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MULTIFRACTAL FORMALISM DERIVED FROM THERMODYNAMICS FOR GENERAL DYNAMICAL SYSTEMS

VAUGHN CLIMENHAGA

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Abstract. We show that under quite general conditions, various multifractal spectra may be obtained as Legendre transforms of functions $T: \mathbb{R} \to \mathbb{R}$ arising in the thermodynamic formalism. We impose minimal requirements on the maps we consider, and obtain partial results for any continuous map f on a compact metric space. In order to obtain complete results, the primary hypothesis we require is that the functions T be continuously differentiable. This makes rigorous the general paradigm of reducing questions regarding the multifractal formalism to questions regarding the thermodynamic formalism. These results hold for a broad class of measurable potentials, which includes (but is not limited to) continuous functions. Applications include most previously known results, as well as some new ones.

1. Introduction and basic definitions

1.1. **The general idea.** Multifractal analysis characterizes a dynamical system in terms of various asymptotic statistical quantities. Chief among these are Birkhoff averages of a real-valued observable function φ, Lyapunov exponents, local entropies, and local dimensions.

In certain extremely uniform and homogeneous systems, every point has the same set of asymptotic quantities; however, this situation is highly atypical, and for "most" systems, these quantities vary from point to point. In fact, the level sets of these quantities are typically dense and possess a fractal-like structure.

The goal of multifractal analysis is to study the dimensional properties of these level sets, using either Hausdorff dimension or topological entropy in the sense of Bowen [2, 19]. The guiding principle toward reaching this goal is the idea that these dimensional quantities can be determined by certain thermodynamic properties of the system, and so one may establish multifractal results via topological pressure, equilibrium states, etc.

This principle has informed many successful multifractal analyses of various specific classes of systems [24, 21, 4, 28, 29, 17, 22, 27, 12, 11, 15, 14]. However, it has always remained a guiding principle rather than a rigorous result. Here we obtain generally applicable results that do not use specific properties of the system

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in question. Part of the multifractal analysis is carried out in complete generality, and the remainder is shown to follow from sufficient knowledge of the system's thermodynamic properties.

1.2. **Definitions.** Given a function $\varphi\colon X \to \mathbb{R}$, we write the sum of φ along an orbit as $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$. The level sets of the Birkhoff averages are

$$K_\alpha^{\mathcal{B}} = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\};$$

the Birkhoff ergodic theorem guarantees that for any ergodic measure μ, one of these level sets has full measure, and the rest have measure 0. Thus we quantify their size not in terms of measure, but in terms of topological entropy. This gives the *entropy spectrum of Birkhoff averages*

$$\mathcal{B}(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{B}}),$$

concerning which we obtain our strongest results. The Birkhoff spectrum provides a simpler setting for arguments which can be used for other multifractal spectra, and is also of interest in its own right, having applications to the theory of large deviations [22, 6].

If f is a conformal map and $\varphi(x) = -\log \|Df(x)\|$ is the geometric potential, the Birkhoff spectrum goes by the name of the *Lyapunov spectrum*. The present results also imply results for the Lyapunov spectrum of conformal maps [8].

If a measure μ has a weak Gibbs property, then the Birkhoff averages are related to the local entropies, whose level sets are

$$K_\alpha^{\mathcal{E}} = \left\{ x \in X \;\middle|\; \lim_{\delta\to 0} \varliminf_{n\to\infty} -\frac{1}{n} \log \mu(B(x,n,\delta)) = \alpha \right\},$$

where $B(x,n,\delta)$ is the Bowen ball containing all points whose orbits δ-shadow the orbit of x for at least n iterates. It was shown by Brin and Katok that if μ is ergodic, then one of the level sets $K_\alpha^{\mathcal{E}}$ has full measure, and the rest have measure 0 [1]; thus we must once again quantify them using a (global) dimensional characteristic, the topological entropy. Upon doing so, we obtain the *entropy spectrum for local entropies*

$$\mathcal{E}(\alpha) = h_{\text{top}}\,(K_\alpha^{\mathcal{E}}).$$

Finally, we may consider the pointwise dimension of μ at x, whose level sets are

$$K_\alpha^{\mathcal{D}} = \left\{ x \in X \;\middle|\; \lim_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} = \alpha \right\}.$$

Many measures of interest are *exact-dimensional*; that is, the pointwise dimension is constant μ-almost everywhere. In particular, this is true of hyperbolic measures (those with non-zero Lyapunov exponents almost everywhere) [5]. For an exact-dimensional measure, one of the $K_\alpha^{\mathcal{D}}$ has full measure, and the rest have measure 0, and so we quantify these sets with the Hausdorff dimension, obtaining the *dimension spectrum for pointwise dimensions*

$$\mathcal{D}(\alpha) = \dim_H K_\alpha^{\mathcal{D}}.$$

Taken together, the various multifractal spectra provide a great deal of information about the map f. In fact, certain classes of systems are known to exhibit *multifractal rigidity*, in which a finite number of multifractal spectra completely characterize a map [4].

We now give the relevant thermodynamic definitions. Fix a compact metric space X, a continuous map $f\colon X \to X$, and a Borel measurable potential function $\varphi\colon X \to \mathbb{R}$ which is bounded above and below. The *(variational) pressure* of φ is

$$P^*(\varphi) = \sup\left\{ h(\mu) + \int \varphi\, d\mu \,\middle|\, \mu \in \mathcal{M}^f(X) \right\}, \tag{1.1}$$

where $\mathcal{M}^f(X)$ is the set of all f-invariant Borel probability measures on X. If $Z \subset X$ is compact and f-invariant, the pressure on Z is the supremum of the quantity $h(\mu) + \int \varphi\, d\mu$ taken over all invariant probability measures supported on Z; we denote this supremum by $P^*_Z(\varphi)$.

As is customary in multifractal formalism, we use the Legendre transform in a slightly non-standard form. Given a convex function $T\colon \mathbb{R} \to [-\infty, +\infty]$, we write

$$T^{L_1}(\alpha) = \inf_{q\in\mathbb{R}} (T(q) - q\alpha). \tag{1.2}$$

Given a concave function $S\colon \mathbb{R} \to [-\infty, +\infty]$, we write

$$S^{L_2}(q) = \sup_{\alpha\in\mathbb{R}} (S(\alpha) + q\alpha). \tag{1.3}$$

The Legendre transform of a convex function is concave, and vice versa. Furthermore, the Legendre transform is self-dual: if T is convex and $T^{L_1} = S$, then $S^{L_2} = T$. Similarly, if S is concave and $S^{L_2} = T$, then $T^{L_1} = S$.

We will consider situations in which the function T is known to be convex (being given in terms of the pressure function), but the function S is one of the multifractal spectra, about which we have no *a priori* knowledge; we will continue to define the Legendre transform of such a function S by (1.3).

Observe also that if $S(x) \geq 0$ for every $x \in \mathbb{R}$, then S^{L_2} is infinite everywhere. Thus for purposes of defining the various multifractal spectra, we adopt the (non-standard) convention that $h_{\text{top}}\, \emptyset = \dim_H \emptyset = -\infty$.

If T is known to be convex, then left and right derivatives exist at every point where T is finite; we will denote these by

$$D^- T(q) = \lim_{q' \to q^-} \frac{T(q) - T(q')}{q - q'}, \qquad D^+ T(q) = \lim_{q' \to q^+} \frac{T(q') - T(q)}{q' - q}.$$

For a given convex function T, define a map from $\mathbb{R}$ to closed intervals in $\mathbb{R}$ by $A(q) = [D^- T(q), D^+ T(q)]$. Extend this in the natural way to a map from subsets of $\mathbb{R}$ to subsets of $\mathbb{R}$; we will again denote this map by A. This map has the following useful property: given any set $I_Q \subset \mathbb{R}$ and $\alpha \in A(I_Q)$, we have

$$T^{L_1}(\alpha) = \sup_{q\in I_Q} (T(\alpha) + q\alpha).$$

This will be important for us in settings where we only have partial information about the functions T and S. We will also make use of a map in the other direction: given a set $I_A \subset \mathbb{R}$ (in the domain of S), we denote the set of corresponding values of q by

$$Q(I_A) = \{q \in \mathbb{R} \mid A(q) \cap I_A \neq \emptyset\}.$$

In particular, if $\alpha = T'(q)$, then $\alpha = A(q)$, and if $q = -S'(\alpha)$, then $q = Q(\alpha)$. If (q_1, q_2) is an interval on which T is affine, then $A((q_1, q_2))$ is the slope of T on that interval; furthermore, T^{L_1} has a point of non-differentiability at $A((q_1, q_2))$.

2. Results: Birkhoff spectrum

Our most general result gives the Birkhoff spectrum as the Legendre transform of the following (convex) function:

$$T_{\mathcal{B}}(q) = P^*(q\varphi), \tag{2.1}$$

Before stating the general result, we describe the class of functions to which it applies. Given a function $\varphi\colon X \to \mathbb{R}$, let $\mathcal{C}(\varphi) \subset X$ denote the set of points at which φ is discontinuous. Then we let $\mathcal{A}_f$ denote the class of Borel measurable functions $\varphi\colon X \to \mathbb{R}$ which satisfy the following conditions:

(A) φ is bounded (both above and below);
(B) $\mu(\overline{\mathcal{C}(\varphi)}) = 0$ for all $\mu \in \mathcal{M}^f(X)$.

In particular, $\mathcal{A}_f$ includes all continuous functions $\varphi \in \mathcal{C}(X,\mathbb{R})$. It also includes all bounded measurable functions φ for which $\mathcal{C}(\varphi)$ is finite and contains no periodic points, and more generally, all bounded measurable functions for which $\overline{\mathcal{C}(\varphi)}$ is disjoint from all its iterates.

Theorem 2.1 (The entropy spectrum for Birkhoff averages)**.** *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi \in \mathcal{A}_f$. Then*

I. $T_{\mathcal{B}}$ is the Legendre transform of the Birkhoff spectrum:

$$T_{\mathcal{B}}(q) = \mathcal{B}^{L_2}(q) = \sup_{\alpha\in\mathbb{R}}(\mathcal{B}(\alpha) + q\alpha) \tag{2.2}$$

for every $q \in \mathbb{R}$.

II. The domain of $\mathcal{B}(\alpha)$ is bounded by the following:

$$\alpha_{\min} = \inf\{\alpha \in \mathbb{R} \mid T_{\mathcal{B}}(q) \geq q\alpha \text{ for all } q\},$$
$$\alpha_{\max} = \sup\{\alpha \in \mathbb{R} \mid T_{\mathcal{B}}(q) \geq q\alpha \text{ for all } q\},$$

That is, $K_\alpha^{\mathcal{B}} = \emptyset$ for every $\alpha < \alpha_{\min}$ and every $\alpha > \alpha_{\max}$.

III. Suppose that $T_{\mathcal{B}}$ is $\mathcal{C}^r$ on (q_1, q_2) for some $r \geq 1$, and that for each $q \in (q_1, q_2)$, there exists a (not necessarily unique) equilibrium state ν_q for the potential function $q\varphi$. Let $\alpha_1 = D^+T_{\mathcal{B}}(q_1)$ and $\alpha_2 = D^-T_{\mathcal{B}}(q_2)$; then

$$\mathcal{B}(\alpha) = T_{\mathcal{B}}^{L_1}(\alpha) = \inf_{q\in\mathbb{R}}(T_{\mathcal{B}}(q) - q\alpha) \tag{2.3}$$

for all $\alpha \in (\alpha_1, \alpha_2)$. In particular, $\mathcal{B}(\alpha)$ is strictly concave on (α_1, α_2), and $\mathcal{C}^r$ except at points corresponding to intervals on which $T_{\mathcal{B}}$ is affine.

Observe that the first two statements hold for *every* continuous map f, without any assumptions on the system, thermodynamic or otherwise. For discontinuous potentials in $\mathcal{A}_f$, these are the first rigorous multifractal results of any sort.

There are many physically interesting systems which display *phase transitions*—that is, values of q at which $T_{\mathcal{B}}$ is non-differentiable. If $T_{\mathcal{B}}$ is differentiable everywhere except q_0, then Theorem 2.1 gives the Birkhoff spectrum on the intervals $(-\infty, \alpha_1)$ and (α_2, ∞), where $\alpha_1 = \lim_{q\to q_0^-} T_{\mathcal{B}}'(q)$ and $\alpha_2 = \lim_{q\to q_0^+} T_{\mathcal{B}}'(q)$, but says nothing about the interval $[\alpha_1, \alpha_2]$, on which $T_{\mathcal{B}}^{L_1}(\alpha) = -q_0\alpha$ is linear. For certain systems, the following condition still holds despite the non-differentiability of $T_{\mathcal{B}}$ [17, 12]:

(A): There exists a sequence of compact f-invariant subsets $X_n \subset X$ such that the pressure function $q \mapsto P^*_{X_n}(q\varphi)$ is continuously differentiable for all $q \in \mathbb{R}$ (and equilibrium states exist), and furthermore,

$$\lim_{n\to\infty} P^*_{X_n}(q\varphi) = P^*(q\varphi). \tag{2.4}$$

Theorem 2.2. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi \in \mathcal{A}_f$ be measurable. If Condition **(A)** holds, then (2.3) holds for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.*

We can also deal with more general discontinuous potentials. Given $h \geq 0$, consider the set

$$I_A(h) = \{\alpha \in \mathbb{R} \mid T_{\mathcal{B}}^{L_1}(\alpha) > h\},$$

and also its counterpart

$$I_Q(h) = Q(I_A(h)).$$

Geometrically, $I_Q(h)$ may be described as the set of values $q \in \mathbb{R}$ such that there is a line through $(q, T_{\mathcal{B}}(q))$ that lies on or beneath the graph of $T_{\mathcal{B}}$ and intersects the y-axis somewhere above $(0, h)$.

We recall that $C\underline{h}_{\text{top}}(Z)$ is the lower capacity topological entropy of a set Z, defined as the asymptotic exponential growth rate of a minimal (n, δ)-spanning subset of Z.

Theorem 2.3. *Let X be a compact metric space, $f\colon X \to X$ be continuous, and $\varphi\colon X \to \mathbb{R}$ be measurable and bounded (above and below). Let $\mathcal{C}(\varphi)$ be the set of discontinuities of φ, and let $h_0 = C\underline{h}_{\text{top}}(\mathcal{C}(\varphi))$. Then*

I. For every $q \in I_Q(h_0)$, we have the following version of (2.2):

$$T_{\mathcal{B}}(q) = \sup_{\alpha \in I_A(h_0)} (\mathcal{B}(\alpha) + q\alpha). \tag{2.5}$$

II. $\mathcal{B}(\alpha) \leq h_0$ for every $\alpha \notin I_A(h_0)$.

III. Suppose that $T_{\mathcal{B}}$ is $\mathcal{C}^r$ on $(q_1, q_2) \subset Q(h_0)$ for some $r \geq 1$, and that for each $q \in (q_1, q_2)$ there exists a (not necessarily unique) equilibrium state ν_q for the potential function $q\varphi$. Then (2.3) holds for all $\alpha \in (\alpha_1, \alpha_2) = A((q_1, q_2))$.

3. Remarks

3.1. **Computability of the pressure and the spectrum.** Theorem 2.1 relates two quantities, the topological pressure and the Birkhoff spectrum, which are typically difficult to compute. Of the two, however, it is in general more feasible to compute the topological pressure and use it to determine the Birkhoff spectrum, rather than vice versa. One reason for this is that in order to determine the sets $K_\alpha^{\mathcal{B}}$ for different values of α, one must know the Birkhoff average (whose computation takes infinitely long in principle) at every point. On the other hand, to determine the potential function $q\varphi$, one takes a known function (φ) and multiplies by a scalar, a relatively easy task.

Furthermore, the computation of topological pressure on a compact invariant set is easier than the computation of (Bowen) topological entropy or Hausdorff dimension on a non-compact set. There are many cases (in particular, when φ is continuous) in which we can use a variational principle to compute $P^*(q\varphi)$ without dealing with invariant measures at all. In these cases, the pressure is revealed as a dimensional characteristic of capacity type (similar to the box dimension), and can be computed as the growth rate of a certain partition function. The quantity $h_{\text{top}}(K_\alpha^{\mathcal{B}})$, on the other hand, is not given as a growth rate but as the critical value of a parameter, similar to the Hausdorff dimension, and can be quite difficult to compute, even if the set $K_\alpha^{\mathcal{B}}$ is known.

3.2. **Differentiability and uniqueness of equilibrium states.** If the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous, then one can show that existence of a *unique* equilibrium state ν_q for $q\varphi$ on an interval (q_1, q_2) implies that $T_{\mathcal{B}}$ is differentiable on (q_1, q_2), with derivative $T_{\mathcal{B}}'(q) = \int_X q\varphi \, d\nu_q$. Thus in order to apply Theorem 2.1, it is sufficient to establish existence of a unique equilibrium state at each q. This is not necessary, however: $T_{\mathcal{B}}$ may be differentiable at q even in the presence of multiple equilibrium states for $q\varphi$, as evidenced by the following example.

Example 3.1. Let X be the disjoint union of two copies of Σ_3, the full two-sided shift on three symbols, and let $f\colon X \to X$ be the map whose restriction to each copy of Σ_3 is the usual shift. Then (X, f) is a topological Markov chain on six symbols $\{1, 2, 3, 1', 2', 3'\}$, where all transitions within $\{1, 2, 3\}$ are allowed, as are all transitions within $\{1', 2', 3'\}$, but no transitions $a \to b'$ or $b' \to a$ are permitted. Note that f is not transitive, and that there are two ergodic measures of maximal entropy, one supported on each of the two copies of Σ_3.

Given two triples $v, w \in \mathbb{R}^3$, define a potential $\varphi\colon X \to \mathbb{R}$ as follows:

$$\varphi(ax) = v_a, \quad \varphi(a'x) = w_a.$$

If we choose v and w such that $\sum_i v_i = \sum_i w_i = 0$, then $T_{\mathcal{B}}$ is differentiable at $q = 0$ with $T_{\mathcal{B}}'(0) = 0$, despite the non-uniqueness of the equilibrium state for $q\varphi$ with $q = 0$. Furthermore, by a suitable choice of v and w, we can arrange things so that for small negative values of q, the equilibrium state for $q\varphi$ sits on one copy of Σ_3, while for small positive values of q, it sits on the other: $v = (2, -1, -1)$ and $w = (-2, 1, 1)$ suffices. Thus there is a sort of phase transition in the nature and location of the equilibrium states, despite the fact that $T_{\mathcal{B}}$ is differentiable.

3.3. **Relationship to other results.** After this article was submitted, it was brought to the author's attention that similar results have been announced by Feng and Huang in [9]. We observe that the results there do not extend to discontinuous potentials, many of which are included in the class $\mathcal{A}_f$. Furthermore, Feng and Huang do not consider the dimension spectrum, which is considered below in Theorem 5.1.

We also point out that the multifractal formalism for some (but not all) of the examples in the next section follows from results of Pfister and Sullivan [20, Proposition 7.1], using different techniques than those here: specifically, they first prove a variational principle using a certain property of the system (the *g-almost product property*), and then derive multifractal results as a corollary.

4. Applications: Birkhoff spectrum

Here we list some of the results that follow from Theorem 2.1. The first two parts of the theorem do not require *any* hypotheses on the map f beyond continuity, and so for every continuous map f and every potential $\varphi \in \mathcal{A}_f$, the pressure function $T_{\mathcal{B}}$ is the Legendre transform of $\mathcal{B}(\alpha)$ (and hence $T_{\mathcal{B}}^{L_1}$ is the concave hull of $\mathcal{B}(\alpha)$), and the domain of the Birkhoff spectrum is the interval $[\alpha_{\min}, \alpha_{\max}]$.

4.1. **Uniform hyperbolicity.** In [3], Bowen showed that if M is a $\mathcal{C}^\infty$ Riemannian manifold and $f\colon M \to M$ is an Axiom A diffeomorphism, then any Hölder continuous potential function $\varphi\colon M \to \mathbb{R}$ has a unique equilibrium state. Since such maps are expansive on the hyperbolic set, this suffices to check the hypotheses of Theorem 2.1, and hence the Birkhoff spectrum is concave and $\mathcal{C}^1$. Versions of this result may be extracted from the results in [28, 22], but Theorem 2.1 provides a more direct proof.

Non-Hölder potentials were studied by Pesin and Zhang in [23] (building on results in [26], see also [17, 13]). They consider a uniformly piecewise expanding full-branched Markov map f of the unit interval, and use inducing schemes and tools from the theory of countable Markov shifts to study the existence and uniqueness of equilibrium states for a large class of potentials. In particular, they give the following example of a non-Hölder potential:

$$\varphi(x) = \begin{cases} -(1-\log x)^{-\alpha} & x \in (0,1], \\ 0 & x = 0. \end{cases} \tag{4.1}$$

It was shown in [23] that for any $\alpha > 1$ and $q \in \mathbb{R}$, the potential $q\varphi$ has a unique equilibrium state. Since f is expansive, one can show that equilibrium states exist and $T_{\mathcal{B}}$ is differentiable, and we have the following result.

Proposition 4.1. *Let f be a uniformly piecewise expanding full-branched Markov map of the unit interval, and let φ be the potential function given in (4.1), $\alpha > 1$. Then the Birkhoff spectrum $\mathcal{B}(\alpha)$ is smooth and concave, has domain $[\alpha_{\min}, \alpha_{\max}]$, and is the Legendre transform of $T_{\mathcal{B}}$.*

We remark that although Proposition 4.1 is stated for a single family of potentials, it actually holds for any potential φ such that each function $q\varphi$ is in the class considered in [23]. Because the precise definition of that class involves a fair amount of technical setup, we refer the reader to that paper for further information.

For the potential φ in (4.1) and values $0 < \alpha \leq 1$, it is shown in [23] that $T_{\mathcal{B}}$ has a phase transition at some value $q_0 > 0$. Applying Theorem 2.1, we obtain a result for the non-linear part of the Birkhoff spectrum; to obtain a complete result, we would need to apply Theorem 2.2 by establishing Condition **(A)**. Although this remains open, one might attempt to do this by using the fact that for a potential with summable variations, the Gurevich pressure on a topologically mixing countable Markov shift X is the supremum of the classical topological pressure over topologically mixing finite Markov subshifts of X [25]; these finite subshifts give natural candidates for the compact invariant sets X_n in Condition **(A)**.

4.2. **Maps with critical points.** By using a suitable modification of Theorem 2.1, one can obtain partial results regarding the Lyapunov spectra (Birkhoff spectra of the geometric potential); however, full results cannot yet be obtained using these techniques. However, the Birkhoff spectrum of other potentials can be studied with Theorem 2.1. Existence and uniqueness of equilibrium states for a certain class of bounded potentials were established in [7]. In particular, let $\mathcal{H}$ denote the collection of topologically mixing $\mathcal{C}^\infty$ interval maps $f\colon [0,1] \to [0,1]$ with hyperbolically repelling periodic points and non-flat critical points; given $f \in \mathcal{H}$, let $\varphi\colon [0,1] \to \mathbb{R}$ be a Hölder continuous potential such that

$$\sup \varphi - \inf \varphi < h_{\text{top}}\, f. \tag{4.2}$$

It was shown in [7] that there exists a unique equilibrium state for φ, and so we have the following result on the Birkhoff spectrum.

Proposition 4.2. *Given a map $f \in \mathcal{H}$ and a Hölder continuous potential φ satisfying (4.2), there exists $q_0 > 1$ such that the hypotheses of Theorem 2.1 are satisfied on the interval $(-q_0, q_0)$. In particular, $T_\mathcal{B}$ is $\mathcal{C}^1$ on this interval, and writing $\alpha_1 = D^+T_\mathcal{B}(-q_0)$, $\alpha_2 = D^-T_\mathcal{B}(q_0)$, we have $\mathcal{B}(\alpha) = T_\mathcal{B}^{L_1}(\alpha) = \inf_{q\in\mathbb{R}}(T_\mathcal{B}(q) - q\alpha)$ for every $\alpha \in (\alpha_1, \alpha_2)$.*

We remark that the multifractal formalism for this example and the example in the previous section (but not in the next section) follows from [20].

4.3. **Non-uniformly expanding maps.** The existence and uniqueness of equilibrium states for a broad class of non-uniformly expanding maps in higher dimensions was studied by Oliveira and Viana [18] and by Varandas and Viana [30]. To the best of the author's knowledge, the multifractal properties of these systems have not been studied at all, and so they provide an ideal application of Theorem 2.1. It does not appear to be known whether or not these systems, which may have contracting regions, satisfy specification or any other property that would imply Pfister and Sullivan's g-almost product property, and so the results of [20] cannot be applied.

A brief description of these results follows (see [30] for full details and for specific examples). Let M be a compact manifold and $f: M \to M$ a local homeomorphism. Let $\mathcal{A} \subset M$ be such that f is uniformly expanding outside of $\mathcal{A}$, and is not too contracting inside of $\mathcal{A}$. Furthermore, suppose M can be covered by domains of injectivity for f such that only $r < e^{h_{\text{top}}(f)}$ of these regions intersect $\mathcal{A}$, and let φ be a Hölder continuous potential such that

$$\sup \varphi - \inf \varphi < h_{\text{top}}(f) - \log r.$$

Then there exists a unique equilibrium state for φ. This allows us to apply the present results as follows: there exists $q_0 > 1$ such that $q\varphi$ satisfies the above condition for all $q \in (-q_0, q_0)$. Thus Theorem 2.1 applies, and we have an analog of Proposition 4.2.

5. Results: Entropy and dimension spectra

Definition 5.1. Given a compact metric space X, a continuous map $f: X \to X$, and a potential $\varphi: X \to \mathbb{R}$ (not necessarily continuous), we say that a Borel probability measure μ is a *weak Gibbs measure* for φ with constant $P \in \mathbb{R}$ if for every $x \in X$ and $\delta > 0$ there exists a sequence $M_n = M_n(x, \delta) > 0$ such that

$$\frac{1}{M_n} \leq \frac{\mu(B(x, n, \delta))}{\exp(-nP + S_n\varphi(x))} \leq M_n \tag{5.1}$$

for every $n \in \mathbb{N}$, where we require the following growth condition on M_n:

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log M_n(x, \delta) = 0. \tag{5.2}$$

There are various definitions in the literature of Gibbs measures of one sort or another; most of these definitions agree with each other and with the present definition in spirit, but differ in some slight details [3, 31, 16, 10, 30, 15]. The key property for our purposes is the following:

$$\left| -\frac{1}{n}\log\mu(B(x,n,\delta)) + \frac{1}{n}S_n\varphi(x) - P \right| \leq \frac{1}{n}\log M_n(x,\delta) \to 0, \tag{5.3}$$

where the limit is taken as $n \to \infty$ and then as $\delta \to 0$. One may show that if φ is continuous, then $P = P^*(\varphi)$, and so writing $\varphi_1(x) = \varphi(x) - P^*(\varphi)$, we observe that $K_\alpha^{\mathcal{B}}(\varphi_1) = K_{-\alpha}^{\mathcal{E}}$. This allows us to obtain the entropy spectrum directly from the Birkhoff spectrum.

For the dimension spectrum, we must use the following version of the Legendre transform:

$$T^{L_3}(\alpha) = \inf_{q\in\mathbb{R}}(T(q) + q\alpha), \qquad S^{L_4}(q) = \sup_{\alpha\in\mathbb{R}}(S(\alpha) - q\alpha).$$

These differ from T^{L_1} and S^{L_2} only in the sign on the second term: this change in sign occurs because the pointwise dimension is related to the local entropy, which is the rate of decay appearing as the first term in (5.3), and which is related to the *negative* of the Birkhoff average for φ.

We assume that f is conformal, and we write $\lambda_n(x) = \frac{1}{n}S_n \log\|Df\|(x)$. We must eliminate points at which the values of $\lambda_n(x)$ cluster around zero along a sequence of times at which the local entropy is also negligible; that is, the following set:

$$\mathbf{Z} = \left\{ x \in X \;\middle|\; \lim_{\delta\to 0} \varliminf_{n\to\infty} \left|\frac{1}{n}\log\mu(B(x,n,\delta))\right| + |\lambda_n(x)| = 0 \right\}. \tag{5.4}$$

The set $\mathbf{Z}$ contains all points x for which $\underline{\lambda}(x) = 0$ but μ has finite pointwise dimension; these are the only points our methods cannot deal with. In many cases we do not lose much by neglecting them; for example, if $\sup\varphi - \inf\varphi < h(\mu)$, then $\mathbf{Z} = \emptyset$. Even in cases when $\mathbf{Z}$ is non-empty, it often has zero Hausdorff dimension [15]. The remaining set of "good" points will be denoted by $X' = X \setminus \mathbf{Z}$. In the definition of $\mathcal{D}(\alpha)$, we adopt the convention that $\mathcal{D}(\alpha) = -\infty$ if $K_\alpha^{\mathcal{D}} \subset \mathbf{Z}$.

We also assume that every point $x \in X$ has bounded contraction:

$$\varliminf_{k\to\infty} \varliminf_{n\to\infty} S_n \log\|Df\|(f^k(x)) > -\infty\}. \tag{5.5}$$

Now consider the centered potential $\varphi_1(x) = \varphi(x) - P^*(\varphi)$, and also the potentials $\varphi_{q,t}(x) = q\varphi_1(x) - t\log\|Df\|(x)$. Define a thermodynamic function $T_{\mathcal{D}}$ by

$$T_{\mathcal{D}}(q) = \inf\{t \in \mathbb{R} \mid P^*(\varphi_{q,t}) \leq 0\}. \tag{5.6}$$

Given $\eta > 0$ and $I_Q \subset \mathbb{R}$, we will need to consider the following region lying just under the graph of $T_{\mathcal{D}}(q)$:

$$R_\eta(I_Q) = \{(q,t) \in \mathbb{R}^2 \mid q \in I_Q, T_{\mathcal{D}}(q) - \eta < t < T_{\mathcal{D}}(q)\}.$$

We can now state a general result regarding the dimension spectrum.

Theorem 5.1 (The dimension spectrum for pointwise dimensions)**.** *Let X be a compact metric space with $\dim_H X < \infty$, and let $f\colon X \to X$ be continuous and conformal with continuous non-vanishing factor $a(x)$. Suppose that (5.5) holds for all $x \in X$ and that $\lambda(\nu) \geq 0$ for every $\nu \in \mathcal{M}^f(X)$. Let $\mu \in \mathcal{M}^f(X)$ be a weak Gibbs measure for a continuous potential φ. Finally, suppose that $\dim_H \mathbf{Z} = 0$. Then we have the following.*

I. $T_{\mathcal{D}}$ is the Legendre transform of the dimension spectrum:

$$T_{\mathcal{D}}(q) = \mathcal{D}^{L_4}(q) = \sup_{\alpha\in\mathbb{R}}(\mathcal{D}(\alpha) - q\alpha) \tag{5.7}$$

for every $q \in \mathbb{R}$.

II. Neglecting points in $\mathbf{Z}$, the domain of $\mathcal{D}$ is bounded by the following:

$$\alpha_{\min} = \inf\{\alpha \in \mathbb{R} \mid T_{\mathcal{D}}(q) \geq -q\alpha \text{ for all } q\},$$
$$\alpha_{\max} = \sup\{\alpha \in \mathbb{R} \mid T_{\mathcal{D}}(q) \geq -q\alpha \text{ for all } q\}.$$

That is, $K_\alpha^{\mathcal{D}} \cap X' = \emptyset$ for every $\alpha < \alpha_{\min}$ and every $\alpha > \alpha_{\max}$.

III. Suppose $I_Q = (q_1, q_2)$ and $\eta > 0$ are such that for every $(q,t) \in R_\eta(I_Q)$, the potential $\varphi_{q,t}$ has a (not necessarily unique) equilibrium state, and that the map $(q,t) \mapsto P^*(\varphi_{q,t})$ is $\mathcal{C}^r$ on $R_\eta(I_Q)$ for some $r \geq 1$. Then we have

$$\mathcal{D}(\alpha) = T_{\mathcal{D}}^{L_3}(\alpha) = \inf_{q\in\mathbb{R}}(T_{\mathcal{D}}(q) + q\alpha) \tag{5.8}$$

for all $\alpha \in (\alpha_2, \alpha_1) = A(I_Q)$; in particular, $\mathcal{D}$ is strictly concave on (α_2, α_1), and $\mathcal{C}^r$ except at points corresponding to intervals on which $T_{\mathcal{D}}$ is affine.

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Department of Mathematics, McAllister Building, Pennsylvania State University, University Park, PA 16802, USA.

E-mail address: climenha@math.psu.edu

URL: http://www.math.psu.edu/climenha/