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ON SUBGROUPS OF THE DIXMIER GROUP AND CALOGERO-MOSER SPACES

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Abstract. We describe the structure of the automorphism groups of algebras Morita equivalent to the first Weyl algebra $A_1(k)$. In particular, we give a geometric presentation for these groups in terms of amalgamated products, using the Bass-Serre theory of groups acting on graphs. A key rôle in our approach is played by a transitive action of the automorphism group of the free algebra $k\langle x, y\rangle$ on the Calogero-Moser varieties $\mathcal{C}_n$ defined in [5]. In the end, we propose a natural extension of the Dixmier Conjecture for $A_1(k)$ to the class of Morita equivalent algebras.

1. Introduction. Let k be an algebraically closed field of characteristic 0. Let $A_1(k) := k\langle x,\, y\rangle/(xy-yx-1)$ be the first Weyl algebra over k, with canonical generators x and y. In his classic paper [11], J. Dixmier described the group $\mathrm{Aut}_k\, A_1$ of automorphisms of $A_1(k)$; specifically, he proved that $\mathrm{Aut}_k\, A_1$ is generated by the following transformations:

$$\Phi_p \,:\, (x,\, y) \mapsto (x,\, y + p(x)) \quad \text{and} \quad \Psi_q \,:\, (x,\, y) \mapsto (x + q(y),\, y), \tag{1}$$

where $p(x) \in k[x]$ and $q(y) \in k[y]$. Using this result of Dixmier, L. Makar-Limanov (see [15, 16]) showed that $\mathrm{Aut}_k\, A_1$ is isomorphic to the group $G_0 \subset \mathrm{Aut}_k\, k\langle x,\, y\rangle$ of 'symplectic' (i.e. preserving $[x, y] := xy - yx$) automorphisms of the free algebra $k\langle x,\, y\rangle$: the corresponding isomorphism

$$G_0 \xrightarrow{\sim} \mathrm{Aut}_k\, A_1 \tag{2}$$

is induced by the canonical projection $k\langle x,\, y\rangle \to A_1(k)$. In [15], Makar-Limanov also gave a presentation for the group G_0, from which one can easily deduce (see, e.g., [10]) that G_0 is the amalgamated free product

$$G_0 = A *_U B\ , \tag{3}$$

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where A is the subgroup of symplectic affine transformations

(4) $\quad (x,\, y) \mapsto (ax + by + e,\, cx + dy + f)\ , \quad a,\, b, \ldots, f \in k\ , \quad ad - bc = 1\ ,$

B is the subgroup of triangular (Jonquières) transformations

(5) $\quad (x,\, y) \mapsto (ax + q(y),\, a^{-1}y + h)\ , \quad a \in k^*,\ h \in k\ , \quad q(y) \in k[y]\ ,$

and U is the intersection of A and B in G_0:

(6) $\quad (x,\, y) \mapsto (ax + by + e,\, a^{-1}y + h)\ , \quad a \in k^*,\ b,\, e,\, h \in k\ .$

Combining (2) and (3), we thus have the decomposition $\mathrm{Aut}_k\, A_1 \cong A *_U B$, which completely describes the structure of $\mathrm{Aut}_k\, A_1$ as a discrete group (cf. [1]).

The aim of the present paper is to generalize the above results to the case when A_1 is replaced by a noncommutative domain D, which is Morita equivalent to A_1 as a k-algebra. This question was originally raised by J. T. Stafford in [19] (see *op. cit.*, p. 636). To explain why it is natural we recall that the algebras D are classified, up to isomorphism, by a single integer $n \geq 0$; the corresponding isomorphism classes are represented by the endomorphism rings $D_n := \mathrm{End}_{A_1}(M_n)$ of the (fractional) ideals $M_n = x^n A_1 + (y + nx^{-1})\, A_1$ and can be realized geometrically as rings of global differential operators on some rational singular curves (see [14, 6] and [9] for a detailed exposition). Thus, the Dixmier group $\mathrm{Aut}_k\, A_1 = \mathrm{Aut}_k\, D_0$ appears naturally as the first member in the family $\{\mathrm{Aut}_k\, D_n \ :\ n \geq 0\}$. Our aim is to describe the 'higher' groups in this family: specifically, to give a presentation of $\mathrm{Aut}_k\, D_n$ for arbitrary $n \geq 0$ in terms of amalgamated products.

The present paper is mostly a research announcement: we focus here on explanation and motivation of our results; detailed proofs and computations will appear elsewhere.

2. Calogero-Moser spaces and the Makar-Limanov isomorphism. Recall that G_0 is the automorphism group of the free algebra $k\langle x,\, y\rangle$ preserving $[x,\, y]$. Now, for $n > 0$, we introduce the groups G_n geometrically, using a natural action of G_0 on the *Calogero-Moser spaces*

(7) $\quad \mathcal{C}_n := \{\, (X,\, Y) \in \mathrm{M}_n(k) \times \mathrm{M}_n(k)\ :\ \mathrm{rk}([X,\, Y] + I_n) = 1\,\}/\, \mathrm{PGL}_n(k)\ ,$

where $\mathrm{M}_n(k)$ denotes the space of $n \times n$ matrices with entries in k, and $\mathrm{PGL}_n(k)$ operates on pairs of matrices $(X,\, Y)$ by simultaneous conjugation. The action of G_0 on $\mathcal{C}_n$ is defined by

(8) $\quad (X,\, Y) \mapsto (\sigma^{-1}(X),\, \sigma^{-1}(Y))\ , \quad \sigma \in G_0\ ,$

where $\sigma^{-1}(X)$ and $\sigma^{-1}(Y)$ are the noncommutative polynomials $\sigma^{-1}(x) \in k\langle x,\, y\rangle$ and $\sigma^{-1}(y) \in k\langle x,\, y\rangle$ evaluated at (X, Y). In [21], G. Wilson showed that $\mathcal{C}_n$ is a smooth irreducible affine variety of dimension $2n$, equipped with a natural symplectic form: $\omega = \mathrm{tr}(dX \wedge dY)$. Knowing that G_0 is generated by triangular transformations (1), it is easy to see that the action (8) is symplectic. Much less obvious is the fact that (8) is *transitive* for all $n \geq 0$. This last fact was proven in [5] (Theorem 1.3), and it plays a crucial role in the present paper.

Now, we define the groups G_n to be the stabilizers of points of $\mathcal{C}_n$ under the action (8): precisely, for each $n \geq 0$, we fix a basepoint $(X_0,\, Y_0) \in \mathcal{C}_n$, with

(9) $\quad X_0 = \sum_{k=1}^{n-1} E_{k+1,k} \quad , \qquad Y_0 = \sum_{k=1}^{n-1} (k-n)\, E_{k,k+1}\ ,$

where $E_{i,j}$ stands for the elementary matrix with (i,j)-entry 1, and let

$$G_n := \mathrm{Stab}_{G_0}(X_0, Y_0)\ , \quad n \geq 0\ .$$

The following result can be viewed as a generalization of the above-mentioned theorem of Makar-Limanov.

Theorem 1. *There is a natural isomorphism of groups* $G_n \xrightarrow{\sim} \mathrm{Aut}_k\, D_n$.

To construct the isomorphism of Theorem 1, we first note that the groups $\mathrm{Aut}_k\, D_n$ can be naturally identified with subgroups of $\mathrm{Aut}_k\, D_0$. To be precise, let $\mathrm{Pic}_k D$ denote the (noncommutative) Picard group of a k-algebra D. By definition, $\mathrm{Pic}_k D$ is the group of k-linear Morita equivalences of the category of D-modules; its elements can be represented by the isomorphism classes of invertible D-bimodules $[P]$ (see, e.g., [3], Ch. II, Sect. 5). There is a natural group homomorphism $\omega_D : \mathrm{Aut}_k\, D \to \mathrm{Pic}_k\, D$, taking $\sigma \in \mathrm{Aut}_k\, D$ to the class of the bimodule $[{}_1 D_\sigma]$, and if D' is a ring Morita equivalent to D, with progenerator M, then there is a group isomorphism $\alpha_M : \mathrm{Pic}_k\, D' \xrightarrow{\sim} \mathrm{Pic}_k\, D$ given by

$$[P] \mapsto [M^* \otimes_{D'} P \otimes_{D'} M]\ .$$

Thus, in our situation, for each $n \geq 0$, we have the following diagram

$$\begin{array}{ccc} \mathrm{Aut}_k\, D_n & \xrightarrow{\ \omega_{D_n}\ } & \mathrm{Pic}_k\, D_n \\ {\scriptstyle i_n}\,\vdots\downarrow & & \downarrow{\scriptstyle \alpha_{M_n}} \\ \mathrm{Aut}_k\, D_0 & \xrightarrow{\ \omega_{D_0}\ } & \mathrm{Pic}_k\, D_0 \end{array} \tag{10}$$

where the vertical map α_{M_n} is an isomorphism and the two horizontal maps are injective. Moreover, since $D_0 = A_1$, a theorem of Stafford (see [19], Theorem 4.7) implies that ω_{D_0} is actually an isomorphism. Inverting this isomorphism, we define the embedding $i_n : \mathrm{Aut}_k\, D_n \hookrightarrow \mathrm{Aut}_k\, D_0$, which makes (10) a commutative diagram. Now, we have group homomorphisms

$$G_n \ \hookrightarrow\ G_0 \xrightarrow{\sim} \mathrm{Aut}_k\, A_1 \xleftarrow{\ i_n\ } \mathrm{Aut}_k\, D_n\ ,$$

where the first map is the canonical inclusion and the second is Makar-Limanov's isomorphism (2). The claim of Theorem 1 is that the image of i_n coincides with the image of G_n in $\mathrm{Aut}_k\, A_1$: this gives the required isomorphism $G_n \xrightarrow{\sim} \mathrm{Aut}_k\, D_n$.

Theorem 1 is a consequence of the main results of [5]. In fact, it is shown in [5] that there is a natural G_0-equivariant bijection between $\bigsqcup_{n\geq 0} \mathcal{C}_n$ and the space of isomorphism classes of right ideals of A_1. This bijection can be described explicitly as follows (see [4]). A point of $\mathcal{C}_n$ is represented by a pair of linear endomorphisms (X, Y) of k^n satisfying the condition that $[X, Y] + I_n$ has rank 1. Factoring $[X, Y] + I_n = ij$ in $\mathrm{End}(k^n)$, with $i \in k^n$ and $j \in \mathrm{Hom}(k^n, k)$, we define $\chi(X, Y) := 1 + j\,(X - x\,I_n)^{-1}(Y - y\,I_n)^{-1} i$ as an element in the quotient field of $A_1(k)$ and assign to (X, Y) the fractional ideal

$$M(X, Y) = \det(X - x\,I_n)\,A_1 \ +\ \chi(X, Y) \cdot \det(Y - y\,I_n)\,A_1\ .$$

The assignment $(X, Y) \mapsto M(X, Y)$ induces a map from $\mathcal{C}_n$ to the set of isomorphism classes of right ideals of A_1; amalgamating such maps for all n yields the required bijection. Notice that $M(X_0, Y_0) = M_n$ for X_0 and Y_0 given by (9), so our basepoints $(X_0, Y_0) \in \mathcal{C}_n$ correspond precisely to the classes of the ideals M_n whose endomorphism rings are the algebras D_n.

3. G_n **as a fundamental group.** We will use Theorem 1 to give a geometric presentation for the groups $\mathrm{Aut}_k\, D_n$. To this end, we associate to each space $\mathcal{C}_n$ a graph Γ_n consisting of orbits of certain subgroups of G_0 and identify G_n with the fundamental group $\pi_1(\mathbf{\Gamma}_n, *)$ of a graph of groups $\mathbf{\Gamma}_n$ defined by stabilizers of those orbits in Γ_n. The Bass-Serre theory of groups acting on graphs will then give an explicit formula for $\pi_1(\mathbf{\Gamma}_n, *)$ in terms of generalized amalgamated products.

To state our results in precise terms we recall the notion of a graph of groups and its fundamental group, following Serre's classic book [17]. The readers unfamiliar with Bass-Serre theory are recommended to skim §5 of Chapter I of [17].

A *graph of groups* $\mathbf{\Gamma} = (\Gamma, G)$ consists of the following data: (1) a connected oriented graph Γ with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$ and incidence maps $i, t : E \to V$, (2) a group G_a assigned to each vertex $a \in V$, (3) a group G_e assigned to each edge $e \in E$, (4) a pair of injective group homomorphisms $\alpha_e : G_e \hookrightarrow G_{i(e)}$ and $\beta_e : G_e \hookrightarrow G_{t(e)}$ defined for each $e \in E$. Associated to $\mathbf{\Gamma}$ is the path group $\pi(\mathbf{\Gamma})$, which is given by the presentation

$$\pi(\mathbf{\Gamma}) := \frac{(*_{a\in V} G_a) * \langle E\rangle}{(e^{-1}\alpha_e(g)\, e = \beta_e(g)\ :\ \forall\, e \in E,\, \forall\, g \in G_e)}\ ,$$

where '$*$' stands for the free product (= coproduct in the category of groups) and $\langle E\rangle$ for the free group with basis set $E = E(\Gamma)$. Now, choosing a maximal tree T in Γ, we define $\pi_1(\mathbf{\Gamma}, T)$, the *fundamental group of* $\mathbf{\Gamma}$ *relative to* T, as a quotient of $\pi(\mathbf{\Gamma})$ by 'contracting the edges of T to a point': precisely,

$$\pi_1(\mathbf{\Gamma}, T) := \pi(\mathbf{\Gamma})/(e = 1\ :\ \forall\, e \in E(T))\ . \tag{11}$$

For different maximal trees $T \subseteq \Gamma$, the groups $\pi_1(\mathbf{\Gamma}, T)$ are isomorphic. Moreover, if $\mathbf{\Gamma}$ is trivial (i. e. $G_a = \{1\}$ for all $a \in V$), then $\pi_1(\mathbf{\Gamma}, T)$ is isomorphic to the usual fundamental group $\pi_1(\Gamma, a_0)$ of the graph Γ viewed as a CW-complex. In general, $\pi_1(\mathbf{\Gamma}, T)$ can be also defined in a topological fashion by introducing an appropriate notion of path and homotopy equivalence of paths in $\mathbf{\Gamma}$.

When its underlying graph is a tree ($\Gamma = T$), $\mathbf{\Gamma}$ can be viewed as a directed system of groups $\{G_{i(e)} \xleftarrow{\alpha_e} G_e \xrightarrow{\beta_e} G_{t(e)}\}$ indexed by the edges of T. In this case, the fundamental group $\pi_1(\mathbf{\Gamma}, T)$ is given by the inductive limit $\varinjlim \mathbf{\Gamma}$, which is called the *tree product* of groups $\{G_a\}$ amalgamated by $\{G_e\}$ along T. For example, if T is a segment with $V(T) = \{0, 1\}$ and $E(T) = \{e\}$, the tree product is the usual amalgamated free product $G_0 *_{G_e} G_1$. In general, abusing notation, we will denote the tree product by

$$G_{a_1} *_{G_{e_1}} G_{a_2} *_{G_{e_2}} G_{a_3} *_{G_{e_3}} \ldots$$

Now, we return to our situation. To define the graph Γ_n we take the subgroups A, B and U of G_0 defined by the transformations (4), (5) and (6), respectively. Restricting the action of G_0 on $\mathcal{C}_n$ to these subgroups, we let Γ_n be the oriented bipartite graph, with vertex and edge sets

$$V(\Gamma_n) := (A\backslash\mathcal{C}_n) \bigsqcup (B\backslash\mathcal{C}_n)\ , \quad E(\Gamma_n) := U\backslash\mathcal{C}_n\ , \tag{12}$$

and the incidence maps $E(\Gamma_n) \to V(\Gamma_n)$ given by the canonical projections $i : U\backslash\mathcal{C}_n \to A\backslash\mathcal{C}_n$ and $t : U\backslash\mathcal{C}_n \to B\backslash\mathcal{C}_n$. Since the elements of A and B generate G_0 and G_0 acts transitively on each $\mathcal{C}_n$, the graph Γ_n is connected.

Now, on each orbit in $A\backslash\mathcal{C}_n$ and $B\backslash\mathcal{C}_n$, we choose a basepoint and elements $\sigma_A \in G_0$ and $\sigma_B \in G_0$ moving these basepoints to the basepoint (X_0, Y_0) of $\mathcal{C}_n$.

Next, on each U-orbit $\mathcal{O}_U \in U\backslash\mathcal{C}_n$, we also choose a basepoint and an element $\sigma_U \in G_0$ moving this basepoint to (X_0, Y_0) such that $\sigma_U \in \sigma_A A \cap \sigma_B B$, where σ_A and σ_B correspond to the (unique) A- and B-orbits containing $\mathcal{O}_U$. Then, we assign to the vertices and edges of Γ_n the stabilizers $A_\sigma = G_n \cap \sigma A \sigma^{-1}$, $B_\sigma = G_n \cap \sigma B \sigma^{-1}$, $U_\sigma = G_n \cap \sigma U \sigma^{-1}$ of the corresponding elements σ in the graph of right cosets of G_0 under the action of G_n. These data together with natural group homomorphisms $\alpha_\sigma : U_\sigma \hookrightarrow A_\sigma$ and $\beta_\sigma : U_\sigma \hookrightarrow B_\sigma$ define a graph of groups $\mathbf{\Gamma}_n$ over Γ_n, and its fundamental group $\pi_1(\mathbf{\Gamma}_n, T)$ relative to a maximal tree $T \subseteq \Gamma_n$ has canonical presentation, cf. (11):

$$\pi_1(\mathbf{\Gamma}_n, T) = \frac{(A_\sigma *_{U_\sigma} B_\sigma * \ldots) * \langle\, E(\Gamma_n \backslash T)\,\rangle}{(\, e^{-1}\alpha_\sigma(g)\, e = \beta_\sigma(g)\ :\ \forall\, e \in E(\Gamma_n \backslash T),\ \forall\, g \in U_\sigma\,)}\ . \tag{13}$$

In (13), the amalgam $(A_\sigma *_{U_\sigma} B_\sigma * \ldots)$ stands for the tree product taken along the edges of T, while $\langle\, E(\Gamma_n \backslash T)\,\rangle$ denotes the free group based on the set of edges of Γ_n in the complement of T.

Our main result is the following

Theorem 2. *For each $n \geq 0$, the group G_n is isomorphic to $\pi_1(\mathbf{\Gamma}_n, T)$. In particular, G_n has an explicit presentation of the form* (13)*.*

Theorems 1 and 2 reduce the problem of describing the groups $\mathrm{Aut}_k\, D_n$ to a purely geometric problem of describing the structure of the orbit spaces of A and B and U on the Calogero-Moser varieties $\mathcal{C}_n$. Using the earlier results of [21] and [5] and some geometric invariant theory, one can obtain much information about these orbits (and hence about the groups G_n). In particular, the graphs Γ_n can be completely described for small n; it turns out that Γ_n is a finite tree for $n = 0, 1, 2$ (see examples below), but has infinitely many cycles for $n \geq 3$.

4. The graphs Γ_n and an adelic Grassmannian. We now explain the origin of the graphs Γ_n by realizing them as quotient graphs of a certain 'universal' tree Γ, on which all the groups $\mathrm{Aut}_k\, D_n$ naturally act. Our construction of Γ is motivated by algebraic geometry: specifically, an application of the Bass-Serre theory in the theory of surfaces (see, e.g., [13], [23]). In that approach, the automorphism group of an affine surface S is described via its action on a tree, whose vertices correspond to certain (admissible) projective compactifications of S. Following a standard philosophy in noncommutative geometry (see, e.g., [20]), we may think of our algebra D as the coordinate ring of a 'noncommutative affine surface'; a 'projective compactification' of D is then determined by a choice of filtration. Thus, we will define Γ by taking as its vertices a certain class of filtrations on the algebra D. It turns out that these filtrations can be naturally parametrized by an infinite-dimensional *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$ introduced in [22] and studied in [21, 5, 8] (in particular, we rely heavily on results of [8]). Our contruction is close in spirit to Serre's classic application of Bruhat-Tits trees for computing arithmetic subgroups of $\mathrm{SL}_2(\mathbb{K})$ over the function fields of curves (see [17], Chap. II, § 2); however, we are not aware of a direct connection.

We begin by briefly recalling the definition of $\mathrm{Gr}^{\mathrm{ad}}$. Let $k[z]$ be the polynomial ring in one variable z. For each $\lambda \in k$, we choose a λ-*primary* subspace in $k[z]$; that is, a k-linear subspace $V_\lambda \subseteq k[z]$ containing a power of the maximal ideal $\mathfrak{m}_\lambda$ at λ. We suppose that $V_\lambda = k[z]$ for all but finitely many λ's. Let $V = \bigcap_\lambda V_\lambda$

(such a subspace V is called *primary decomposable* in $k[z]$) and, finally, let

$$W = \prod_{\lambda} (z-\lambda)^{-n_\lambda} V \subset k(z)\,,$$

where n_λ is the codimension of V_λ in $k[z]$. By definition, $\mathrm{Gr}_k^{\mathrm{ad}}$ consists of all subspaces $W \subset k(z)$ obtained in this way. For each $W \in \mathrm{Gr}_k^{\mathrm{ad}}$ we set

$$A_W := \{f \in k[z] \,:\, fW \subseteq W\}\,.$$

Taking Spec of A_W then gives a rational curve X, the inclusion $A_W \hookrightarrow k[z]$ corresponds to normalization $\pi : \mathbb{A}_k^1 \to X$ (which is set-theoretically a bijective map), and the A_W-module W defines a rank 1 torsion-free coherent sheaf $\mathcal{L}$ over X. In this way, the points of $\mathrm{Gr}_k^{\mathrm{ad}}$ correspond bijectively to isomorphism classes of triples $(\pi, X, \mathcal{L})$ (see [22]).

Now, following [5], for $W \in \mathrm{Gr}_k^{\mathrm{ad}}$ we define[1]

$$D(W) := \{\mathcal{D} \in k(z)[\partial_z] \,:\, \mathcal{D}W \subseteq W\}\,, \tag{14}$$

where $k(z)[\partial_z]$ is the ring of rational differential operators in the variable z. This last ring carries two natural filtrations: the standard filtration, in which both generators z and ∂_z have degree 1, and the differential filtration, in which $\deg(z) = 0$ and $\deg(\partial_z) = 1$. These filtrations induce two different filtrations on the algebra $D(W)$, which we denote by $\{D_\bullet^A(W)\}$ and $\{D_\bullet^B(W)\}$ respectively.

Now, let D be a fixed domain Morita equivalent to $A_1(k)$. Following [8], we consider the set[2] $\mathrm{Gr}_k^{\mathrm{ad}}(D)$ of all algebra isomorphisms $\sigma_W : D(W) \to D$, where $W \in \mathrm{Gr}_k^{\mathrm{ad}}$ (more precisely, $\mathrm{Gr}_k^{\mathrm{ad}}(D)$ is the set of all pairs (W, σ_W), where $W \in \mathrm{Gr}_k^{\mathrm{ad}}$ and σ_W is an isomorphism as above). Each $\sigma_W \in \mathrm{Gr}_k^{\mathrm{ad}}(D)$ maps the two distinguished filtrations $\{D_\bullet^A(W)\}$ and $\{D_\bullet^B(W)\}$ into the algebra D: we call their images the *admissible* filtrations on D of type A and type B, respectively. Let $\mathbb{P}_A(D)$ and $\mathbb{P}_B(D)$ denote the sets of all such filtrations coming from various $\sigma_W \in \mathrm{Gr}_k^{\mathrm{ad}}(D)$. By definition, we have then two natural projections

$$\mathbb{P}_A(D) \xleftarrow{\pi_A} \mathrm{Gr}_k^{\mathrm{ad}}(D) \xrightarrow{\pi_B} \mathbb{P}_B(D)\;. \tag{15}$$

We say that (W, σ_W) and (W', σ'_W) are *equivalent* in $\mathrm{Gr}_k^{\mathrm{ad}}(D)$ if their images under π_A and π_B coincide. Writing $\mathrm{Gr}_k^{\mathrm{ad}}(D)/\!\sim$ for the set of equivalence classes under this relation, we define an oriented graph Γ by

$$V(\Gamma) := \mathbb{P}_A(D) \bigsqcup \mathbb{P}_B(D)\;, \quad E(\Gamma) := \mathrm{Gr}_k^{\mathrm{ad}}(D)/\!\sim\;,$$

with incidence maps $E(\Gamma) \to V(\Gamma)$ induced by the projections (15). Observe that the group $\mathrm{Aut}_k\, D$ acts naturally on the set $\mathrm{Gr}_k^{\mathrm{ad}}(D)$ (by composition), and this action induces an action of $\mathrm{Aut}_k\, D$ on the graph Γ via (15). We write $\mathrm{Aut}_k(D)\backslash\Gamma$ for the corresponding quotient graph.

Theorem 3. (*a*) *Γ is a tree, which is independent of D (up to isomorphism).*

(*b*) *For each $n \geq 0$, the graph $\mathrm{Aut}_k(D_n)\backslash\Gamma$ is naturally isomorphic to Γ_n.*

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[1] In geometric terms, $D(W)$ can be thought of as the ring $D_{\mathcal{L}}(X)$ of twisted differential operators on X with coefficients in $\mathcal{L}$.

[2] More generally, we may think of $\mathrm{Gr}_k^{\mathrm{ad}}$ as a groupoid, in which the objects are the W's and the arrows are given by the algebra isomorphisms $D(W) \to D(W')$. For $D = D(W)$, the set $\mathrm{Gr}_k^{\mathrm{ad}}(D)$ is then a costar in $\mathrm{Gr}_k^{\mathrm{ad}}$, consisting of all arrows with target at W. In [8], this set was denoted by `Grad` D.

Theorem 3 can be viewed as a generalization of the main results of [8]. Indeed, this last paper is concerned with a description of the maximal abelian ad-nilpotent (*mad*) subalgebras of D_n : its main theorems (Theorem 1.5 and Theorem 1.6) say that the space $\mathrm{Mad}(D_n)$ of all mad subalgebras of D_n is independent of D_n and its quotient modulo the natural action of $\mathrm{Aut}_k\, D_n$ is isomorphic to the orbit space $B\backslash\mathcal{C}_n$. Now, every admissible filtration of type B determines a mad subalgebra of D_n, which is simply the degree zero component of that filtration. (Indeed, by definition, a type B filtration comes through an isomorphism from the usual differential filtration on some $D(W)$, but the degree zero component of the differential filtration is just A_W, which is certainly a mad subalgebra of $D(W)$.) Thus, we have a well-defined map $\mathbb{P}_B(D_n) \to \mathrm{Mad}(D_n)$. This map is injective, since each type B filtration is maximal ad-nilpotent and hence determined by its degree zero component. On the other hand, Theorem 1.4 of [8] says that *every* mad subalgebra of D_n arises from some $W \in \mathrm{Gr}^{\mathrm{ad}}$: this implies the surjectivity of the above map. Summing up, we have a natural bijection $\mathbb{P}_B(D_n) \cong \mathrm{Mad}(D_n)$, which is equivariant under the action of $\mathrm{Aut}_k\, D_n$. This means that $\mathbb{P}_B(D_n)$ does not depend on D_n, which is part of Theorem 3(a), and

$$\mathrm{Aut}_k(D_n)\backslash\mathbb{P}_B(D_n) \;\cong\; \mathrm{Aut}_k(D_n)\backslash\mathrm{Mad}(D_n) \;\cong\; B\backslash\mathcal{C}_n\ ,$$

which is part of Theorem 3(b). The main part of Theorem 3 does not follow directly from the results of [8], although its proof relies on techniques of that paper.

In the end, we should mention that, for $D = A_1(k)$, our construction of the tree Γ agrees with the one given in [1].

5. Examples. We now look at the graphs Γ_n and groups G_n for small n. For $n = 0$, the space $\mathcal{C}_0$ is just a point, and so are a fortiori its orbit spaces. The graph Γ_0 is thus a segment, and the corresponding graph of groups $\mathbf{\Gamma}_0$ is given by $[\,A \xrightarrow{U} B\,]$. Formula (13) then says that $G_0 = A *_U B$, which agrees, of course, with Makar-Limanov's isomorphism (3), and G_0 is generated by its subgroups

$$\begin{aligned} G_{0,x} &:= \{\,\Phi_p \in G_0 \;:\; p \in k[x]\,\}\ ,\\ G_{0,y} &:= \{\,\Psi_q \in G_0 \;:\; q \in k[y]\,\}\ , \end{aligned}$$

which is the Dixmier result cited in the Introduction.

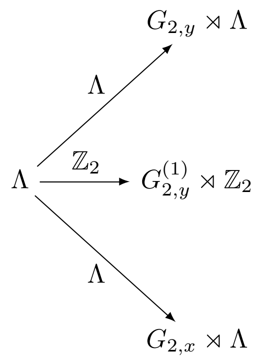
For $n = 1$, we have $\mathcal{C}_1 \cong \mathbb{A}^2_k$, with (X_0, Y_0) corresponding to the origin. Since each of the groups A, B and U contains translations $(x, y) \mapsto (x+a, y+b)$, $a, b \in k$, they act transitively on $\mathcal{C}_1$. So again Γ_1 is just the segment, and $\mathbf{\Gamma}_1$ is given by $[\,A_1 \xrightarrow{U_1} B_1\,]$, where $A_1 := G_1 \cap A$, $B_1 := G_1 \cap B$ and $U_1 := G_1 \cap U$. Since, by definition, G_1 consists of all $\sigma \in G_0$ preserving $(0, 0)$, the groups A_1, B_1 and U_1 are obvious:

$$\begin{aligned} A_1 :&\quad (x,\, y) \mapsto (ax + by,\, cx + dy)\ ,\quad a,\, b,\, c,\, d \in k\ ,\ ad - bc = 1\ ,\\ B_1 :&\quad (x,\, y) \mapsto (ax + q(y),\, a^{-1}y)\ ,\quad a \in k^*\ ,\ q \in k[y]\ ,\ q(0) = 0\ ,\\ U_1 :&\quad (x,\, y) \mapsto (ax + by,\, a^{-1}y)\ ,\quad a \in k^*\ ,\ b \in k\ . \end{aligned}$$

It follows from (13) that $G_1 = A_1 *_{U_1} B_1$. In particular, G_1 is generated by its subgroups

$$\begin{aligned} G_{1,x} &:= \{\,\Phi_p \in G_0 \;:\; p \in k[x]\ ,\ p(0) = 0\,\}\ ,\\ G_{1,y} &:= \{\,\Psi_q \in G_0 \;:\; q \in k[y]\ ,\ q(0) = 0\,\}\ . \end{aligned}$$

Now, for $n = 2$, the situation is more interesting. A simple calculation shows that U has three orbits in $\mathcal{C}_2$: two closed orbits of dimension 3 and one open orbit of dimension 4. Moreover, the B-orbits coincide with the U-orbits. Combinatorially, this means that the group A acts transitively, and the graph Γ_2 is a tree with one nonterminal and three terminal vertices corresponding to the A-orbit and the B-orbits, respectively. In this case, the graph of groups $\mathbf{\Gamma}_2$ is given by



where $\Lambda \subset G_0$ is the subgroup of scaling transformations $(x, y) \mapsto (\lambda x, \lambda^{-1} y)$, $\lambda \in k^*$, and the groups $G_{2,x},\ G_{2,y}, G_{2,y}^{(1)}$ are defined in terms of generators (1) by

$$
\begin{aligned}
G_{2,x} &:= \{\, \Phi_p \in G_0 \ : \ p \in k[x]\ ,\ p(0) = p'(0) = 0 \,\}\ ,\\
G_{2,y} &:= \{\, \Psi_q \in G_0 \ : \ q \in k[y]\ ,\ q(0) = q'(0) = 0 \,\}\ ,\\
G_{2,y}^{(1)} &:= \{\, \Phi_{-x}\, \Psi_q\, \Phi_x \in G_0 \ : \ q \in k[y]\ ,\ q(\pm 1) = 0 \,\}\ .
\end{aligned}
$$

Formula (13) yields the presentation

$$
G_2 = (G_{2,x} \rtimes \Lambda) *_{\Lambda} (G_{2,y} \rtimes \Lambda) *_{\mathbb{Z}_2} (G_{2,y}^{(1)} \rtimes \mathbb{Z}_2)\ . \tag{16}
$$

In particular, G_2 is generated by its subgroups $G_{2,x},\ G_{2,y},\ G_{2,y}^{(1)}$ and Λ.

Using the above presentations, it is easy to see that *the groups* G_0, G_1 *and* G_2 *are pairwise non-isomorphic.* First of all, G_0 and G_1 are perfect groups, since they are generated by the triangular subgroups $G_{0,x}$, $G_{0,y}$ and $G_{1,x}$, $G_{1,y}$ respectively, while $G_{i,x} = [\Lambda,\, G_{i,x}]$ and $G_{i,y} = [\Lambda,\, G_{i,y}]$ for $i = 0, 1$. Hence, neither G_0 nor G_1 have nontrivial one-dimensional linear representations (characters) over k. On the other hand, from our presentation (16) it follows that the natural isomorphism $\Lambda \xrightarrow{\sim} k^*$ can be extended to a homomorphism $G_2 \to k^*$ (by simply mapping $G_{2,x},\ G_{2,y}$ and $G_{2,y}^{(1)}$ to the identity). Hence G_2 has at least one nontrivial character, and thus $G_2 \not\cong G_0$ and $G_2 \not\cong G_1$ as abstract groups. Now, the fact that $G_0 \not\cong G_1$ is probably known. One way to see it is to notice that G_0 has no nontrivial two-dimensional representations over k, while G_1 does. Indeed, in every linear representation of G_0 on k^2 the translations Φ_a and Ψ_b $(a, b \in k)$ must act trivially, but then the obvious relations $\Phi_a\, \Psi_{q(y)} = \Psi_{q(y+a)}\, \Phi_a$ and $\Psi_b\, \Phi_{p(x)} = \Phi_{p(x-b)}\, \Psi_b$ between the generators (1) imply that the whole G_0 acts trivially. On the other hand, the group G_1 is the stabilizer of a point (X_0, Y_0) under the action of G_0 on $\mathcal{C}_1$; as explained in [5], Sect. 11, this action is algebraic, hence G_1 acts linearly on the tangent space at (X_0, Y_0). Since $\mathcal{C}_1$ is smooth of dimension 2, we get a two-dimensional linear representation of G_1, which is certainly nontrivial.

6. Questions. We end this paper with some questions and conjectures.

1. It is proved in [15] that G_0 is isomorphic to the group $\mathrm{SAut}\,\mathbb{A}^2_k$ of symplectic automorphisms of the affine plane $\mathbb{A}^2_k$ (as in the case of the Weyl algebra, the isomorphism $G_0 \cong \mathrm{SAut}\,\mathbb{A}^2_k$ is induced by the canonical projection $k\langle x, y\rangle \to k[x, y]$). Thus, the groups G_n can be naturally identified with subgroups of $\mathrm{Aut}\,\mathbb{A}^2_k$. Do these last subgroups have a geometric interpretation?

2. In this paper, we have described the structure of G_n and $\mathrm{Aut}_k\, D_n$ as discrete groups. However, these two groups carry natural *algebraic* structures and can be viewed as infinite-dimensional algebraic groups (in the sense of Shafarevich [18]). Despite being isomorphic to each other as discrete groups, they are not isomorphic as algebraic groups (for $n = 0$, this phenomenon was observed in [5].) A natural question is to explicitly describe the algebraic structures on G_n and $\mathrm{Aut}_k\, D_n$; in particular, to compute the corresponding (infinite-dimensional) Lie algebras. The last question was an original motivation for our work. For G_0, the answer is known (see [12]).

3. Compute the homology of the groups G_n for all n. Again, for $n = 0$, the answer is known (see [2]): $H_*(G_0, \mathbb{Z}) \cong H_*(\mathrm{SL}_2(k), \mathbb{Z})$. One may wonder whether the groups $H_*(G_n, \mathbb{Z})$ are strong enough invariants to distinguish the algebras D_n up to isomorphism. Unfortunately, the answer is 'no': in fact, it follows from our description of G_1 that $H_*(G_1, \mathbb{Z}) \cong H_*(\mathrm{SL}_2(k), \mathbb{Z})$. However, for $n \geq 2$, it seems that the groups $H_*(G_n, \mathbb{Z})$ are neither isomorphic to $H_*(\mathrm{SL}_2(k), \mathbb{Z})$ nor to each other, so they may provide interesting invariants.

4. Finally, we would like to propose an extension of the well-known *Dixmier Conjecture* for $A_1(k)$ (see [11], Problème 11.1) to the class of Morita equivalent algebras. We recall that if D is a domain Morita equivalent to A_1, then there is a unique integer $n \geq 0$ such that $D \cong D_n$, where D_n is the endomorphism ring of the right ideal $M_n = x^n A_1 + (y + nx^{-1})\, A_1$. For two unital k-algebras A and B, we denote by $\mathrm{Hom}_k(A, B)$ the set of all unital k-algebra homomorphisms $A \to B$.

Conjecture 1. For all $n, m \geq 0$, we have

$$\mathrm{Hom}_k(D_n, D_m) = \begin{cases} \emptyset & \text{if} \quad n \neq m \\ \mathrm{Aut}_k\, D_n & \text{if} \quad n = m \end{cases}$$

Formally, Conjecture 1 is a strengthening of the Dixmier Conjecture for A_1: in fact, in our notation, the latter says that $\mathrm{Hom}_k(D_0, D_0) = \mathrm{Aut}_k\, D_0$. Does the Dixmier Conjecture actually imply Conjecture 1?

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