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HÖLDER COCYCLES AND ERGODIC INTEGRALS FOR TRANSLATION FLOWS ON FLAT SURFACES

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Abstract. The main results announced in this note are an asymptotic expansion for ergodic integrals of translation flows on flat surfaces of higher genus (Theorem 1) and a limit theorem for such flows (Theorem 2). Given an abelian differential on a compact oriented surface, consider the space $\mathfrak{B}^+$ of Hölder cocycles over the corresponding vertical flow that are invariant under holonomy by the horizontal flow. Cocycles in $\mathfrak{B}^+$ are closely related to G.Forni's invariant distributions for translation flows [10]. Theorem 1 states that ergodic integrals of Lipschitz functions are approximated by cocycles in $\mathfrak{B}^+$ up to an error that grows more slowly than any power of time. Theorem 2 is obtained using the renormalizing action of the Teichmüller flow on the space $\mathfrak{B}^+$. A symbolic representation of translation flows as suspension flows over Vershik's automorphisms allows one to construct cocycles in $\mathfrak{B}^+$ explicitly. Proofs of Theorems 1, 2 are given in [5].

1. Hölder cocycles over translation flows.

Let $\rho \geq 2$ be an integer, let M be a compact oriented surface of genus ρ, and let ω be a holomorphic one-form on M. Denote by $\mathbf{m} = i(\omega \wedge \overline{\omega})/2$ the area form induced by ω and assume that $\mathbf{m}(M) = 1$.

Let h_t^+ be the *vertical* flow on M (i.e., the flow corresponding to $\Re(\omega)$); let h_t^- be the *horizontal* flow on M (i.e., the flow corresponding to $\Im(\omega)$). The flows h_t^+, h_t^- preserve the area $\mathbf{m}$. Take $x \in M$, $t_1, t_2 \in \mathbb{R}_+$ and assume that the closure of the set

$$\{h^+_{\tau_1} h^-_{\tau_2} x, 0 \leq \tau_1 < t_1, 0 \leq \tau_2 < t_2\} \tag{1}$$

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does not contain zeros of the form ω. The set (1) is then called an *admissible rectangle* and denoted $\Pi(x, t_1, t_2)$. Let $\overline{\mathfrak{C}}$ be the semi-ring of admissible rectangles. Consider the linear space $\mathfrak{B}^+$ of Hölder cocyles $\Phi^+(x,t)$ over the vertical flow h_t^+ that are invariant under horizontal holonomy. More precisely, a function $\Phi^+(x,t) : M \times \mathbb{R} \to \mathbb{R}$ belongs to the space $\mathfrak{B}^+$ if it satisfies:

Assumption 1. (1) *$\Phi^+(x, t+s) = \Phi^+(x,t) + \Phi^+(h_t^+ x, s)$;*

(2) *There exists $t_0 > 0$, $\theta > 0$ such that $|\Phi^+(x,t)| \le t^\theta$ for all $x \in M$ and all $t \in \mathbb{R}$ satisfying $|t| < t_0$;*

(3) *If $\Pi(x, t_1, t_2)$ is an admissible rectangle, then $\Phi^+(x, t_1) = \Phi^+(h^-_{t_2} x, t_1)$.*

For example, a cocycle Φ_1^+ defined by $\Phi_1^+(x,t) = t$ belongs to $\mathfrak{B}^+$.

In the same way define the space of $\mathfrak{B}^-$ of Hölder cocyles $\Phi^-(x,t)$ over the horizontal flow h_t^- which are invariant under vertical holonomy, and set $\Phi_1^-(x,t) = t$.

Given $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\overline{\mathfrak{C}}$ of admissible rectangles is introduced by the formula

$$\Phi^+ \times \Phi^-(\Pi(x, t_1, t_2)) = \Phi^+(x, t_1) \cdot \Phi^-(x, t_2). \tag{2}$$

In particular, for $\Phi^- \in \mathfrak{B}^-$, set $m_{\Phi^-} = \Phi_1^+ \times \Phi^-$:

$$m_{\Phi^-}(\Pi(x, t_1, t_2)) = t_1 \Phi^-(x, t_2). \tag{3}$$

For any $\Phi^- \in \mathfrak{B}^-$ the measure m_{Φ^-} satisfies $(h_t^+)_* m_{\Phi^-} = m_{\Phi^-}$ and is an invariant distribution in the sense of G. Forni [9], [10]. For instance, $m_{\Phi_1^-} = \mathbf{m}$.

An $\mathbb{R}$-linear pairing between $\mathfrak{B}^+$ and $\mathfrak{B}^-$ is given, for $\Phi^+ \in \mathfrak{B}^+$, $\Phi^- \in \mathfrak{B}^-$, by the formula

$$\langle \Phi^+, \Phi^- \rangle = \Phi^+ \times \Phi^-(M). \tag{4}$$

Take an abelian differential $\mathbf{X} = (M, \omega)$. The space $\mathfrak{B}^+_{\mathbf{X}} = \mathfrak{B}^+(M,\omega)$ can be mapped to $H^1(M, \mathbb{R})$ in the following way. A continuous closed curve γ on M is called *rectangular* if

$$\gamma = \gamma_1^+ \sqcup \cdots \sqcup \gamma_{k_1}^+ \bigsqcup \gamma_1^- \sqcup \cdots \sqcup \gamma_{k_2}^-,$$

where γ_i^+ are arcs of the flow h_t^+, γ_i^- are arcs of the flow h_t^-. For $\Phi^+ \in \mathfrak{B}^+$ define $\Phi^+(\gamma) = \sum_{i=1}^{k_1} \Phi^+(\gamma_i^+)$. It is easy to show that if γ is homologous to γ', then $\Phi^+(\gamma) = \Phi^+(\gamma')$. Using a similar construction for $\mathfrak{B}^-_{\mathbf{X}} = \mathfrak{B}^-(M,\omega)$, we obtain maps

$$\check{\mathcal{I}}^+_{\mathbf{X}} : \mathfrak{B}^+_{\mathbf{X}} \to H^1(M, \mathbb{R}),\ \check{\mathcal{I}}^-_{\mathbf{X}} : \mathfrak{B}^-_{\mathbf{X}} \to H^1(M, \mathbb{R}). \tag{5}$$

For a generic abelian differential, the image of $\mathfrak{B}^+_{\mathbf{X}}$ under $\check{\mathcal{I}}^+_{\mathbf{X}}$ is the strictly unstable space of the Konstevich-Zorich cocycle over the Teichmüller flow.

More precisely, let $\kappa = (\kappa_1, \dots, \kappa_\sigma)$ be a nonnegative integer vector such that $\kappa_1 + \cdots + \kappa_\sigma = 2\rho - 2$. Denote by $\mathcal{M}_\kappa$ the moduli space of pairs (M, ω), where M is a Riemann surface of genus ρ and ω is a holomorphic differential of area 1 with singularities of orders $\kappa_1, \dots, \kappa_\sigma$. The space $\mathcal{M}_\kappa$ is often called the *stratum* in the moduli space of abelian differentials.

The Teichmüller flow $\mathbf{g}_s$ on $\mathcal{M}_\kappa$ sends the modulus of a pair (M, ω) to the modulus of the pair (M, ω'), where $\omega' = e^s \Re(\omega) + i e^{-s} \Im(\omega)$; the new complex structure on M is uniquely determined by the requirement that the form ω' be holomorphic. As shown by Veech, the space $\mathcal{M}_\kappa$ need not be connected; let $\mathcal{H}$ be a connected component of $\mathcal{M}_\kappa$.

Let $\mathbb{H}^1(\mathcal{H})$ be the fibre bundle over $\mathcal{H}$ whose fibre at a point (M,ω) is the cohomology group $H^1(M,\mathbb{R})$. The bundle $\mathbb{H}^1(\mathcal{H})$ carries the *Gauss-Manin connection* which declares continuous integer-valued sections of our bundle to be flat and is uniquely defined by that requirement. Parallel transport with respect to the Gauss-Manin connection along the orbits of the Teichmüller flow yields a cocycle over the Teichmüller flow, called the *Kontsevich-Zorich cocycle* and denoted $\mathbf{A} = \mathbf{A}_{KZ}$.

Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$. By definition, the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$ satisfies the assumptions of the Oseledets Theorem with respect to $\mathbb{P}$. For $\mathbf{X} \in \mathcal{H}$, $\mathbf{X} = (M,\omega)$, denote by $E^u_{\mathbf{X}} \subset H^1(M,\mathbb{R})$ the space spanned by vectors corresponding to the positive Lyapunov exponents of $\mathbf{A}_{KZ}$, and by $E^s_{\mathbf{X}} \subset H^1(M,\mathbb{R})$ the space spanned by vectors corresponding to the negative Lyapunov exponents of $\mathbf{A}_{KZ}$. As before, let $\mathfrak{B}^+_{\mathbf{X}}$, $\mathfrak{B}^-_{\mathbf{X}}$ be the spaces of Hölder cocycles corresponding to the vertical and the horizontal flows of $\mathbf{X}$.

Proposition 1. *For $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$ the map $\check{\mathcal{I}}^+_{\mathbf{X}}$ takes $\mathfrak{B}^+_{\mathbf{X}}$ isomorphically onto $E^u_{\mathbf{X}}$, and the map $\check{\mathcal{I}}^-_{\mathbf{X}}$ takes $\mathfrak{B}^-_{\mathbf{X}}$ isomorphically onto $E^s_{\mathbf{X}}$.*

The pairing $\langle , \rangle$ is nondegenerate and is taken by the isomorphisms $\mathcal{I}^+_{\mathbf{X}}$, $\mathcal{I}^-_{\mathbf{X}}$ to the cup-product in the cohomology group $H^1(M,\mathbb{R})$.

Remark. In particular, if $\mathbb{P}$ is the Masur-Veech "smooth" measure [17], [18], then almost surely, with respect to $\mathbb{P}$, we have

$$\dim \mathfrak{B}^+_{\mathbf{X}} = \dim \mathfrak{B}^-_{\mathbf{X}} = \rho.$$

Remark. The isomorphisms $\check{\mathcal{I}}^+_{\mathbf{X}}$, $\check{\mathcal{I}}^-_{\mathbf{X}}$ are analogues of G. Forni's isomorphism between his space of invariant distributions and the unstable space of the Kontsevich-Zorich cocycle.

Remark. Cocycles in $\mathfrak{B}^+$ can be interpreted, in the spirit of Bonahon [4], as finitely-additive holonomy-invariant Hölder transverse measures on oriented measured foliations and also as finitely-additive invariant measures for interval exchange transformations. See the preprint [5] for details.

Consider the inverse isomorphisms $\mathcal{I}^+_{\mathbf{X}} = \left(\check{\mathcal{I}}^+_{\mathbf{X}}\right)^{-1}$, $\mathcal{I}^-_{\mathbf{X}} = \left(\check{\mathcal{I}}^-_{\mathbf{X}}\right)^{-1}$. Let

$$1 = \theta_1 > \theta_2 > \cdots > \theta_l > 0$$

be the distinct positive Lyapunov exponents of the Kontsevich-Zorich cocycle $\mathbf{A}_{KZ}$, and let

$$E^u_{\mathbf{X}} = \bigoplus_{i=1}^{l} E^u_{\mathbf{X},\theta_i}$$

be the corresponding Oseledets decomposition at $\mathbf{X}$.

Proposition 2. *Let $v \in E^u_{\mathbf{X},\theta_i}$, $v \neq 0$, and denote $\Phi^+ = \mathcal{I}^+_{\mathbf{X}}(v)$. Then for any $\varepsilon > 0$ the cocycle Φ^+ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$ and for any $x \in M(\mathbf{X})$ we have*

$$\limsup_{T\to\infty} \frac{\log |\Phi^+(x,T)|}{\log T} = \theta_i.$$

2. Approximation of weakly Lipschitz functions.

The space of Lipschitz functions is not invariant under h_t^+, and a larger function space $Lip_w^+(M,\omega)$ of weakly Lipschitz functions is introduced as follows. A bounded measurable function f belongs to $Lip_w^+(M,\omega)$ if there exists a constant C, depending only on f, such that for any admissible rectangle $\Pi(x,t_1,t_2)$ we have

$$\left|\int_0^{t_1} f(h_t^+x)dt - \int_0^{t_1} f(h_t^+(h_{t_2}^-x))dt\right| \le C. \tag{6}$$

Let C_f be the infimum of all C satisfying (6). We norm $Lip_w^+(M,\omega)$ by setting

$$||f||_{Lip_w^+} = \sup_M |f| + C_f.$$

By definition, the space $Lip_w^+(M,\omega)$ contains all Lipschitz functions on M and is invariant under h_t^+. We denote by $Lip_{w,0}^+(M,\omega)$ the subspace of $Lip_w^+(M,\omega)$ of functions whose integral with respect to $\mathbf{m}$ is 0.

For any $f \in Lip_w^+(M,\omega)$ and any $\Phi^- \in \mathfrak{B}^-$, the integral $\int_M f dm_{\Phi^-}$ can be defined as the limit of Riemann sums.

If the pairing $\langle , \rangle$ induces an isomorphism between $\mathfrak{B}^+$ and the dual $(\mathfrak{B}^-)^*$, then one can assign to a function $f \in Lip_w^+(M,\omega)$ the cocycle $\Phi_f^+ \in \mathfrak{B}^+$ by the formula

$$\langle \Phi_f^+, \Phi^- \rangle = \int_M f dm_{\Phi^-},\ \ \Phi^- \in \mathfrak{B}^-. \tag{7}$$

By definition, $\Phi^+_{f\circ h_t^+} = \Phi_f^+$.

Theorem 1. *Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$. For any $\varepsilon > 0$ there exists a constant C_ε depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, any $f \in Lip_w^+(\mathbf{X})$, any $x \in M$ and any $T > 0$ we have*

$$\left|\int_0^T f \circ h_t^+(x)dt - \Phi_f^+(x,T)\right| \le C_\varepsilon ||f||_{Lip_w^+}(1+T^\varepsilon).$$

Consider the case in which the Lyapunov spectrum of the Kontsevich-Zorich cocycle is simple in restriction to the space E^u (as, by the Avila-Viana theorem [2], is the case with the Masur-Veech smooth measure). Let $l_0 = \dim E^u$ and let $1 = \theta_1 > \theta_2 > \cdots > \theta_{l_0}$ be the corresponding simple expanding Lyapunov exponents.

Let Φ_1^+ be given by the formula $\Phi_1^+(x,t) = t$. Introduce a basis $\Phi_1^+, \Phi_2^+, \dots, \Phi_{l_0}^+$ in $\mathfrak{B}_{\mathbf{X}}^+$ in such a way that $\check{\mathcal{I}}_{\mathbf{X}}^+(\Phi_i^+)$ lies in the Lyapunov subspace with exponent θ_i. By Proposition 2, for any $\varepsilon > 0$ the cocycle Φ_i^+ satisfies the Hölder condition with exponent $\theta_i - \varepsilon$, and for any $x \in M(\mathbf{X})$ we have

$$\limsup_{T\to\infty} \frac{\log |\Phi_i^+(x,T)|}{\log T} = \theta_i.$$

Let $\Phi_1^-, \dots, \Phi_{l_0}^-$ be the dual basis in $\mathfrak{B}_{\mathbf{X}}^-$. Clearly, $\Phi_1^-(x,t) = t$.

By definition, we have

$$\Phi_f^+ = \sum_{i=1}^{l_0} m_{\Phi_i^-}(f) \cdot \Phi_i^+. \tag{8}$$

Noting that by definition we also have $m_{\Phi_1^-} = \mathbf{m}$, we derive from Theorem 1 the following corollary.

Corollary 1. *Assume that $\mathbb{P}$ is an invariant ergodic probability measure for the Teichmüller flow such that, with respect to $\mathbb{P}$, all positive Lyapunov exponents of the Kontsevich-Zorich cocycle are simple.*

Then for any $\varepsilon > 0$ there exists a constant C_ε depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X} \in \mathcal{H}$, any $f \in Lip_w^+(\mathbf{X})$, any $x \in \mathbf{X}$ and any $T > 0$ we have

$$\left| \int_0^T f \circ h_t^+(x)dt - T \cdot \int_M f d\mathbf{m} - \sum_{i=2}^{l_0} m_{\Phi_i^-}(f) \cdot \Phi_i^+(x,T) \right| \leq C_\varepsilon ||f||_{Lip_w^+}(1 + T^\varepsilon).$$

Remark. If $\mathbb{P}$ is the Masur-Veech smooth measure on $\mathcal{H}$, then the work of G.Forni [9], [10], [11] and S. Marmi, P. Moussa, J.-C. Yoccoz [16] implies that the left-hand side is bounded for any $f \in C^{1+\varepsilon}(M)$ (in fact, for any f in the Sobolev space $H^{1+\varepsilon}$). In particular, if $f \in C^{1+\varepsilon}(M)$ and $\Phi_f^+ = 0$, then f is a coboundary.

3. Limit Theorems for Translation Flows.

3.1. Time integrals as random variables. As before, (M, ω) is an abelian differential, and h_t^+, h_t^- are, respectively, its vertical and horizontal flows. Take $\tau \in [0, 1]$, $s \in \mathbb{R}$, a real-valued $f \in Lip_{w,0}^+(M, \omega)$ and introduce the function

$$\mathfrak{S}[f, s; \tau, x] = \int_0^{\tau \exp(s)} f \circ h_t^+(x)dt. \tag{9}$$

For fixed f, s and x the quantity $\mathfrak{S}[f, s; \tau, x]$ is a continuous function of $\tau \in [0, 1]$; therefore, as x varies in the probability space $(M, \mathbf{m})$, we obtain a random element of $C[0, 1]$. In other words, we have a random variable $\mathfrak{S}[f, s] : (M, \mathbf{m}) \to C[0, 1]$ defined by the formula (9).

For any fixed $\tau \in [0, 1]$ the formula (9) yields a real-valued random variable $\mathfrak{S}[f, s; \tau] : (M, \mathbf{m}) \to \mathbb{R}$ whose expectation, by definition, is zero. Our first aim is to estimate the growth of its variance as $s \to \infty$. Without loss of generality, one may take $\tau = 1$.

3.2. The growth rate of the variance. Let $\mathbb{P}$ be an invariant ergodic probability measure for the Teichmüller flow such that, with respect to $\mathbb{P}$, the second Lyapunov exponent θ_2 of the Kontsevich-Zorich cocycle is positive and simple (recall that, as Veech and Forni showed, the first one, $\theta_1 = 1$, is always simple [20, 10] and that, by the Avila-Viana theorem [2], the second one is simple for the Masur-Veech smooth measure).

For an abelian differential $\mathbf{X} = (M, \omega)$, denote by $E_{2,\mathbf{X}}^+$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the second Lyapunov exponent θ_2, and let $\mathfrak{B}_{2,\mathbf{X}}^+ = \mathcal{I}_{\mathbf{X}}^+(E_{2,\mathbf{X}}^+)$. Similarly, denote by $E_{2,\mathbf{X}}^-$ the one-dimensional subspace in $H^1(M, \mathbb{R})$ corresponding to the Lyapunov exponent $-\theta_2$, and let $\mathfrak{B}_{2,\mathbf{X}}^- = \mathcal{I}_{\mathbf{X}}^-(E_{2,\mathbf{X}}^-)$. Recall that the space $H^1(M, \mathbb{R})$ is endowed with the Hodge norm $|\cdot|_H$; the isomorphisms $\mathcal{I}_{\mathbf{X}}^{\pm}$ take the Hodge norm to a norm on $\mathfrak{B}_{\mathbf{X}}^{\pm}$; slightly abusing notation, we denote the latter norm by the same symbol.

Introduce a multiplicative cocycle $H_2(s, \mathbf{X})$ over the Teichmüller flow $\mathbf{g}_s$ by taking $v \in E^+_{2,\mathbf{X}}$, $v \neq 0$, and setting

$$H_2(s, \mathbf{X}) = \frac{|\mathbf{A}(s, \mathbf{X})v|_H}{|v|_H}. \tag{10}$$

By definition, we have $\lim_{s\to\infty} \frac{\log H_2(s,\mathbf{X})}{s} = \theta_2$.

Now take $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^- \in \mathfrak{B}^-_{2,\mathbf{X}}$ in such a way that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

Proposition 3. *There exists a constant $\alpha > 0$ depending only on $\mathbb{P}$ and positive measurable functions*

$$C : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+,\ V : \mathcal{H} \to \mathbb{R}_+,\ s_0 : \mathcal{H} \to \mathbb{R}_+$$

such that the following is true for $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$. If $f \in Lip^+_{w,0}(\mathbf{X})$ satisfies $m_{\Phi_2^-}(f) \neq 0$, then for all $s \geq s_0(\mathbf{X})$ we have

$$\left| \frac{Var_{\mathbf{m}}\mathfrak{S}[f, s; 1]}{V(\mathbf{g}_s\mathbf{X})(m_{\Phi_2^-}(f)|\Phi_2^+|(H_2(\mathbf{X}, s)))^2} - 1 \right| \leq C(\mathbf{X}, \mathbf{g}_s\mathbf{X}) \exp(-\alpha s). \tag{11}$$

Remark. Observe that the quantity $(m_{\Phi_2^-}(f)|\Phi^+|)^2$ does not depend on the specific choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^- \in \mathfrak{B}^-_{2,\mathbf{X}}$ such that $\langle \Phi_2^+, \Phi_2^- \rangle = 1$.

Proposition 3 is based on

Proposition 4. *There exists a positive measurable function $V : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ such that for $\mathbb{P}$-almost all $\mathbf{X} \in \mathcal{H}$, we have*

$$Var_{\mathbf{m}(\mathbf{X})}\Phi_2^+(x, e^s) = V(\mathbf{g}_s\mathbf{X})|\Phi_2^+|^2(H_2(\mathbf{X}, s))^2. \tag{12}$$

In particular, we have $Var_{\mathbf{m}}\Phi_2^+(x, e^s) \neq 0$ for any $s \in \mathbb{R}$. The function $V(\mathbf{X})$ is given by $V(\mathbf{X}) = \frac{Var_{\mathbf{m}(\mathbf{X})}\Phi_2^+(x,1)}{|\Phi_2^+|^2}$ (observe that the right-hand side does not depend on a particular choice of $\Phi_2^+ \in \mathfrak{B}^+_{2,\mathbf{X}}$, $\Phi_2^+ \neq 0$).

3.3. **The limit theorem.** Go back to the $C[0,1]$-valued random variable $\mathfrak{S}[f, s]$ and denote by $\mathfrak{m}[f, s]$ the distribution of the normalized random variable

$$\frac{\mathfrak{S}[f, s]}{\sqrt{Var_{\mathbf{m}}\mathfrak{S}[f, s; 1]}}. \tag{13}$$

By definition, $\mathfrak{m}[f, s]$ is a Borel probability measure on $C[0,1]$; furthermore, if $\xi = \xi(t) \in C[0,1]$, then we have $\xi(0) = 0$ almost surely with respect to $m[f, s]$; $\mathbb{E}_{\mathfrak{m}[f,s]}\xi(\tau) = 0$ for any fixed $\tau \in [0,1]$; $Var_{\mathfrak{m}[f,s]}\xi(1) = 1$.

We are interested in the weak accumulation points of $\mathfrak{m}[f, s]$ as $s \to \infty$. Consider the space $\mathcal{H}'$ given by the formula

$$\mathcal{H}' = \{\mathbf{X}' = (M, \omega, v), v \in E_2^+(M, \omega), |v|_H = 1\}.$$

By definition, the space $\mathcal{H}'$ is a $\mathbb{P}$-almost surely defined two-to-one cover of the space $\mathcal{H}$. The skew-product flow of the Kontsevich-Zorich cocycle over the Teichmüller flow yields a flow $\mathbf{g}'_s$ on $\mathcal{H}'$ given by the formula

$$\mathbf{g}'_s(\mathbf{X}, v) = \left(\mathbf{g}_s\mathbf{X}, \frac{\mathbf{A}(s, \mathbf{X})v}{|\mathbf{A}(s, \mathbf{X})v|_H}\right).$$

Given $\mathbf{X}' \in \mathcal{H}'$, set $\Phi^+_{2,\mathbf{X}'} = \mathcal{I}^+(v)$. Take $\tilde{v} \in E_2^-(M,\omega)$ such that $\langle v, \tilde{v}\rangle = 1$ and set $\Phi^-_{2,\mathbf{X}'} = \mathcal{I}^-(v)$, $m^-_{2,\mathbf{X}'} = m^-_{\Phi_{2,\mathbf{X}'}}$. By Proposition 4 for any $\tau > 0$ we have $Var_{\mathbf{m}} \Phi^+_{2,\mathbf{X}'}(x,\tau) \neq 0$.

Let $\mathfrak{M}$ be the space of all probability distributions on $C[0,1]$. Introduce a $\mathbb{P}$-almost surely defined map $\mathcal{D}_2^+ : \mathcal{H}' \to \mathfrak{M}$ by setting $\mathcal{D}_2^+(\mathbf{X}')$ to be the distribution of the $C[0,1]$-valued normalized random variable

$$\frac{\Phi^+_{2,\mathbf{X}'}(x,\tau)}{\sqrt{Var_{\mathbf{m}} \Phi^+_{2,\mathbf{X}'}(x,1)}},\ \tau \in [0,1].$$

By definition, $\mathcal{D}_2^+(\mathbf{X}')$ is a Borel probability measure on the space $C[0,1]$; it is a compactly supported measure as its support consists of equibounded Hölder functions with exponent $\theta_2/\theta_1 - \varepsilon$.

Consider the set $\mathfrak{M}_1$ of probability measures $\mathfrak{m}$ on $C[0,1]$ satisfying, for $\xi \in C[0,1]$, $\xi = \xi(t)$, the conditions:

(1) the equality $\xi(0) = 0$ holds $\mathfrak{m}$-almost surely;
(2) for any $\tau \in [0,1]$ we have $\mathbb{E}_{\mathfrak{m}}\xi(\tau) = 0$;
(3) we have $Var_{\mathfrak{m}}\xi(1) = 1$ and for any $\tau \neq 0$ we have $Var_{\mathfrak{m}}\xi(\tau) \neq 0$.

By Proposition 4 we have $\mathcal{D}_2^+(\mathcal{H}') \subset \mathfrak{M}_1$.

Consider a semi-flow J_s on the space $C[0,1]$ defined by the formula

$$J_s\xi(t) = \xi(e^{-s}t),\ s \geq 0.$$

Introduce a semi-flow G_s on $\mathfrak{M}_1$ by the formula

$$G_s\mathfrak{m} = \frac{(J_s)_*\mathfrak{m}}{Var_{\mathfrak{m}}(\xi(e^{-s}))}, \mathfrak{m} \in \mathfrak{M}_1. \tag{14}$$

By definition, the diagram

$$\begin{array}{ccc}
\mathcal{H}' & \xrightarrow{\mathcal{D}_2^+} & \mathfrak{M}_1 \\
\big\downarrow \mathbf{g}_s & & \big\uparrow G_s \\
\mathcal{H}' & \xrightarrow{\mathcal{D}_2^+} & \mathfrak{M}_1
\end{array}$$

is commutative.

Let d_{LP} be the Lévy-Prohorov metric on $\mathfrak{M}$ (see, e.g., [3]).

Theorem 2. *Let $\mathbb{P}$ be a $\mathbf{g}_s$-invariant ergodic probability measure on $\mathcal{H}$ such that the second Lyapunov exponent of the Kontsevich-Zorich cocycle is positive and simple with respect to $\mathbb{P}$.*

There exists a positive measurable function $C : \mathcal{H}' \times \mathcal{H}' \to \mathbb{R}_+$ and a positive constant α depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $\mathbf{X}' \in \mathcal{H}'$ and any $f \in Lip^+_{w,0}(\mathbf{X})$ satisfying $m^-_{2,\mathbf{X}'}(f) > 0$ we have

$$d_{LP}(\mathfrak{m}[f,s], \mathcal{D}_2^+(\mathbf{g}'_s\mathbf{X}')) \leq C(\mathbf{X}', g'_s\mathbf{X}')\exp(-\alpha s). \tag{15}$$

Fix $\tau \in \mathbb{R}$ and let $\mathfrak{m}_2(\mathbf{X}',\tau)$ be the distribution of the $\mathbb{R}$-valued random variable

$$\frac{\Phi^+_{2,\mathbf{X}'}(x,\tau)}{\sqrt{Var_{\mathbf{m}} \Phi^+_{2,\mathbf{X}'}(x,\tau)}}.$$

For brevity, write $\mathfrak{m}_2(\mathbf{X}',1) = \mathfrak{m}_2(\mathbf{X}')$.

Proposition 5. *For $\mathbb{P}$-almost any $\mathbf{X}' \in \mathcal{H}'$, the measure $\mathfrak{m}_2(\mathbf{X}', \tau)$ admits atoms for a dense set of $\tau \in \mathbb{R}$.*

By definition, $\mathfrak{m}_2(\mathbf{X}')$ is always compactly supported; the following Proposition shows, however, that the family $\{\mathfrak{m}_2(\mathbf{X}'), \mathbf{X}' \in \mathcal{H}'\}$ is in general not closed. Let δ_0 stand for the delta-measure at zero.

Proposition 6. *Let $\mathcal{H}$ be endowed with the Masur-Veech smooth measure. Then the measure δ_0 is an accumulation point for the set $\{\mathfrak{m}_2(\mathbf{X}'), \mathbf{X}' \in \mathcal{H}'\}$ in the weak topology.*

4. A symbolic coding for translation flows.

By Vershik's Theorem [21], every ergodic automorphism of a Lebesgue probability space can be represented as a Vershik automorphism of a Markov compactum. For an interval exchange transformation, an explicit representation is obtained using Rohlin towers given by Rauzy-Veech induction (see [12]). Passing to Veech's zippered rectangles and their bi-infinite Rauzy-Veech expansions, one represents a minimal translation flow as a flow along the leaves of the asymptotic foliation of a bi-infinite Markov compactum. In this representation, the cocycles in $\mathfrak{B}^+$ become finitely-invariant measures on the asymptotic foliation of a Markov compactum.

Thus, after passage to a finite cover (namely, the Veech space of zippered rectangles), the moduli space of abelian differentials is represented as a space of Markov compacta. The Teichmüller flow and the Kontsevich-Zorich cocycle admit a simple description in terms of this symbolic representation, and the cocycles in $\mathfrak{B}^+$ are constructed explicitly. Theorems 1, 2 are then derived from their symbolic counterparts. Detailed proofs are given in [5].

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