark that for convex sets with non empty interior $K_1, K_2 \subseteq \mathbb{R}^n$ such that either $K_1 \neq \mathbb{R}^n$ or $K_2 \neq \mathbb{R}^n$, an order reversing isomorphism $\mathcal{T} : Cvx(K_1) \to Cvx(K_2)$ does not exist.

We turn to the study of geometric convex functions. First, let us quote the known theorem for the whole class $Cvx_0(\mathbb{R}^n)$, as given in [3].

**Theorem 12.** *Let $n \geq 2$. Any order isomorphism $\mathcal{T} : Cvx_0(\mathbb{R}^n) \to Cvx_0(\mathbb{R}^n)$ is either of the form $\mathcal{T}f = C_0 f \circ B$ or of the form $\mathcal{T}f = C_0(\mathcal{J}f) \circ B$ for some $B \in GL_n$ and $C_0 > 0$.*

Although we see a partition into two types of transforms, this can be somewhat hidden using the fact that transforms of both types are induced by fractional linear maps on the epi-graphs. The next theorem (for windows) is stated this way:

**Theorem 13.** *Let $n \geq 2$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior such that $0 \in K_1, 0 \in K_2$. If $\mathcal{T} : Cvx_0(K_1) \to Cvx_0(K_2)$ is an order preserving isomorphism, then there exists a bijective fractional linear map $F : K_1 \times \mathbb{R}^+ \to K_2 \times \mathbb{R}^+$ such that $\mathcal{T}$ is given by*

$$epi(\mathcal{T}f) = F(epi(f)).$$

However, the investigation of fractional linear maps which may actually appear ("permissible" inducing maps) reveals that the separation into two classes still exists. We define this separation by the action of $F$ on one specific half line; $0 \times \mathbb{R}^+$. This fiber above 0 is the epi-graph of the maximal element and is thus mapped to itself by $F$. The map $g : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $F(0, y) = (0, g(y))$, and there are two options: $g$ is either increasing or decreasing. If $g$ is increasing, we say that $\mathcal{T}$ is of $\mathcal{I}$-type (since this is the case for the identity mapping $I$), and if it is decreasing, we say that $\mathcal{T}$ is of $\mathcal{J}$-type (and then it is very different from $I$). In [1], it is shown that if $0 \in int(K_i)$ and $K_i \neq \mathbb{R}^n$ for some $i$, then there are no $\mathcal{J}$-type transforms.

For the characterization of order reversing isomorphisms, we recall the geometric duality $\mathcal{A}$, which was re-introduced in [4], having first appeared in the book [7] (chapter 15, p. 136) but remained unnoticed in geometric analysis for several decades:

$$(2) \qquad (\mathcal{A}f)(x) = \left\{ \begin{array}{ll} \sup_{\{y \in \mathbb{R}^n : f(y) > 0\}} \frac{\langle x, y\rangle - 1}{f(y)} & \text{if } x \in \{f^{-1}(0)\}^\circ \\ +\infty & \text{if } x \notin \{f^{-1}(0)\}^\circ \end{array} \right\}$$

(with the convention $\sup \emptyset = 0$). The order reversing property of $\mathcal{A}$ can be verified directly or from noting that $\mathcal{A} = \mathcal{J} \circ \mathcal{L}$, which also implies $\mathcal{A}^{-1} = \mathcal{A}$. The (closure of the) epi-graph of $\mathcal{A}f$ is obtained by taking the dual set to $epi(f)$ (in $\mathbb{R}^{n+1}$), and then reflecting it back to $\mathbb{R}^n \times \mathbb{R}^+$. While $\mathcal{A}$ is an order reversing isomorphism from $Cvx_0(\mathbb{R}^n)$ to itself, it does not map $Cvx_0(K)$ to itself if $K \neq \mathbb{R}^n$, and so the following class becomes relevant to us:

**Definition 14.** Let $P \subseteq K \subseteq \mathbb{R}^n$ be closed convex sets. We denote by $Cvx_P(K)$ the subclass of $Cvx(\mathbb{R}^n)$ consisting of non-negative functions which attain 0 on $P$, and $+\infty$ outside $K$. By denoting $1_K^\infty$ and $1_P^\infty$ the convex indicator functions of $K$ and $P$ (zero on the set and $+\infty$ outside) we get

$$Cvx_P(K) = \{f \in Cvx(\mathbb{R}^n) : 1_K^\infty \leq f \leq 1_P^\infty\}.$$

The following theorem is a direct corollary of Theorem 13 (an injective version may also be stated and proved):

**Corollary 15.** *Let $n \geq 2$, and $K \subsetneq \mathbb{R}^n$ a convex set with $0 \in int(K)$. If $\mathcal{T} : Cvx_0(K) \to Cvx_{K^\circ}(\mathbb{R}^n)$ is an order reversing isomorphism, then, up to linear terms, and a fractional linear change of variables, $\mathcal{T}$ is the geometric duality $\mathcal{A}$, i.e. there exist a bijective fractional linear map $F : K \to K$ and affine linear functionals $L_0, L_1 : \mathbb{R}^n \to \mathbb{R}$, such that*

$$(\mathcal{A} \circ \mathcal{T})f = \left(\frac{f}{L_0}\right) \circ F + L_1.$$

We conclude this note by stating the corresponding theorems for $Cvx_P(K)$.

**Theorem 16.** *Let $n \geq 2$, and $P \subseteq int(K)$ compact convex sets in $\mathbb{R}^n$. If $\mathcal{T} : Cvx_P(K) \to Cvx_P(K)$ is an order preserving isomorphism, then up to linear terms and a fractional linear change of variables, $\mathcal{T}$ is the identity map.*

In fact, one may generalize to $\mathcal{T} : Cvx_P(K) \to Cvx_{P'}(K')$, suitably changing the assumptions. The following is an easy consequence of the theorem (which again the reader is invited to state more generally).

**Corollary 17.** *Let $n \geq 2$ and $K, P$ be compact convex sets in $\mathbb{R}^n$ such that $0 \in int(P)$ and $P \subseteq int(K)$. If $\mathcal{T} : Cvx_P(K) \to Cvx_{K^\circ}(P^\circ)$ is an order reversing isomorphism, then, up to linear terms and a fractional linear change of variables, $\mathcal{T}$ is the geometric duality $\mathcal{A}$.*

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