ELECTRONIC RESEARCH ANNOUNCEMENTS
IN MATHEMATICAL SCIENCES
Volume 17, Pages 122–124 (October 14, 2010)
S 1935-9179 AIMS (2010)
doi:10.3934/era.2010.17.122

CURVATURE BOUNDED BELOW: A DEFINITION A LA BERG–NIKOLAEV

NINA LEBEDEVA AND ANTON PETRUNIN

(Communicated by Dmitri Burago)

ABSTRACT. We give a new characterization of spaces with nonnegative curvature in the sense of Alexandrov.

1. INTRODUCTION

In this note we give a new characterization of spaces with curvature $\geqslant 0$ in the sense of Alexandrov. Our work is inspired by [1] and [6], where an analogous definition was given for curvature ≤ 0.

1.1. Main theorem. *Let $\mathcal{X}$ be a complete space with intrinsic metric. Then $\mathcal{X}$ is an Alexandrov space with curvature $\geqslant 0$ if and only if any quadruple $p, x, y, z \in \mathcal{X}$ satisfies the following inequality*

$$|px|^2 + |py|^2 + |pz|^2 \geqslant \tfrac{1}{3} \cdot \big(|xy|^2 + |yz|^2 + |zx|^2\big). \tag{1}$$

The inequality (1) is quite weak. For example, one can[1] construct a metric space $\mathcal{F}$ with 4 points which satisfies (1) for each relabeling by p, x, y, z, such that $\mathcal{F}$ does not admit an isometric embedding into any Alexandrov space with curvature $\geqslant 0$.

Similar, previously known conditions simply describe all 4-point sets in a nonnegatively curved space. For instance the following inequality for model angles:

$$\tilde\measuredangle(p\,{}^x_y) + \tilde\measuredangle(p\,{}^y_z) + \tilde\measuredangle(p\,{}^z_x) \leq 2 \cdot \pi. \tag{2}$$

In fact, if a 4-point metric space satisfies (2) for each relabeling, then it can be isometrically embedded into the Euclidean plane or a 2-sphere of some radius $R > 0$ (the proof is left to the reader).

Why do we care. Since the condition (1) is so weak, it should be useful as a test to check that a given space has curvature $\geqslant 0$ in the sense of Alexandrov. However, we are not aware of a single case when it makes life easier.

To explain the real reason why we are interested in this topic, we need to reformulate our Main theorem using language similar that used in [4, Section 1.19_+].

Received by the editors August 19, 2010 and, in revised form, September 9, 2010.
2000 *Mathematics Subject Classification.* Primary: 51K10; Secondary: 53C45.
Key words and phrases. Alexandrov space, curvature.
We want to thank Alexander Lytchak for an interesting discussion in Münster's botanical garden.

[1]Say, consider the metric on $\{p, x, y, z\}$ defined as $|px| = py| = |pz| = 1$, $|xy| = |xz| = 2$ and $|yz| = \varepsilon$ where $\varepsilon > 0$ is sufficiently small; it satisfies (1) for each relabeling but not (2).

Denote by $\mathbf{M}^4$ the set of isometry classes of 4-point metric spaces. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the sets of isometry classes in $\mathbf{M}^4$ which satisfy, respectively, (1) and (2) for all relabelings of points by p, x, y, z. (As is mentioned above, $\mathfrak{B} \subsetneq \mathfrak{A}$.) Further, given a metric space $\mathcal{X}$, denote by $\mathbf{M}^4(\mathcal{X})$ the set of isometry classes of 4-point subspaces in $\mathcal{X}$.

The main theorem says that if the space $\mathcal{X}$ has intrinsic metric and $\mathbf{M}^4(\mathcal{X}) \subset \mathfrak{A}$ then $\mathbf{M}^4(\mathcal{X}) \subset \mathfrak{B}$. From above, the set $\mathfrak{B}$ is the smallest set which satisfies the above property for any $\mathcal{X}$.

It would be interesting to find a general pattern of such phenomena. Assume you start with arbitrary $\mathfrak{A} \subset \mathbf{M}^4$, can you figure out the corresponding $\mathfrak{B}$, or can one describe the properties of $\mathfrak{B}$ which might appear this way?

Note that the Globalization theorem (see [2]) as well as the Berg–Nikolaev characterization of CAT(0) spaces both admit interpretations in the above terms. Also, in [3], it was shown that set defined by the Ptolemy inequality can appear as $\mathfrak{B}$.

2. The proof

The "only if" part follows directly from the generalized Kirszbraun's theorem, see [5]. One only has to check that the inequality (1) holds in the model plane. (Alternatively, one can prove (2) $\Rightarrow$ (1) directly.)

"if" part. We may assume that $\mathcal{X}$ is geodesic, otherwise pass to its ultrapower.

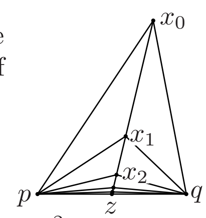
It is sufficient to show that if z is a midpoint of geodesic $[pq]$ in $\mathcal{X}$ then

$$2 \cdot |xz|^2 \geqslant |xp|^2 + |xq|^2 - \tfrac{1}{2} \cdot |pq|^2, \tag{3}$$

for any $x \in \mathcal{X}$.

Directly from (1) we have the following weaker estimate

$$3 \cdot |xz|^2 \geqslant |xp|^2 + |xq|^2 - \tfrac{1}{2} \cdot |pq|^2 \tag{4}$$



Set $x_0 = x$ and consider a sequence of points $x_0, x_1, \dots$ on the geodesic $[xz]$ such that $|x_n z| = \frac{1}{3^n} \cdot |xz|$. Let α_n be a sequence of real numbers such that

$$\alpha_n \cdot |x_n z|^2 = |x_n p|^2 + |x_n q|^2 - \tfrac{1}{2} \cdot |pq|^2.$$

Applying (1), we get

$$|x_{n+1}p|^2 + |x_{n+1}q|^2 + |x_{n+1}x_n|^2 \geqslant \tfrac{1}{3} \cdot (|x_n p|^2 + |x_n q|^2 + |pq|^2).$$

Subtract $\frac{1}{2} \cdot |pq|^2$ from both sides; after simplification you get

$$\alpha_{n+1} \geqslant 3 \cdot \alpha_n - 4.$$

Now assume (3) does not hold; i.e., $\alpha_0 > 2$ then $\alpha_n \to \infty$ as $n \to \infty$. On the other hand, from (4), we get $\alpha_n \leq 3$, a contradiction. □

P.S.: Arbitrary curvature bound

One can obtain analogous characterization of Alexandrov spaces with curvature $\geqslant \kappa$ for any $\kappa \in \mathbb{R}$.

Here are the inequalities for the cases $\kappa = 1$ and -1 which correspond to (1) for quadruple p, x^1, x^2, x^3:

$$\left(\sum_{i=1}^{3} \cos |px^i|\right)^2 \leq \sum_{i,j=1}^{3} \cos |x^i x^j|. \tag*{$(1)^+$}$$

$$\left(\sum_{i=1}^{3}\cosh|px^i|\right)^2 \geqslant \sum_{i,j=1}^{3}\cosh|x^i x^j|; \qquad (1)^-$$

Note that in both cases we have equality if p is the point of intersections of medians of the triangle $[x^1x^2x^3]$ in the corresponding model plane. (In the model planes, medians also pass through the incenter.)

The proof goes along the same lines. The case $\kappa = 1$ also follows from the Main theorem applied to the cone over the space.

References

[1] I. D. Berg and I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov spaces*, Geom. Dedicata, **133** (2008), 195–218. MR MR2390077

[2] Yu. Burago, M. Gromov and G. Perelman, *A. D. Aleksandrov spaces with curvatures bounded below*, (Russian) Uspekhi Mat. Nauk, **47** (1992), 3–51, 222; translation in Russian Math. Surveys **47** (1992), 1–58. MR MR1185284

[3] T. Foertsch, A. Lytchak and V. Schroeder, *Nonpositive curvature and the Ptolemy inequality*, Int. Math. Res. Not. (IMRN), **2007** (2007), Art. ID rnm100, 15 pp. MR MR2376212

[4] M. Gromov, “Metric Structures for Riemannian and Non-Riemannian Spaces,” Progress in Mathematics, vol. **152**, Birkhäuser Boston, Inc., Boston, MA, 1999. MR MR1699320

[5] U. Lang and V. Schroeder, *Kirszbraun's theorem and metric spaces of bounded curvature*, Geom. funct. anal., Vol. **7** (1997), 535–560. MR MR1466337

[6] Takashi Sato, *An alternative proof of Berg and Nikolaev's characterization of* CAT(0)*-spaces via quadrilateral inequality*, Arch. Math. (Basel), **93** (2009), 487–490. MR MR2563595

St. Petersburg Department of V.A. Steklov Mathematical Institute, Fontanka 27, 191023, St. Petersburg, Russia

E-mail address: `lebed@pdmi.ras.ru`

Department of Mathematics, Penn State University, University Park, PA 16802, USA

E-mail address: `petrunin@math.psu.edu`