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PINNED REPETITIONS IN SYMBOLIC FLOWS: PRELIMINARY RESULTS

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Abstract. We consider symbolic flows over finite alphabets and study certain kinds of repetitions in these sequences. Positive and negative results for the existence of such repetitions are given for codings of interval exchange transformations and codings of quadratic polynomials.

1. **Introduction.** Fix some finite alphabet $\mathcal{A}$ and consider the compact space $\mathcal{A}^{\mathbb{Z}}$ of two-sided sequences over this alphabet. Here we endow $\mathcal{A}$ with the discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with the product topology.

We want to study repetitions in sequences $\omega \in \mathcal{A}^{\mathbb{Z}}$ that either begin at the origin or are centered at the origin and hence are "pinned." Moreover, we also want to study how many times arbitrarily long subwords are repeated in this way. Thus, for $\omega \in \mathcal{A}^{\mathbb{Z}}$, we define

$$R_n(\omega) = 1 + \frac{1}{n} \sup\{m : \omega_k = \omega_{k+n} \text{ for } 1 \le k \le m\},$$

$$T_n(\omega) = 1 + \frac{1}{n} \sup\{m : \omega_k = \omega_{k+n} \text{ and } \omega_{n+1-k} = \omega_{1-k} \text{ for } 1 \le k \le m\},$$

and

$$R(\omega) = \limsup_{n\to\infty} R_n(\omega),$$
$$T(\omega) = \limsup_{n\to\infty} T_n(\omega).$$

For definiteness, we will declare $\sup \emptyset = 0$. Notice that $R(\omega), T(\omega)$ may be infinite and that we always have $T(\omega) \le R(\omega)$.

Repetitions that begin at the origin have been studied, for example, by Berthé et al. [1]. (In their terminology, $R(\omega) = \text{ice}(\omega)$, the *initial critical exponent* of ω).

Repetitions that are centered at the origin are of interest in the study of Schrödinger operators; compare the survey articles [4, 5]. As explained there, if $T(\omega)$ is sufficiently large, one can prove results about the continuity of spectral measures of an associated Schrödinger operator using the Cayley-Hamilton theorem.

Denote the shift transformation on $\mathcal{A}^{\mathbb{Z}}$ by S, that is, $(S\omega)_k = \omega_{k+1}$. A subshift Ω is a closed, S-invariant subset of $\mathcal{A}^{\mathbb{Z}}$. A subshift Ω is called minimal if the S-orbit of every $\omega \in \Omega$ is dense in Ω. We denote by $\mathcal{W}(\omega)$ the set of all finite words over

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$\mathcal{A}$ that occur somewhere in ω. If Ω is minimal, then there is a set $\mathcal{W}(\Omega)$ such that $\mathcal{W}(\omega) = \mathcal{W}(\Omega)$ for every $\omega \in \Omega$. Finally, we let $\mathcal{W}_n(\Omega) = \mathcal{W}(\Omega) \cap \mathcal{A}^n$.

Proposition 1. *Suppose that Ω is a minimal subshift. Then, $R_{\max}(\Omega) = \max_{\omega\in\Omega} R(\omega)$ and $T_{\max}(\Omega) = \max_{\omega\in\Omega} T(\omega)$ both exist, as elements of $[1,\infty]$. Moreover, the sets $\{\omega \in \Omega : R(\omega) = R_{\max}(\Omega)\}$ and $\{\omega \in \Omega : T(\omega) = T_{\max}(\Omega)\}$ are residual.*

Proof. We prove the two statements for two-sided pinned repetitions. The one-sided case may be treated analogously.

Note that it suffices to show that, for any $\hat\omega \in \Omega$, the set $M(\hat\omega) = \{\omega \in \Omega : T(\omega) \ge T(\hat\omega)\}$ is residual. Indeed, once this is shown, one may choose a sequence in $\{\hat\omega^{(k)}\} \subset \Omega$ such that $T(\hat\omega^{(k)}) \to \sup_{\omega\in\Omega} T(\omega)$ and then consider the residual set $M = \bigcap_{k\ge1} M(\hat\omega^{(k)})$. By construction, $T(\omega) = \sup_{\omega\in\Omega} T(\omega)$ for every $\omega \in M$, and hence the sup is a max, and since the set $\{\omega \in \Omega : T(\omega) = T_{\max}(\Omega)\}$ contains M, it is residual.

So let $\hat\omega \in \Omega$ be given. Choose a finite or countable strictly increasing sequence $t_1, t_2, \ldots$ with $T(\hat\omega) = \sup_m t_m$. Fix m. By minimality, for each length l, there is N_l such that every word in $\mathcal{W}_{N_l}(\Omega)$ contains all words from $\mathcal{W}_l(\Omega)$ as subwords. Thus, it is possible to find, for each $w \in \mathcal{W}_{2n+1}(\Omega)$ a word $E(w,m) \in \mathcal{W}(\Omega)$ of odd length that has w as its central subword of length $2n+1$ and obeys[1]

$$T(E(w,m)) \ge t_m - \frac{1}{n}. \tag{1}$$

We may simply shift $\hat\omega$ until the first occurrence of w is centered at the origin and then take a sufficiently long finite piece that is centered at the origin as well. By the consequence of minimality mentioned above, it is clearly possible to ensure an estimate of the form (1) that is uniform in $w \in \mathcal{W}_{2n+1}(\Omega)$.

Consider the open set

$$\bigcup_{w\in\mathcal{W}_{2n+1}(\Omega)} [E(w,m)],$$

where, for a word x of odd length, $[x] = \{\omega \in \Omega : \omega_{-\frac{|x|-1}{2}} \ldots \omega_{\frac{|x|-1}{2}} = x\}$. Notice that

$$\mathcal{T}_m = \bigcup_{n\ge m} \bigcup_{w\in\mathcal{W}_{2n+1}(\Omega)} [E(w,m)],$$

is dense in Ω. Consequently,

$$\mathcal{T} = \bigcap_{m\ge1} \mathcal{T}_m$$

is a dense G_δ subset of Ω and for every $\omega \in \mathcal{T}$, we have by construction $T(\omega) \ge T(\hat\omega)$. ∎

Proposition 2. *Suppose that μ is an S-ergodic probability measure on Ω. Then there exist R_μ and T_μ such that $R(\omega) = R_\mu$ and $T(\omega) = T_\mu$ for μ-almost every $\omega \in \Omega$.*

Proof. Clearly, $T(\cdot)$ is invariant and hence it is μ-almost surely constant. While $R(\cdot)$ may not be globally invariant, we always have the inequality $R(S\omega) \ge R(\omega)$ for every $\omega \in \Omega$. This implies that $R(S\omega) = R(\omega)$ for μ-almost every ω and hence $R(\omega)$ is μ-almost surely constant. ∎

[1] While T was defined above only for two-sided infinite words, a completely analogous definition can be given for finite words, where the central position plays the role of the origin.

Remark. There are some related results in [1]. They prove that, for minimal subshifts, $R(\cdot)$ attains its maximum (without showing that it does so on a residual set). They also establish the almost sure constancy of $R(\cdot)$ with respect to any ergodic measure under the assumption of minimality and sublinear block complexity ([1], Proposition 2.1). Our result (Proposition 2) holds in complete generality, and its proof is very short.

In the remainder of the paper, we study pinned repetitions in symbolic flows that are generated by coding certain specific transformations of finite-dimensional tori. In fact, given the space allotment, we will focus on two such classes – interval exchange transformations and quadratic polynomials arising in the study of skew-shifts. We intend to continue our study of pinned repetitions in symbolic flows in a future work. We would also like to point out that the present study is related to our recent papers [2, 3] on the repetition property for dynamical systems on general compact metric spaces (which are not necessarily totally disconnected).

2. **Preliminaries.**

In this section, we present several results that will be useful later in our study of pinned repetitions in certain specific models.

2.1. **Irrational Rotations of the Circle.**

For $x \in \mathbb{R}$, we write $\langle x \rangle = \operatorname{dist}(x, \mathbb{Z})$. Notice that $d(x, y) = \langle x - y \rangle$ is a metric on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Recall that the Farey sequence of order n is the sequence of reduced fractions between 0 and 1 which have denominators less than or equal to n, arranged in order of increasing size. Thus, for example, $F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$, $F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$, $F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$, $F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$. If $\frac{a}{b}$ and $\frac{c}{d}$ are neighbors in a Farey sequence, then $b + d > n$ and $|\frac{a}{b} - \frac{c}{d}| = \frac{1}{bd}$; see, for example, [9].

Lemma 1. *Let $x, y \in \mathbb{T}$ and $u > 0$ be such that for some $n \geq 2$, the set $\{x+y, 2x+y, 3x+y, \ldots, nx+y\}$ does not intersect an arc $J \subseteq \mathbb{T}$ of length $|J| = u$. Then for some positive integer $q < \min\left\{n, \frac{2}{u}\right\}$, we have $\langle qx \rangle < \frac{1}{n}$.*

Proof. Of course, we can assume without loss of generality that $y = 0$. Denote $S = \left\{kx : 1 \leq k \leq \left\lfloor \frac{2}{u} \right\rfloor\right\}$ and consider the position of x relative to the Farey sequence of order $n-1$: $\frac{r_1}{s_1} \leq x \leq \frac{r_2}{s_2}$. By the well-known properties of the Farey sequence of order $n-1$ recalled above, it follows that $s_j < n$ for $j = 1, 2$ and $s_1 + s_2 \geq n$.

The point $\frac{r_1+r_2}{s_1+s_2}$ subdivides the interval $[\frac{r_1}{s_1}, \frac{r_2}{s_2}]$. Suppose $x \in [\frac{r_1}{s_1}, \frac{r_1+r_2}{s_1+s_2}]$. Then,

$$\left| x - \frac{r_1}{s_1} \right| \leq \frac{r_1 + r_2}{s_1 + s_2} - \frac{r_1}{s_1} = \frac{1}{s_1(s_1 + s_2)} < \frac{1}{s_1 n}.$$

The case $x \in [\frac{r_1+r_2}{s_1+s_2}, \frac{r_2}{s_2}]$ is analogous. Consequently, for $\frac{p}{q}$ equal to either $\frac{r_1}{s_1}$ or $\frac{r_2}{s_2}$, we have that

$$\left| x - \frac{p}{q} \right| < \frac{1}{qn}. \tag{2}$$

In particular, we must have $q < n$ and $\langle qx \rangle < \frac{1}{n}$. Moreover, since $\frac{p}{q}$ is a reduced fraction, the estimate (2) implies that the maximal gap (in $\mathbb{T}$) of $\{kx : 1 \leq k \leq q\}$ is bounded from above by $\frac{2}{q}$. By assumption, we therefore must have $u < \frac{2}{q}$, that is, $q < \frac{2}{u}$. □

2.2. **Continued Fraction Expansion.** Let us recall some basic results from the theory of continued fractions; compare [9, 12]. Given an irrational number $\alpha \in \mathbb{T}$, there are uniquely determined $a_n \in \mathbb{Z}_+$, $n \geq 1$ such that

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Truncation of this infinite continued fraction expansion after k steps yields the k-th *convergent* $\frac{p_k}{q_k}$. We have the following two-sided estimate for the quality of approximation of α by the k-th convergent:

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.$$

The numerators and denominators of the convergents obey the following recursive relations:

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2} \text{ for } k \geq 2, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2} \text{ for } k \geq 2. \end{aligned}$$

2.3. **Discrepancy Estimates for Quadratic Polynomials.** In this subsection we discuss uniform distribution properties of quadratic polynomials. More precisely, given $\alpha, \beta, \gamma \in \mathbb{T}$, we consider the points

$$x_n = \alpha n^2 + \beta n + \gamma \in \mathbb{T}. \tag{3}$$

The numbers

$$D_N = D_N(\alpha, \beta, \gamma) = \sup_{\text{intervals } I \subseteq \mathbb{T}} \left| \frac{1}{N} \#\{n : 1 \leq n \leq N, \ x_n \in I\} - \mathrm{Leb}(I) \right|$$

measure the quality of uniform distribution of the given sequence and are called its *discrepancy*.

Theorem 1. *Suppose $\alpha \in \mathbb{T}$ is irrational and*

$$\alpha = \frac{p}{q} + \frac{\theta}{q^2},$$

where p and q are relatively prime and $|\theta| \leq 1$. Then, for $\beta, \gamma \in \mathbb{T}$ and $\varepsilon > 0$ arbitrary, we have

$$D_q(\alpha, \beta, \gamma) < C_\varepsilon q^{-\frac{1}{3} - \varepsilon}$$

with some constant C_ε that only depends on ε. In particular, the set $\{x_1, \dots, x_q\}$ intersects every interval $I \subseteq \mathbb{T}$ of length at least $C_\varepsilon q^{-\frac{1}{3} - \varepsilon}$.

We will use the following version of the Erdös-Turán Theorem, which holds in fact for arbitrary real numbers $x_1, \dots, x_N$; compare [11, pp. 112–114].

Theorem 2 (Erdös-Turán 1948)**.** *There is a universal constant $\tilde{C}$ such that for every $m \in \mathbb{Z}_+$,*

$$D_N \leq \tilde{C} \left(\frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right).$$

This estimate relates discrepancy bounds to bounds for exponential sums. Thus the following lemma, which is closely related to a lemma given on p. 43 of [12], is of interest.

Lemma 2. *Suppose $\alpha \in \mathbb{T}$ is irrational, $\beta, \gamma \in \mathbb{T}$ are arbitrary, and x_n is given by (3). Then, we have for $N \in \mathbb{Z}_+$,*

$$\left| \sum_{n=1}^{N} e^{2\pi i x_n} \right|^2 \le N + \sum_{n=1}^{N} \min \left(2N, \frac{1}{2\langle n\alpha \rangle} \right).$$

Proof. This can be proved by a slight variation of the argument given on pp. 43–44 of [12]. □

Proof of Theorem 1. By the Erdös-Turán Theorem, we have

$$D_q \lesssim \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \frac{1}{q} \sum_{n=1}^{q} e^{2\pi i h x_n} \right|$$

for every $m \in \mathbb{Z}_+$.[2] Take $m = \lfloor q^\delta \rfloor$ for some $\delta \in (0,1)$. Then, together with Lemma 2, we find for any $\varepsilon 0$,

$$\begin{aligned} D_q &\lesssim q^{-\delta} + \sum_{h=1}^{\lfloor q^\delta \rfloor} \frac{1}{h} \left| \frac{1}{q} \sum_{n=1}^{q} e^{2\pi i h x_n} \right| \\ &\le q^{-\delta} + \frac{1}{q} \sum_{h=1}^{\lfloor q^\delta \rfloor} \frac{1}{h} \left(q + \sum_{n=1}^{q} \min \left(2q, \frac{1}{2\langle hn\alpha \rangle} \right) \right)^{1/2} \\ &\lesssim q^{-\delta} + \frac{1}{q} \sum_{h=1}^{\lfloor q^\delta \rfloor} \frac{1}{h} \left(q + \frac{h}{\varepsilon} \left(1 + q^{\delta - 1}\right) q^{1+\varepsilon} \right)^{1/2} \\ &\lesssim q^{-\delta} + q^{\frac{1}{2}\delta - \frac{1}{2} + \frac{\varepsilon}{2}}. \end{aligned}$$

Here, we applied [10, Eqn. (153) on p. 75] in the third step. In the last line, the exponents coincide when $\delta = \frac{1-\varepsilon}{3}$. □

Given an irrational α, Theorem 1 gives a discrepancy estimate for those values of N that appear as denominators in the convergents associated with α. This will be sufficient for our purpose. In some cases (e.g., α's of Roth type), it is possible to use [12, Theorem 6 on p. 45] to modify the proof so as to cover all values of N. Moreover, if one is only interested in a metric result, the following improved discrepancy estimate is of relevance.

Theorem 3. *Suppose $\alpha \in \mathbb{T}$ is irrational. Then, for Lebesgue almost every $\beta \in \mathbb{T}$ and every $\gamma \in \mathbb{T}$, we have*

$$D_N(\alpha, \beta, \gamma) < C_\varepsilon N^{-\frac{1}{2}} (\log N)^{\frac{5}{2}+\varepsilon}.$$

Since we will not use this result, we do not prove it. It may be derived from [7, Theorem 1.158 on p. 157].

3. Pinned Repetitions in Codings of Interval Exchange Transformations.

In this section we study pinned repetitions occurring in codings of interval exchange transformations. Interval exchange transformations are maps from an interval to itself that are obtained by partitioning the interval and then permuting the subintervals.

[2] We write $a \lesssim b$ for positive a, b if there is a universal constant C such that $a \le Cb$.

More explicitly, let $m > 1$ be a fixed integer and denote

$$\Lambda_m = \{\lambda \in \mathbb{R}^m : \lambda_j > 0,\ 1 \le j \le m\}$$

and, for $\lambda \in \Lambda_m$,

$$\beta_j(\lambda) = \begin{cases} 0 & j = 0 \\ \sum_{i=1}^j \lambda_i & 1 \le j \le m \end{cases},\ I_j^\lambda = [\beta_{j-1}(\lambda), \beta_j(\lambda)),\ |\lambda| = \sum_{i=1}^m \lambda_i,\ I^\lambda = [0, |\lambda|).$$

Denote by $\mathcal{S}_m$ the group of permutations on $\{1, \dots, m\}$, and set $\lambda_j^\pi = \lambda_{\pi^{-1}(j)}$ for $\lambda \in \Lambda_m$ and $\pi \in \mathcal{S}_m$. With these definitions, the (λ, π)-interval exchange map $T_{\lambda,\pi}$ is given by

$$T_{\lambda,\pi} : I^\lambda \to I^\lambda,\ x \mapsto x - \beta_{j-1}(\lambda) + \beta_{\pi(j)-1}(\lambda^\pi) \text{ for } x \in I_j^\lambda,\ 1 \le j \le m.$$

A permutation $\pi \in \mathcal{S}_m$ is called irreducible if $\pi(\{1, \dots, k\}) = \{1, \dots, k\}$ implies $k = m$. We denote the set of irreducible permutations by $\mathcal{S}_m^0$.

Veech proved the following theorem in [13].

Theorem 4 (Veech 1984)**.** *Let $\pi \in \mathcal{S}_m^0$. For Lebesgue almost every $\lambda \in \Lambda_m$ and every $\varepsilon > 0$, there are $N \ge 1$ and an interval $J \subseteq I^\lambda$ such that*

(i) $J \cap T^l J = \emptyset$, $1 \le l < N$,
(ii) T *is linear on* $T^l J$, $0 \le l < N$,
(iii) $|\bigcup_{l=0}^{N-1} T^l J| > (1 - \varepsilon)|\lambda|$,
(iv) $|J \cap T^N J| > (1 - \varepsilon)|J|$.

An interval exchange transformation $T = T_{\lambda,\pi}$ is said to satisfy property V if for every $\varepsilon > 0$, there are $N \ge 1$ and an interval $J \subseteq I^\lambda$ such that the four conditions of Theorem 4 are satisfied. This convention is motivated by the following result.

Theorem 5. *Let $T = T_{\lambda,\pi}$ satisfy property V. Then the coding s of the T-orbit of Lebesgue almost every $x \in I^\lambda$ with respect to any finite partition of I^λ obeys $R(s) = T(s) = \infty$.*

Proof. Fix any finite partition $I^\lambda = J_1 \sqcup \cdots \sqcup J_N$.

If $\limsup_{\varepsilon \to 0+} N(\varepsilon) < \infty$, it is not hard to see that $T_{\lambda,\pi}$ is a "rational rotation" and the assertion of the theorem holds trivially.

Consider the other case and let $\varepsilon_n \to 0$ be such that $N(\varepsilon_n) \to \infty$. By passing to a suitable subsequence $\{\varepsilon_{n_m}\}$, we can ensure that the Lebesgue measure of those points $x \in I^\lambda$ that do not have m repetitions in both directions is bounded by 2^{-m}. Thus, by Borel-Cantelli, almost every point has unbounded repetitions in both directions, which implies the assertion. □

Corollary 1. *Let $\pi \in \mathcal{S}_m^0$. For Lebesgue almost every $\lambda \in \Lambda_m$, the coding s of the $T_{\lambda,\pi}$-orbit of Lebesgue almost every $x \in I^\lambda$ with respect to any finite partition of I^λ obeys $R(s) = T(s) = \infty$.*

4. **Pinned Repetitions in Codings of Quadratic Polynomials.** In this section we study sequences s of the following type. Suppose $\{J_l : 1 \le l \le N\}$ is a partition of $\mathbb{T}$ into finitely many intervals, $(\alpha, \beta, \gamma) \in \mathbb{T}^3$, and s denotes the coding of $\alpha n^2 + \beta n + \gamma$ with respect to the partition J. For example, assigning distinct numbers λ_l to the partition intervals, we may write

$$s_n = \sum_{l=1}^N \lambda_l \chi_{J_l}(\alpha n^2 + \beta n + \gamma).$$

Sometimes we make the dependence of s on the parameters explicit and write $s(\alpha,\beta,\gamma)$ or $s(\alpha,\beta,\gamma,J)$. As explained in [8], there is close connection between codings of quadratic polynomials and codings of orbits of the skew-shift on $\mathbb{T}^2$.

We are interested in identifying the numbers $R(s)$ and $T(s)$ for such sequences s. We present a number of results regarding this problem. Roughly speaking, these numbers may take on the extreme values 1 and ∞ and they grow with the quality with which α can be approximated by rational numbers.

4.1. **Absence of Repetitions.** We first consider the case where α is not well approximated by rational numbers and show that there indeed are no repetitions in the sense that $R(s) = T(s) = 1$. To make the argument more transparent, we begin by considering α's with bounded partial quotients. This means that the coefficients $\{a_n\}$ in the continued fraction expansion are bounded; equivalently, $\inf_{q\in\mathbb{Z}_+} q\langle q\alpha\rangle > 0$. The set of such α's has zero Lebesgue measure.

Theorem 6. *Suppose that $\alpha \in \mathbb{T}$ has bounded partial quotients and each partition interval J_l has length strictly less than $1/2$. Then, $R(s(\alpha,\beta,\gamma,J)) = T(s(\alpha,\beta,\gamma,J)) = 1$ for every $(\beta,\gamma)\in\mathbb{T}^2$.*

Proof. Assume that $R(s) > 1+\nu$ for some $\nu\in(0,1)$. Then, s has infinitely many $(1+\nu)$-repetitions starting at the origin. Let n be the length of such a prefix of $s|_{\mathbb{Z}_+}$ that is $(1+\nu)$-repeated.

For $1 \le k \le \nu n$, write

$$\begin{aligned} y_k &= \alpha k^2 + \beta k + \gamma \\ z_k &= \alpha(n+k)^2 + \beta(n+k) + \gamma \\ d_k &= z_k - y_k = \alpha n^2 + \beta n + 2\alpha n k. \end{aligned}$$

Choose $0 < l < 1/2$ such that each interval of the partition under consideration has length less than l. Then, for $1 \le k \le \nu n$, y_k and z_k must fall in the same interval of the partition. In particular, $\langle d_k\rangle$ is bounded above by l for each such k. Consequently, d_k avoids an arc $J \subseteq \mathbb{T}$ of length $u = 1-2l > 0$.

Applying Lemma 1 with $x = 2\alpha n$ and $y = \alpha n^2 + \beta n$, we find that there is a positive integer $q < \frac{2}{u}$ such that $\langle qx\rangle < \frac{1}{n}$. On the other hand, there is $c = c(u,\alpha) > 0$ such that $\langle qx\rangle = \langle 2\alpha q n\rangle > \frac{c}{n}$ since α has bounded partial quotients. Combining the two estimates, we have that

$$\frac{c}{n} < \langle qx\rangle < \frac{1}{n}. \tag{4}$$

Now assume in addition that $n > \frac{2}{\nu u}$ and let $m = \lfloor \frac{\nu n}{q} \rfloor$, so that $1 \le \frac{\nu n}{q} \le m \le \nu n$. Consider the points $\{d_{qk} : 1 \le k \le m\}$. It follows from (4) that the diameter of this set belongs to the interval $(\frac{c\nu}{q}, \frac{\nu}{q})$.

For n sufficiently large, we obtain a contradiction because, as we saw above, the points y_{sk} are well distributed on $\mathbb{T}$ and will come close to the partition points. Addition of the difference d_{qk} for k with $1 \le k \le m$ sufficiently large will then go "across" such a partition point, which contradicts the assumption the y_{qk} and z_{qk} belong to the same interval of the partition. It follows that $R(s) = 1$, which also yields $T(s) = 1$. ∎

As pointed out above, this result covers only a set of α's that has zero Lebesgue measure. Let us extend it to a larger set. Recall that $\alpha\in\mathbb{T}$ is a Roth number if for every $\varepsilon > 0$, there is a constant $c(\varepsilon)$ such that $\langle q\alpha\rangle > \frac{c(\varepsilon)}{q^{1+\varepsilon}}$, for every $q \in \mathbb{Z}_+$.

If we replace the qualitative well-distribution property with the quantitative discrepancy bound established above, virtually the same argument proves the following result, which covers a set of α's that has full Lebesgue measure.

Theorem 7. *Suppose that $\alpha \in \mathbb{T}$ is a Roth number and each partition interval J_l has length strictly less than $1/2$. Then, $R(s(\alpha,\beta,\gamma,J)) = T(s(\alpha,\beta,\gamma,J)) = 1$ for every $(\beta,\gamma) \in \mathbb{T}^2$.*

Proof. The only change that needs to be made to the argument above is the following. Assuming α to be Roth, instead of (4), we can prove

$$\frac{c_\varepsilon}{n^{1+\varepsilon}} < \langle qx \rangle < \frac{1}{n} \tag{5}$$

for any $\varepsilon > 0$. Applying Theorem 1, we see that for every $\tilde{\varepsilon} > 0$, the points $x_1, x_2, \ldots, x_n$ are $C_{\tilde{\varepsilon}} q_k^{-\frac{1}{3}-\tilde{\varepsilon}}$-dense in $\mathbb{T}$, where k is chosen such that $q_k \le n < q_{k+1}$. Since the Roth condition also implies that the q_k's associated with α obey $q_{k+1} \lesssim q_k^{1+\varepsilon}$, it follows that $x_1, x_2, \ldots, x_n$ is $C_{\tilde{\varepsilon}} n^{-(1+\varepsilon)(\frac{1}{3}+\tilde{\varepsilon})}$-dense in $\mathbb{T}$.

Now we can conclude the proof as before by considering the points $\{d_{qk} : 1 \le k \le m\}$. The addition of one of them to the corresponding y_{qk} will take the point across a partition point by the estimates just obtained. □

4.2. **Infinite Repetitions.** Let us now turn to the other extreme and start off by studying the case of rational α. We will see that in this case, there are infinite repetitions for almost every pair (β,γ). The next step will then be to identify situations in which we have infinite repetitions for irrational α's that are well approximated by rational numbers in a suitable sense.

Denote by $\mathbb{T}_w$ the subset of irrational numbers in $\mathbb{T}$ with unbounded partial quotients. It is well known that $\mathbb{T}_w$ is a set of full Lebesgue measure in $\mathbb{T}$.

First we address the case $\alpha = 0$.

Lemma 3. *Suppose that $\beta \in \mathbb{T}_w$. Then, for Lebesgue a. a. $\gamma \in \mathbb{T}$,*

$$R(s(0,\beta,\gamma,J)) = T(s(0,\beta,\gamma,J)) = \infty.$$

Proof. In this case $s = s(0,\beta,\gamma,J)$ is a coding of $\beta n + \gamma$ with respect to the fixed partition J. The claim of the proposition follows from the fact that the 2-interval exchange transformation $T = T_{\lambda,\pi}$ with $\lambda = (1-\beta,\beta),\ \pi = (21)$ is Veech if $\beta \in \mathbb{T}_w$ (see Theorems 4 and 5 above).

For an alternative, more direct argument consult [6]. □

Theorem 8. *Suppose that $\alpha \in \mathbb{T}$ is rational. Then, for every $\beta \in \mathbb{T}_w$,*

$$R(s(\alpha,\beta,\gamma,J)) = T(s(\alpha,\beta,\gamma,J)) = \infty \tag{6}$$

for Lebesgue a. a. $\gamma \in \mathbb{T}$. In particular, (6) holds for Lebesgue a. a. $(\beta,\gamma) \in \mathbb{T}^2$.

Proof. Write $\alpha = \frac{p}{q}$. Notice that αn^2 is q-periodic. This serves as a motivation to begin with a study of repetitions along arithmetic progressions of step-length q. Put differently, we regard $\beta(qn)$ as $(\beta q)n$ and then add the constant $\alpha(qn)^2 + \gamma$.

For every $\beta \in \mathbb{T}_w$, $\beta q \in T_w$ as well. By Lemma 3, there exists a sequence $Q_k \to \infty$ and a set $\tilde{G}_\beta \subset \mathbb{T}$ of full measure such that for $\tilde{\gamma} \in \tilde{G}_\beta$, the coding of $(\beta q)n + \tilde{\gamma}$ with respect to the given partition of $\mathbb{T}$ has k repetitions of length Q_k to both sides, for every $k \ge 1$.

To piece together these repetitions along arithmetic progressions of step-length q, define $\mathcal{G}'_\beta = \bigcap_{k=1}^\infty \bigcap_{n=1}^{Q_k} \{\gamma \in \mathbb{T} : \alpha(qn)^2 + \beta(qn) + \gamma \in \tilde{G}_\beta\}$.

As a countable intersection of sets of full measure, $\mathcal{G}'_\beta$ has full measure. We find that for $\alpha = \frac{p}{q}$ rational, $\beta \in \mathbb{T}_w$ and $\gamma \in \mathcal{G}'_\beta$ (and hence for almost every $(\beta, \gamma) \in \mathbb{T}^2$), $R(s) = T(s) = \infty$. ◻

Theorem 9. *There is a dense G_δ set $\mathcal{R} \subset \mathbb{T}$ such that for $\alpha \in \mathcal{R}$, we have $R(s(\alpha, \beta, \gamma, J)) = T(s(\alpha, \beta, \gamma, J)) = \infty$ for Lebesgue almost every $(\beta, \gamma) \in \mathbb{T}^2$.*

Proof. Let $r_1, r_2, r_3, \ldots$ be a sequence of rational numbers that contains each fixed $\frac{p}{q} \in \mathbb{Q} \cap (0,1)$ infinitely many times.

Fix k and consider the coding of $r_k n^2 + \beta n + \gamma$ with respect to the given partition, denoted by s_k. By Theorem 8 we have that $T(s_k) = \infty$ for almost every β, γ.

For $m \in \mathbb{Z}_+$, we can therefore choose a set $\mathcal{B}_{m,k} \subset \mathbb{T}^2$ such that

- $B_{m,k}$ is open,
- $\mathrm{Leb}(\mathcal{B}_{m,k}) > 1 - \frac{1}{2^m}$,
- for $(\beta, \gamma) \in \mathcal{B}_{m,k}$, s_k has m repetitions in both directions at least once,
- for $(\beta, \gamma) \in \mathcal{B}_{m,k}$, the itinerary $r_k n^2 + \beta n + \gamma$ does not contain any partition point for n's from the finite interval on which we observe the m repetitions in both directions.

Next, choose $\mathcal{K}_{m,k} \subset \mathcal{B}_{m,k}$ such that

- $\mathcal{K}_{m,k}$ is compact,
- $\mathrm{Leb}(\mathcal{K}_{m,k}) > 1 - \frac{1}{2^m}$.

By compactness of $\mathcal{K}_{m,k}$, we have that for $(\beta, \gamma) \in \mathcal{K}_{m,k}$, the itinerary $r_k n^2 + \beta n + \gamma$ for n's from the finite interval on which we observe the m repetitions in both directions has a uniform positive distance from the partition points.

Consequently, we can perturb r_k slightly and not change the coding on the (large) finite interval that supports the two-sided repetition in question. In other words, there is an open set $\mathcal{U}_k$ containing r_k such that for $\alpha \in \mathcal{U}_k$ and $(\beta, \gamma) \in \mathcal{K}_{m,k}$, s has m repetitions in both directions at least once.

Define $\mathcal{R} = \bigcap_{m \geq 1} \bigcup_{k \geq m} \mathcal{U}_k$. Since $r_1, r_2, r_3, \ldots$ contains each fixed $\frac{p}{q} \in \mathbb{Q} \cap (0,1)$ infinitely many times, the open set $\bigcup_{k \geq m} \mathcal{U}_k$ is dense.

Therefore, $\mathcal{R}$ is a dense G_δ set.

By construction, we have that for $\alpha \in \mathcal{R}$ and Lebesgue almost every $(\beta, \gamma) \in \mathbb{T}^2$, $R(s) = T(s) = \infty$. Indeed, if $\alpha \in \mathcal{R}$, then α belongs to some $\mathcal{U}_k$. By Borel-Cantelli and the measure estimates for the sets $\mathcal{K}_{m,k}$, we have that for almost every $(\beta, \gamma) \in \mathbb{T}^2$, s has m repetitions to both sides for any m. ◻

The residual set obtained in Theorem 9 is not explicit. If we are willing to settle for infinite repetitions for just one pair $(\beta, \gamma) \in \mathbb{T}^2$, then the following result is of interest. It also has the advantage that the argument we give can treat general polynomials, not merely quadratic polynomials, and hence we state and prove the result in this more general setting. For $\tau > 0$, denote

$$\mathcal{S}_\tau = \{\alpha \in \mathbb{T} : \langle q\alpha \rangle < q^{-\tau} \text{ for infinitely many odd positive integers } q\}.$$

Clearly, $\mathcal{S}_\tau$ is a residual subset of $\mathbb{T}$.

Theorem 10. *Let r be a positive integer and $\varepsilon > 0$. Then, for every $\alpha \in \mathcal{S}_{r+\varepsilon}$, the sequence s given by $s_n = \chi_{[0,1/2)}(\alpha n^r + 1/4)$ obeys $R(s) = T(s) = \infty$.*

Proof. Fix an integer $m \geq 2$ and let q be an odd integer with $\langle q\alpha \rangle < q^{-r-\varepsilon}$. Let $p \in \mathbb{Z}$ be such that $\langle q\alpha \rangle = |q\alpha - p|$. Then, for $|n| \leq mq$ we have, on the one hand,

$$\langle \alpha(n+q)^r - \alpha n^r \rangle = \left\langle \sum_{j=0}^{r-1} \binom{r}{j} \alpha n^j q^{r-j} \right\rangle \le \sum_{j=0}^{r-1} \binom{r}{j} n^j q^{r-1-j} \langle q\alpha \rangle$$
$$\le \sum_{j=0}^{r-1} \binom{r}{j} m^j q^{r-1} \frac{1}{q^{r+\varepsilon}} = \frac{1}{q^{1+\varepsilon}} \sum_{j=0}^{r-1} \binom{r}{j} m^j$$

and, on the other hand,

$$\langle \alpha n^r \pm 1/4 \rangle \ge \left\langle \frac{p}{q} n^r \pm \frac{1}{4} \right\rangle - \frac{n^r}{q^{1+r+\varepsilon}} \ge \frac{1}{4q} - \frac{m^r}{q^{1+\varepsilon}}.$$

Since for large enough q, we have that

$$\frac{1}{4q} - \frac{m^r}{q^{1+\varepsilon}} > \frac{1}{q^{1+\varepsilon}} \sum_{j=0}^{r-1} \binom{r}{j} m^j,$$

and hence $T(s) \ge m$. Since m was arbitrary, this shows $T(s) = \infty$, which also implies that $R(s) = \infty$. $\square$

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