

REALIZATION OF JOINT SPECTRAL RADIUS VIA ERGODIC THEORY

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(Communicated by Boris Hasselblatt)

ABSTRACT. Based on the classic multiplicative ergodic theorem and the semi-uniform subadditive ergodic theorem, we show that there always exists at least one ergodic Borel probability measure such that the joint spectral radius of a finite set of square matrices of the same size can be realized almost everywhere with respect to this Borel probability measure. The existence of at least one ergodic Borel probability measure, in the context of the joint spectral radius problem, is obtained in a general setting.

1. INTRODUCTION

According to classical matrix theory, the spectral radius of a single square matrix $S \in \mathbb{C}^{d \times d}$ is defined as

$$\rho(S) = \max \{ |\lambda| : \lambda \text{ is an eigenvalue of } S \},$$

which satisfies

$$\rho(S) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|S^n\|}.$$

For any finite set $\mathbf{S} = \{S_1, \dots, S_K\} \subset \mathbb{C}^{d \times d}$ of complex square matrices of the same size, the generalized spectral radius of \mathbf{S} was introduced by Daubechies and Lagarias [19] and is defined as

$$\rho(\mathbf{S}) = \limsup_{n \rightarrow +\infty} \sqrt[n]{\rho_n(\mathbf{S})}, \quad \text{where } \rho_n(\mathbf{S}) := \max \{ \rho(A) : A \in \mathbf{S}^n \}.$$

The generalized spectral radius plays a critical role in a variety of applications such as in the study of sensor networks[27], linear switched dynamical systems [15, 13], compactly supported wavelets [11, 24], capacity of codes [7, 30], combinatorics on words [7], solutions of two-scale dilation equations [17, 18], stochastic processes

Received by the editors August 22, 2010 and, in revised form, March 8, 2011.

2010 *Mathematics Subject Classification.* Primary 15A18, 37N40; Secondary 15A69, 65F15.

Key words and phrases. The finiteness conjecture, random product of matrices, joint spectral radius.

The authors would like to thank the anonymous reviewers for their very helpful suggestions which lead to the improvement of this manuscript. This project was supported partly by the National Natural Science Foundation of China (Grant Nos. 11071112 and 11071263) and the Natural Science Foundation of Guangdong, and in part by NSF 0605181 and 1021203 of the United States.

associated with probability transition matrices [1, 26], and so on. For example, for any given discrete-time switched linear system

$$x_n = x_{n-1}S_{i_n}, \quad x_0 \in \mathbb{C}^{1 \times d} \text{ and } n \geq 1,$$

over a switching signal $i : \mathbb{N} \rightarrow \{1, \dots, K\}$, the generalized spectral radius $\rho = \rho(\mathbf{S})$ is the smallest number such that for every trajectory $(x_n)_{n=0}^{+\infty}$, there is a constant C for which

$$\|x_n\| \leq C\rho^n,$$

provided that \mathbf{S} is irreducible, i.e., there are no common, nontrivial, proper and S_i -invariant linear subspaces of $\mathbb{C}^{1 \times d}$ for each $1 \leq i \leq K$.

Another critical characterization of all infinite products of the matrices of \mathbf{S} is the so called *joint spectral radius* of \mathbf{S} , which is defined as

$$\hat{\rho}(\mathbf{S}) = \lim_{n \rightarrow +\infty} \sqrt[n]{\hat{\rho}_n(\mathbf{S})}, \quad \text{where } \hat{\rho}_n(\mathbf{S}) := \max \left\{ \|S_{i_1} \cdots S_{i_n}\| : \text{each } S_{i_k} \in \mathbf{S} \right\}$$

and $\|\cdot\|$ can be any matrix norm satisfying the submultiplicativity (also called ring) property, i.e., $\|AB\| \leq \|A\|\|B\|$. The quantity $\hat{\rho}(\mathbf{S})$ is well-defined and is independent of the matrix norm $\|\cdot\|$ chosen. The notion of joint spectral radius initially appeared in Rota and Strang [35].

As is shown in Berger and Wang [4], there holds the Gel'fand type spectral radius formula $\rho(\mathbf{S}) = \hat{\rho}(\mathbf{S})$; proofs of this fact can be found in, for example, [21, 37, 9, 12]. Moreover, the generalized spectral radius and joint spectral radius satisfy the following inequalities

$$\sqrt[n]{\rho_n(\mathbf{S})} \leq \rho(\mathbf{S}) = \hat{\rho}(\mathbf{S}) \leq \sqrt[n]{\hat{\rho}_n(\mathbf{S})}$$

for any $n \geq 1$, see [4, 19, 20]. It should be noted that all of the equalities and inequalities presented above are also valid for any bounded matrix family with an infinite number of matrices.

In order to study the rate of convergence of all possible infinite products of a finite set of matrices, it is essential to determine whether there is an effectively computable procedure for the computation of $\rho(\mathbf{S})$ or $\hat{\rho}(\mathbf{S})$. Daubechies and Lagarias [19] proposed the following conjecture:

Conjecture (Finiteness Conjecture). *For each finite set \mathbf{S} of $d \times d$ real matrices there is some finite n such that*

$$\rho(\mathbf{S}) = \sqrt[n]{\rho_n(\mathbf{S})}.$$

If the finiteness conjecture were true, then such an algorithm exists, i.e., for $n \geq 1$ compute $\sqrt[n]{\rho_n(\mathbf{S})}$ and $\sqrt[n]{\hat{\rho}_n(\mathbf{S})}$, calculate the difference $\sqrt[n]{\hat{\rho}_n(\mathbf{S})} - \sqrt[n]{\rho_n(\mathbf{S})}$ and stop when it reaches any desired accuracy. For example, let $\mathbf{S} = \{S_1, S_2\}$ defined by

$$S_1 = \alpha^k \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_2 = \alpha^{-1} \begin{bmatrix} \cos \frac{\pi}{2k} & \sin \frac{\pi}{2k} \\ -\sin \frac{\pi}{2k} & \cos \frac{\pi}{2k} \end{bmatrix},$$

where $1 < \alpha < (\cos \frac{\pi}{2k})^{-1}$ and k is a fixed natural number. For $n \leq k$ we have $\rho_n(\mathbf{S}) < 1$, $\hat{\rho}_n(\mathbf{S}) > 1$ and $\rho_{k+1}(\mathbf{S}) = 1 = \hat{\rho}_{k+1}(\mathbf{S})$.

Given a row-vector norm $\|\cdot\|$ on $\mathbb{C}^{1 \times d}$, the induced operator norm, also written as $\|\cdot\|$, on $\mathbb{C}^{d \times d}$ of all $d \times d$ complex matrices is

$$\|A\| = \sup_{x \in \mathbb{C}^{1 \times d}, \|x\|=1} \|xA\|.$$

Lagarias and Wang [29] proposed the following so called normed finiteness conjecture, which is related to the finiteness conjecture:

Conjecture (Normed Finiteness Conjecture). *Let $\|\cdot\|$ be a given vector norm on $\mathbb{R}^{1 \times d}$. Suppose that $\mathbf{S} = \{S_1, \dots, S_K\}$ is a finite set of $d \times d$ real matrices with joint spectral radius $\hat{\rho}(\mathbf{S}) = 1$, for which $\|S_i\| \leq 1$ for each $1 \leq i \leq K$, in the operator norm induced from $\|\cdot\|$. Then there exists a finite n such that*

$$\hat{\rho}(\mathbf{S}) = \sqrt[n]{\rho_n(\mathbf{S})}.$$

Gurvits proved that the normed finiteness conjecture is true for norms whose unit ball is a polytype [25]. Lagarias and Wang showed more generally that it is true for all piecewise analytic norms in $\mathbb{R}^{1 \times d}$ [29], and they also showed that the finiteness conjecture is equivalent to the case that the normed finiteness conjecture is true for all operator norms on all real $d \times d$ matrices. However, the finiteness conjecture has recently been disproved, respectively, by Bousch and Mairesse [10], by Blondel, Theys and Vladimirov [5], and by Kozyakin [28]; all offered the existence of counter-examples. An explicit counterexample can be found in the recent work of Hare *et al.* [23].

In fact, even for the case of two 47×47 matrices with nonnegative rational entries, the problem of checking $\rho(\mathbf{S}) \leq 1$ is algorithmically undecidable (cf. [6, 8, 41]). Specifically, unless $P=NP$, there is no approximation algorithm that is polynomial with respect to the desired accuracy if $\rho(\mathbf{S}) \leq 1$ [41]. Thus the problems of deciding whether $\rho(\mathbf{S}) \leq 1$ or $\rho(\mathbf{S}) < 1$ are NP-hard in general. Today, the list of general matrix sets \mathbf{S} that have polynomial-time computable joint spectral radius is very short. It includes the case where \mathbf{S} contains only triangular matrices of identical orientation, the case of normal matrices, and the case of a symmetric set \mathbf{S} , i.e. $S_i \in \mathbf{S}$ implying $S_i^T \in \mathbf{S}$. In the last case, the joint spectral radius is given by the largest singular value of the matrices in the set; for example, see [33, 40].

Although the finiteness conjecture has been shown to be false, the surrounding ideas are still attractive and important since algorithms that compute the joint spectral radius must be implemented using finite arithmetic.

It is natural to ask why the finiteness conjecture fails to be true. The principal reason is the randomness present in the multiplication order of the matrices. However, the complexity can still be addressed by utilizing the joint spectral radius (or the generalized spectral radius), where all possible multiplications have to be taken into consideration. In [3], Bell showed that for a finite set of $d \times d$ matrices $\mathbf{S} = \{S_1, \dots, S_K\}$, either there is some constant $c > 1$ such that $\hat{\rho}_n(\mathbf{S}) > c^n$ for all n sufficiently large, or $\hat{\rho}_n(\mathbf{S}) = O(n^{d-1})$. Moreover, $\hat{\rho}_n(\mathbf{S}) = O(n^{d-1})$ if and only if the eigenvalues of every matrix in the multiplicative semigroup generated by \mathbf{S} are all on or inside the unit circle. In fact, both the joint spectral radius and generalized spectral radius characterize the **worst-case** operation count, which in general grows faster than any polynomial of the matrix sizes [22].

In this note, we will realize the joint/generalized spectral radius of a finite set of matrices $\mathbf{S} = \{S_1, \dots, S_K\}$ via an ergodic probability measure μ_* , called an “extremal measure” of \mathbf{S} , on the space of all switching sequences $i: \mathbb{N} \rightarrow \{1, \dots, K\}$. See Theorem 3.1 stated in Section 3. For this realization, our main tools are the classical multiplicative ergodic theorem and a semi-uniform subadditive ergodic theorem.

2. PRELIMINARIES

We equip the set $\{1, \dots, K\}$ with the discrete topology, where $K \geq 2$. Let Σ_K^+ be the one-sided symbolic space that consists of all the infinite (switching) sequences

$$i: \mathbb{N} \rightarrow \{1, \dots, K\}, \quad \text{where } \mathbb{N} = \{1, 2, \dots\}.$$

Since Σ_K^+ is in one-to-one correspondence with the Descartes product space, $\{1, \dots, K\}^{\mathbb{N}}$, we can endow Σ_K^+ with the product topology. According to the Tychonoff product theorem, Σ_K^+ is a compact topological space. Moreover, we can introduce a compatible metric on the space Σ_K^+ as follows:

$$d_\varrho(i, i') = \sum_{n=1}^{+\infty} \varrho^{-n} d(i_n, i'_n), \quad \text{where } d(k, k') = \begin{cases} 1 & \text{if } k = k', \\ 0 & \text{if } k \neq k'. \end{cases}$$

Here $\varrho > 1$ is a preassigned constant.

Then, the classical one-sided Markov shift transformation

$$\theta: \Sigma_K^+ \rightarrow \Sigma_K^+; \quad i. \mapsto i_{+1}$$

is continuous and surjective. We denote by \mathcal{F} the Borel σ -field of the symbolic space Σ_K^+ . A probability measure \mathbb{P} on the Borel measurable space $(\Sigma_K^+, \mathcal{F})$ is said to be θ -invariant, if $\mathbb{P} = \mathbb{P} \circ \theta^{-1}$, i.e., $\mathbb{P}(\theta^{-1}(B)) = \mathbb{P}(B)$ for all $B \in \mathcal{F}$. A θ -invariant measure \mathbb{P} is called *ergodic* if the only members B of \mathcal{F} with $\mathbb{P}((B \setminus \theta^{-1}(B)) \cup (\theta^{-1}(B) \setminus B)) = 0$ satisfy $\mathbb{P}(B) = 0$ or $\mathbb{P}(B) = 1$.

By $\mathcal{M}_{erg}(\Sigma_K^+, \theta)$, we denote the set of all θ -ergodic probability measures on $(\Sigma_K^+, \mathcal{F})$. In ergodic theory, a basic result is stated as follows:

Krylov-Bogoliubov's Theorem ([31, 42]). *There always exists at least one ergodic Borel probability measure for a continuous transformation T of a compact metric space X .*

Thus, for the one-sided Markov shift $\theta: \Sigma_K^+ \rightarrow \Sigma_K^+$, there always exist ergodic probability measures. Therefore, $\mathcal{M}_{erg}(\Sigma_K^+, \theta) \neq \emptyset$.

3. REALIZATION OF JOINT SPECTRAL RADIUS

An exact description of the joint spectral radius $\hat{\rho}$ is known to be an overwhelming task beyond today's computing capacity, and substantial research is required to remedy this situation. Following the history of mathematical development, it is natural to consider whether we can discard some exceptional multiplying sequences in looking for typical properties within the majority of multiplying sequences. In other words, could we ask for less but still be able to maintain a general setting, such as in the study of probability theory? Let us give a simple example to illustrate the idea presented in this paper.

Let $\mathbf{S} = \{S_1, \dots, S_K\}$ be a set of complex $d \times d$ matrices, with $\|S_i\| < 1$ for $i = 1, \dots, K-1$ and $\|S_K\| = 1$. For any switching signal $i. = (i_n)_{n=1}^{+\infty} \in \Sigma_K^+$, it is easy to see that as long as at least one index j , with $1 \leq j \leq K-1$, appears infinitely many times in $i.$, we have

$$\prod_{n=1}^N S_{i_n} \rightarrow \mathbf{0}_{d \times d} \quad \text{as } N \rightarrow \infty,$$

where and in what follows, $\mathbf{0}_{d \times d}$ stands for the origin of $\mathbb{C}^{d \times d}$. In other words, multiplying sequences with $j, 1 \leq j \leq K-1$, that appear only finitely many times

may not converge to zero; however, the number of such sequences is only countable and its appearance has zero probability (in an appropriate sense of probability measure on Σ_K^+). Thus, we can describe the infinite products of \mathbf{S} as “ $\mathbf{0}_{d \times d}$ almost surely,” which characterizes the typical property of infinite products of \mathbf{S} . To compute the joint spectral radius $\hat{\rho}(\mathbf{S})$, we only need to focus on those multiplying sequences which do not converge to zero. In this case, it is easy to see that $\hat{\rho}(\mathbf{S}) = 1 = \|S_K\|$.

This example provides us with some interesting hints. On one hand, if we focus on the characterization of all possible infinite products of matrices from \mathbf{S} , we know that these products are zero “almost surely.” On the other hand, if we want to characterize those multiplying sequences that do not approach zero, is there a full probability measure that describes them? In this note, we will answer this question in detail.

Now we are ready to state the following realization theorem of the joint / generalized spectral radius:

Theorem 3.1. *Let $\mathbf{S} = \{S_1, \dots, S_K\} \subset \mathbb{C}^{d \times d}$ be an arbitrary set. Then, there is at least one ergodic Borel probability measure μ_* of $\theta: \Sigma_K^+ \rightarrow \Sigma_K^+$, called an “extremal measure” of \mathbf{S} , such that for μ_* -a.e. $i. \in \Sigma_K^+$*

$$\hat{\rho}(\mathbf{S}) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|S_{i_1} \cdots S_{i_n}\|}.$$

Hence, there exists a θ -invariant Borel set $\widehat{\Sigma} \subset \Sigma_K^+$ with the property $\mu_(\widehat{\Sigma}) = 1$ such that the above limit holds for each $i. \in \widehat{\Sigma}$.*

The significant point of the above result is that the joint spectral radius $\hat{\rho}(\mathbf{S})$ of \mathbf{S} can be realized almost everywhere with respect to this Borel probability extremal measure μ_* . If one can identify a *single multiplying sequence* in $\widehat{\Sigma}$, then we can approximate $\hat{\rho}(\mathbf{S})$ to any accuracy in finite arithmetic. The proof of this result is based on a semi-uniform subadditive ergodic theorem, independently due to Schreiber [36] and Sturman and Stark [39]:

Semi-Uniform Subadditive Ergodic Theorem ([36, 39]). *Let*

$$\{f_\ell\}_{\ell=1}^{+\infty}: \Sigma_K^+ \rightarrow \mathbb{R} \cup \{-\infty\}$$

be a sequence of continuous functions satisfying the subadditivity condition:

$$f_{\ell+m}(i.) \leq f_\ell(i_{.+m}) + f_m(i.) \quad \forall i. \in \Sigma_K^+, \quad \text{for all } \ell, m \geq 1.$$

If there is a constant α such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} f_n(i.) < \alpha \quad \mu\text{-a.e. } i., \quad \forall \mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta),$$

then there exists an $N \geq 1$ such that for any $n \geq N$ there holds

$$\frac{1}{n} f_n(i.) < \alpha$$

for every $i. \in \Sigma_K^+$.

An elementary proof of the above theorem can be found in [14].

Next, we present an important connection between μ -almost sure exponential stability and the joint spectral radius.

Lemma 3.2. *Let $\mathbf{S} = \{S_1, \dots, S_K\} \subset \mathbb{C}^{d \times d}$ be arbitrary. If for any $\mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$, we have*

$$\limsup_{n \rightarrow +\infty} \sqrt[n]{\|S_{i_1} \cdots S_{i_n}\|} < 1 \quad \text{for } \mu\text{-a.e. } i. \in \Sigma_K^+,$$

then $\hat{\rho}(\mathbf{S}) < 1$ holds.

Proof. We denote $\mathbb{K} = \{1, \dots, K\}$. Let

$$\text{Pr}_\ell: \Sigma_K^+ \rightarrow \mathbb{K}^\ell; i. \mapsto (i_1, \dots, i_\ell) \quad \text{and} \quad \mathbf{S}(\text{Pr}_\ell(i.)) = S_{i_1} \cdots S_{i_\ell}$$

for all $\ell \geq 1$, be the natural projections. Clearly, the function $\mathbf{S}(\text{Pr}_1(\cdot))$ satisfies the integrability condition $\log^+ \|\mathbf{S}(\text{Pr}_1(\cdot))\| \in L^1(\mu)$ for all $\mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$, where $Y^+ = \max\{Y, 0\}$, as well as the subadditive property:

$$\log \|\mathbf{S}(\text{Pr}_{\ell+m}(i.))\| \leq \log \|\mathbf{S}(\text{Pr}_\ell(i_{.+m}))\| + \log \|\mathbf{S}(\text{Pr}_m(i.))\|$$

for all $\ell, m \geq 1$ and $i. \in \Sigma_K^+$. Thus, according to the Multiplicative Ergodic Theorem (cf. [26, 32]), one can always find a θ -invariant Borel subset Γ of Σ_K^+ with total measure 1 (i.e. $\mu(\Gamma) = 1 \forall \mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$) such that

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|\mathbf{S}(\text{Pr}_\ell(i.))\| = \boldsymbol{\lambda}(i.) \quad \forall i. \in \Gamma.$$

For any $\ell \geq 1$, let

$$f_\ell(\cdot) = \log \|\mathbf{S}(\text{Pr}_\ell(\cdot))\|: \Sigma_K^+ \rightarrow \mathbb{R} \cup \{-\infty\}.$$

By assumption, the infinite product of matrices in \mathbf{S} is exponentially stable μ -almost surely for any $\mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$, so we have

$$\boldsymbol{\lambda}(i.) < 0 \quad \forall i. \in \Gamma.$$

Thus, by the Semi-Uniform Subadditive Ergodic Theorem and the compactness of Σ_K^+ there exists some $N > 0$ such that

$$\|\mathbf{S}(\text{Pr}_N(i.))\| \leq \gamma < 1 \quad \forall i. \in \Sigma_K^+.$$

This implies that

$$\|S_{i_1} \cdots S_{i_n}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \forall i. \in \Sigma_K^+$$

which completes the proof of Lemma 3.2. \square

We now turn to proving the core result of this note.

Proof of Theorem 3.1. If $\hat{\rho}(\mathbf{S}) = 0$, then

$$\rho(S_k) = 0 = \lim_{n \rightarrow +\infty} \sqrt[n]{\|S_k^n\|} \quad \text{for } 1 \leq k \leq K.$$

So, without loss of any generality, we may assume the joint spectral radius $\hat{\rho}(\mathbf{S}) = 1$ by replacing \mathbf{S} with $\mathbf{S}/\hat{\rho}(\mathbf{S})$ if necessary. Then, for each $\mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$, there exists a constant $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mu)$ such that for μ -a.e. $i. \in \Sigma_K^+$, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log \|S_{i_1} \cdots S_{i_n}\| = \boldsymbol{\lambda} \leq 0$$

according to the Multiplicative Ergodic Theorem (see [26, 32]).

Next, we claim that there exists at least one $\mu_* \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$ such that

$$\boldsymbol{\lambda}(\mu_*) = 0 = \log \hat{\rho}(\mathbf{S});$$

that is to say

$$\lim_{n \rightarrow +\infty} \sqrt[n]{\|S_{i_1} \cdots S_{i_n}\|} = \hat{\rho}(\mathbf{S})$$

for μ_* -a.e. $i. \in \Sigma_K^+$.

Suppose, by contradiction, that this were not true; then all infinite products of matrices in \mathbf{S} are μ -a.s. exponentially stable since $\lambda(\mu) < 0$ for all $\mu \in \mathcal{M}_{erg}(\Sigma_K^+, \theta)$. Thus, from Lemma 3.2 it follows that $\hat{\rho}(\mathbf{S}) < 1$. This contradicts the assumption that $\hat{\rho}(\mathbf{S}) = 1$.

The proof of Theorem 3.1 is thus completed. \square

We notice here that Theorem 3.1 only claims the existence of the extremal measures. One can easily construct a finite set \mathbf{S} that has more than one extremal measure. In addition, an extremal measure μ_* is independent of the choice of norm $\|\cdot\|$ used on the state space $\mathbb{C}^{1 \times d}$.

Recall that a switching sequence $i. \in \Sigma_K^+$ is said to be periodic with period $\pi \geq 1$ if $i_{\ell+m\pi} = i_\ell$ for all $1 \leq \ell \leq \pi$ and any $m \geq 0$. For $\mathbf{S} = \{S_1, \dots, S_K\} \subset \mathbb{C}^{d \times d}$, it is called ‘‘periodically switched stable’’ [34, 38, 16], provided that $\prod_{n=1}^N S_{i_n}$ converges to $\mathbf{0}_{d \times d}$ as $N \rightarrow +\infty$ for all periodical switching signals $i. \in \Sigma_K^+$. This is equivalent to having $\rho(S_{i_1} \cdots S_{i_n}) < 1$ for all $i. \in \Sigma_K^+$ and any $n \geq 1$. Thus, if \mathbf{S} is periodically switched stable with the finiteness property, then $\hat{\rho}(\mathbf{S}) < 1$ by Berger-Wang formula, and further, \mathbf{S} is absolutely asymptotically exponentially stable according to [2] or [37]. Unfortunately, there already exist counterexamples that are periodically switched stable, but do not to have the finiteness property [10, 5, 28, 23]. For this case, \mathbf{S} is periodically switched stable with $\hat{\rho}(\mathbf{S}) = 1$, and moreover, \mathbf{S} is exponentially stable almost surely over any (\mathbf{p}, \mathbf{P}) -Markovian probabilities from [16]. On the other hand, from Theorem 3.1, it follows that there exists at least one θ -ergodic probability measure μ_* on Σ_K^+ which is extremal for \mathbf{S} . Therefore, if \mathbf{S} is periodically switched stable with $\hat{\rho}(\mathbf{S}) = 1$, then every (\mathbf{p}, \mathbf{P}) -Markovian probability is not extremal for \mathbf{S} .

4. FINAL REMARKS

In this short note, we have shown that there always exists at least one ergodic Borel probability measure such that the joint spectral radius of a finite set of matrices can be realized. This indicates that for the index sequences/switching signals belonging to the support of the obtained invariant ergodic measure, there are correctly defined frequencies of many matrix products for which

$$\hat{\rho}(\mathbf{S}) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|S_{i_1} \cdots S_{i_n}\|}$$

is valid. How to identify the specific properties of such invariant measures remains unknown, and a further study will be reported elsewhere.

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