DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **828–837**

GENERAL UNIQUENESS RESULTS AND EXAMPLES FOR BLOW-UP SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. In this paper, we establish the blow-up rate of the large positive solution of the singular boundary value problem

$$\begin{cases} -\triangle u = \lambda u - b(x)uf(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases}$$

where Ω is a ball domain and b is a radially symmetric function on the domain, $f(u) \in C^1[0,\infty)$ satisfies $f(0) = 0$, $f'(u) > 0$ for all $u > 0$, and $f(u) \sim Fu^{p-1}$ for sufficiently large u with $F > 0$ and $p > 1$. Naturally, the blow-up rate of the problem equals its blow-up rate for the very special, but important case, when $f(u) = Fu^{p-1}$. Some examples are given to illustrate how the blow-up rate depends on the asymptotic behavior of b near the boundary. b can decay to zero as a polynomial, an exponential function, or a function which is not monotone near the boundary.

1. **Introduction and Main Results.** Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We are interested in the uniqueness and asymptotic behavior of the blow-up solutions to the elliptic equation

$$-\triangle u = \lambda u - b(x)uf(u) \qquad \text{in } \Omega, \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter and $b(x)$ is continuous and positive in Ω and non-negative on $\partial\Omega$.

A solution of (1) is called a large (or explosive) solution which is understood as a strong solution u such that

$$u(x) \to \infty \qquad \text{as } d(x, \partial\Omega) \to 0. \tag{2}$$

The problem (1) together with the boundary blow-up condition is called a singular boundary value problem. The subject of blow-up solutions has obtained much attention starting with the pioneering papers ([1],[2],[3], [9], [14], [17]-[26]) and the reference therein. Singular boundary value problem (1) arises naturally from a number of different areas and have a long history. If $b > 0$ in $\bar{\Omega}$ and $f(u) = u^{p-1}$ ($p > 1$), then (1) is known as the logistic equation. This equation is a basic population model and it is also related to some prescribed curvature problems in Riemannian geometry ([3], [8],[22]). In 1916 Bieberbach [2] studied the large solutions for the particular case $-\triangle u = -b(x)\exp(x)$, $b(x) \equiv 1$ and $N = 2$. He showed that there

2000 *Mathematics Subject Classification.* Primary: 35J25, 35J65, 35J67.

Key words and phrases. Elliptic equation, uniqueness, blow-up rate, large positive solution, sub solution, super solution.

The author is partially supported by RIG Grant (Code 2137) 2008-2009 from Virginia State University.

exists a unique solution of equation (1) such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$. Motivated by a problem in mathematical physics, Rademacher [28] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$. In 1990's, Bandle-Essen [1] and Lazer-Mckenna [14] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniformal external sphere condition. It is shown that the problem exhibits a unique solution in a smooth domain together with an estimate of the form $u = \log d^{-2} + o(d)$ in [14](where $b(x) \geq b_0 > 0$ as $d \to 0$) and in [1](where $b \equiv 1$).

Recently, the uniqueness and blow-up rates of large solutions for $f(u) = u^{p-1}$ are treated in [9]-[30] and in many other works. Under the assumption

$$\lim_{d(x,\partial\Omega)\to 0} \frac{b(x)}{d(x,\partial\Omega)^{\gamma}} = \zeta$$

with $\gamma > 0$ and $\zeta > 0$, an explicit expression for the blow-up rates of (1) has been recently proved in [9](1999) and [11](2001) as $u = (\alpha(\alpha+1)/\zeta)^{1/(p-1)} d^{-\alpha}(1+o(d))$, $\alpha = (\gamma+2)/(p-1)$. Further improvements of this result can be found in [17], [19], [25], [26] and the reference therein. A localization method used in [17], [26] shows that (1) (with $f(u) = u^{p-1}, p > 1$) has at most one blow-up solution for the case when γ and ζ vary along $\partial\Omega$.

S. Cano-Casanova and J. López-Gómez [5] published their very recent uniqueness results, which dealt with the same problem under different assumptions on $b(x)$ and $f(u)$. If Ω is a ball or an annulus, $b(x)$ is a positive and non-decreasing radially continuous function, $f \in \mathcal{C}[0,\infty) \bigcap \mathcal{C}^2(0,\infty)$ satisfies $f(0)$, $f'(u) > 0$, $f''(u) > 0$ and $f(u) \sim Fu^{p-1}$ as $u \to \infty$, then in [5] they proved that (1) possesses a unique positive large solution and the exact blow-up rate of the large solution is estimated. If the results in [5] and [20] are combined, they would only require the monotonicity of b and the concavity of $f(u)$ and would not require that $f(u) \sim Fu^{p-1}$ as $u \to \infty$. Z. Xie [29] extended the study in a more general domain with different type of nonlinear function $f(u)$.

The main purpose of this paper is to provide a more general version of (1) for a class of function of $f(u)$ and a wide range of $b(x)$, in the spirit of [19], [25] and [26]. Consider the singular boundary value problem:

$$\begin{cases} -\triangle u = \lambda u - b(x)uf(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega = B_R(x_0)$ is the ball of radius R centered at x_0. The weight function $b(x)$ satisfies:

(Assumption $\mathcal{B}$) (1) $b(x) = b(r)$ is a radially continuous function in the ball, where $r = \|x - x_0\|$. Define $B(r) = \int_r^R b(s)ds$ and $b^*(r) = \int_r^R B(s)ds$.

(2) $b(r) \in C([0,R];[0,\infty))$ satisfies $b(r) > 0$ for $r \in [0,R)$. $\frac{B(r)}{b(r)} \in C^1([0,R])$, $\lim_{r\to R} \frac{B(r)}{b(r)} = 0$, and

$$C_0 := \lim_{r\to R} \frac{((b^*(r))')^2}{b^*(r)(b^*(r))''} \geq 1. \tag{4}$$

The nonlinear function $f(u)$ satisfies:

(Assumption $\mathcal{F}$) $f(u) \in C^1[0,\infty)$ satisfies $f(0) = 0$, $f'(u) > 0$ for all $u > 0$; and, for some $p > 1$,

$$F = \lim_{u\to\infty} \frac{f(u)}{u^{p-1}} > 0. \tag{5}$$

The main result can be stated as follows.

Theorem 1.1. *Suppose that the weight function $b(x)$ and the nonlinear function $f(u)$ satisfy the assumption $\mathcal{B}$ and $\mathcal{F}$ respectively. Then the problem (3) has a unique solution u satisfying*

$$\lim_{d(x)\to 0} \frac{u(x)}{KF^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} = 1$$

where $d(x) = dist(x, \partial\Omega)$ and K is a constant defined by

$$K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \beta = \frac{1}{p-1}.$$

The rest of the paper is organized as follows. In section 2, we give the proof of theorem 1.1. In Section 3, we make use of theorem 1.1 and construct some examples which give some very interesting asymptotic behavior.

2. Proof of the Main Result.

Proof of Theorem 1.1. We first consider the corresponding singular problem (3) in one dimension

$$\begin{cases} -\psi'' - \frac{N-1}{r}\psi' & = \lambda\psi - b(r)\psi f(\psi) \text{ in } (0,R), \\ \lim_{r\to R}\psi(r) & = \infty, \\ \psi'(0) & = 0. \end{cases} \tag{6}$$

We claim that for each $\epsilon > 0$, the problem (6) possesses a positive large solution ψ_ϵ such that

$$1-\epsilon \le \liminf_{r\to R} \frac{\psi_\epsilon(r)}{KF^{-\beta}(b^*(r))^{-\beta}} \le \limsup_{r\to R} \frac{\psi_\epsilon(r)}{KF^{-\beta}(b^*(r))^{-\beta}} \le 1+\epsilon \tag{7}$$

where we have denoted

$$\beta = \frac{1}{p-1}, \quad b^*(r) = \int_r^R \int_s^R b(t)dtds, \quad K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}, \tag{8}$$

and C_0 is given by (4) and H is constant as in (5).
Therefore, for each $x_0 \in \mathbb{R}^N$, the function

$$u_\epsilon(x) := \psi_\epsilon(r); \qquad r := \|x-x_0\|$$

provides us with a radially symmetric positive large solution of (3) with the assumptions in theorem 1.1 and the solution satisfies

$$1-\epsilon \le \liminf_{d(x)\to 0} \frac{u_\epsilon(x)}{KF^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} \le \limsup_{d(x)\to 0} \frac{u_\epsilon(x)}{KF^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} \le 1+\epsilon. \tag{9}$$

To prove the claim, we first construct a supersolution of (6) for each $\epsilon > 0$. Let

$$\bar{\psi}_\epsilon(r) = A + B_+\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta}, \tag{10}$$

where $A > 0$ and $B_+ > 0$ have to be determined later. Then $\bar{\psi}_\epsilon(r)$ is a supersolution if

$$-\bar{\psi}_\epsilon''(r) - \frac{N-1}{r}\bar{\psi}_\epsilon'(r) \ge \lambda\bar{\psi}_\epsilon(r) - b(r)\bar{\psi}_\epsilon(r)f(\bar{\psi}_\epsilon(r)). \tag{11}$$

By the assumption (5), for the same $\epsilon > 0$,

$$(1-\epsilon)F\bar{\psi}_\epsilon^p(r) \leq \bar{\psi}_\epsilon(r)f(\bar{\psi}_\epsilon(r)) \leq (1+\epsilon)F\bar{\psi}_\epsilon^p(r) \tag{12}$$

for all $r \in [0, R)$ by choosing A sufficiently large, say $A \geq A_0$. The inequality (11) holds if

$$-\bar{\psi}_\epsilon''(r) - \frac{N-1}{r}\bar{\psi}_\epsilon'(r) \geq \lambda\bar{\psi}_\epsilon(r) - b(r)(1-\epsilon)F\bar{\psi}_\epsilon^p(r). \tag{13}$$

Multiplying both sides of this inequality by $\frac{(b^*(r))^{p\beta}}{b(r)}$ and taking into consideration that $p\beta = \beta + 1$.

$$\begin{aligned} &-2N\frac{B_+}{R^2}\frac{b^*(r)}{b(r)} + [N+3]\beta B_+\frac{r}{R^2}\frac{(b^*(r))'}{b(r)} - \\ &\beta(\beta+1)B_+\left(\frac{r}{R}\right)^2\frac{[(b^*(r))']^2}{b^*(r)b(r)} + \beta B_+\left(\frac{r}{R}\right)^2\frac{(b^*(r))''}{b(r)} \\ &\geq \lambda\frac{b^*(r)}{b(r)}\left[A(b^*(r))^\beta + B_+\left(\frac{r}{R}\right)^2\right] - (1-\epsilon)F\left[A(b^*(r))^\beta + B_+\left(\frac{r}{R}\right)^2\right]^p. \end{aligned}$$

Since when $r \to R$, $\frac{b^*(r)}{b(r)} \to 0$, $\frac{(b^*(r))'}{b(r)} \to 0$, $\frac{[(b^*(r))']^2}{b^*(r)b(r)} \to C_0 \geq 1$ and $\frac{(b^*(r))''}{b(r)} \to 1$ by assumption $\mathcal{B}$, then the above inequality becomes into

$$-\beta(\beta+1)B_+C_0 + \beta B_+ \geq -(1-\epsilon)F(B_+)^p$$

as $r \to R$, which is

$$B_+ \geq \frac{[\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}}{[(1-\epsilon)F]^{\frac{1}{p-1}}}.$$

Let $B_+ = (1+\epsilon)(1-\epsilon)^{-\beta}F^{-\beta}\left[\beta((\beta+1)C_0-1)\right]^\beta = (1+\epsilon)(1-\epsilon)^{-\beta}F^{-\beta}K$. Therefore, by making the choice B_+, the inequality (13) is satisfied in a left neighborhood of $r = R$, say $(R-\delta, R]$, for some $\delta = \delta(\epsilon) > 0$. Finally, by choosing A sufficiently large (larger than A_0) it is clear that the inequality is satisfied in the whole interval $[0, R]$ since $p > 1$ and $b^*(r)$ is bounded away from zero in $[0, R-\delta]$. Then $\bar{\psi}_\epsilon$ is our required supersolution of problem 6.

Next, we construct a subsolution with the same blow-up rate as the above supersolution. For doing this we shall distinguish two different cases according to the sign of the parameter λ. First, we assume $\lambda \geq 0$. Due to the assumption (5) on f, for $u \geq A_0$ large,

$$(1-\epsilon)Fu^p \leq uf(u) \leq (1+\epsilon)Fu^p.$$

For each $A_0 > 0$ and $0 < R_0 < R$, we consider the auxiliary problem

$$\begin{cases} -\psi'' - \frac{N-1}{r}\psi' &= \lambda\psi - b(r)\psi f(\psi) \text{ in } (0, R_0), \\ \psi(R_0) &= A_0, \\ \psi'(0) &= 0. \end{cases} \tag{14}$$

By the assumption on b and f, we have

$$\min_{r\in[0,R_0]} b(r) > 0, f(0) = 0, \text{ and } f(u) \to \infty \text{ as } u \to \infty.$$

Then it is easy to know that

$$\underline{\psi}_{A_0} := 0, \qquad \bar{\psi}_{A_0} := A_0$$

provides us with an ordered sub-supersolution pair of (14). Thus (14) possesses a solution ψ_{A_0} such that $\psi_{A_0}(r) \in [0, A_0]$ for all $r \in [0, R_0]$. For each $\epsilon > 0$ sufficiently small, we claim that there exists $0 < C < A_0$ for which the function

$$\underline{\psi}_\epsilon(r) = \begin{cases} \psi_{A_0}(r), & r \in [0, R_0], \\ \max\{A_0, C + B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta}\}, & r \in (R_0, R], \end{cases}$$

provides a subsolution, where R_0 and C are to be determined later and

$$B_- = (1-\epsilon)(1+\epsilon)^{-\beta} F^{-\beta} \left[\beta((\beta+1)C_0 - 1)\right]^{\beta} = (1-\epsilon)(1+\epsilon)^{-\beta} F^{-\beta} K.$$

In fact, denoting $g_C(r) = C + B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta}$ we have

$$g'_C(r) = 2B_- \frac{r}{R^2}(b^*(r))^{-\beta} + \beta B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1} \int_r^R b(s)ds$$

which is strictly bigger than zero in $(0, R)$. It follows that $g_C(r)$ is increasing and

$$\lim_{r\to R} g_C(r) = +\infty, \qquad \lim_{r\to 0} g_C(r) = C < A_0.$$

By the continuity of $g_C(r)$ and the intermediate-value theorem, there exists a unique $Z = Z(C) \in (0, R)$ such that

$$C + B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta} < A_0 \text{ when } r \in [0, Z(C))$$

$$C + B_- \left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta} \geq A_0 \text{ when } r \in [Z(C), R]$$

Moreover, $Z(C)$ is decreasing and

$$\lim_{C\to-\infty} Z(C) = R, \qquad \lim_{C\to A_0} Z(C) = 0.$$

Let $R_0 = Z(C)$. From the definition of $\underline{\psi}_\epsilon(r)$ and R_0, $\underline{\psi}_\epsilon(r) \equiv \psi_{A_0}(r)$ in $[0, Z(C)]$, and then the inequality $-\underline{\psi}''_\epsilon - \frac{N-1}{r}\underline{\psi}'_\epsilon \leq \lambda\underline{\psi}_\epsilon - b(r)h(\underline{\psi}_\epsilon)$ holds in $[0, Z(C)]$. So $\underline{\psi}_\epsilon(r)$ is a subsolution if the following inequality is satisfied in $[Z(C), R]$

$$-\underline{\psi}''_\epsilon(r) - \frac{N-1}{r}\underline{\psi}'_\epsilon(r) \leq \lambda\underline{\psi}_\epsilon(r) - b(r)\underline{\psi}_\epsilon(r)f(\underline{\psi}_\epsilon(r)). \tag{15}$$

By direct computation and by using the fact $\underline{\psi}_\epsilon(r)f(\underline{\psi}_\epsilon(r)) \leq (1+\epsilon)F\underline{\psi}_\epsilon(r)^p$ in $[Z(C), R]$, (15) holds if

$$-\beta(\beta+1)B_- \left(\frac{r}{R}\right)^2 \frac{[(b^*(r))']^2}{b^*(r)b(r)} + \beta B_- \left(\frac{r}{R}\right)^2 \leq -(1+\epsilon)H \left[C(b^*(r))^{\beta} + B_- \left(\frac{r}{R}\right)^2\right]^p$$

for each $r \in [Z(C), R]$. At $r = R$, it becomes $-\beta(\beta+1)B_-C_0 + \beta B_- \leq -(1+\epsilon)FB_-^p$. That is $B_- \leq [(1+\epsilon)F]^{-\frac{1}{p-1}}[\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}$. By making the choice $B_- = (1-\epsilon)[(1+\epsilon)F]^{-\frac{1}{p-1}}[\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}}$ and using the continuity, it is easy to see that a constant $\delta = \delta(\epsilon) > 0$ exists for which the inequality is satisfied in $[R-\delta, R)$, then we choose C such that $Z(C) = R - \delta(\epsilon)$ (i.e. $R_0 = R - \delta(\epsilon)$). For this choice of C, it readily follows that $\underline{\psi}_\epsilon$ is a sub solution to the problem.

A subsolution for $\lambda < 0$ can be constructed by a similar argument as in ([25]). Now we have a subsolution and a supersolution with the same blow-up rate of the problem (6). So there exists a solution $\Psi_\epsilon(r)$ of (6) such that

$$1 - \epsilon \leq \liminf_{r\to R} \frac{\psi_\epsilon(r)}{KF^{-\beta}(b^*(r))^{-\beta}} \leq \limsup_{r\to R} \frac{\psi_\epsilon(r)}{KF^{-\beta}(b^*(r))^{-\beta}} \leq 1 + \epsilon.$$

Proof of uniqueness. Let u be an arbitrary solution of (3). We can show that

$$\lim_{d(x)\to 0} \frac{u(x)}{KF^{-\beta}(b^*(\|x-x_0\|))^{-\beta}} = 1.$$

Consequently, for any pair of solutions u, v of (3)

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1.$$

Thus, for every $\epsilon > 0$, we can find $\delta > 0$(as small as we please) such that

$$(1-\epsilon)v(x) \le u(x) \le (1+\epsilon)v(x)$$

when $0 < d(x) \le \delta$. On the other hand, because $f((1-\epsilon)v) \le f(v)$ and $f((1+\epsilon)v) \ge f(v)$ for any $v > 0$, $\underline{w} = (1-\epsilon)v(x)$ and $\bar{w} = (1+\epsilon)v(x)$ are sub and super solutions respectively to

$$\begin{cases} -\triangle w &= \lambda w - b(x)wf(w) \qquad \text{in } B_{R-\delta}(x_0), \\ w &= u \qquad \text{on } \partial B_{R-\delta}(x_0). \end{cases} \tag{16}$$

The unique solution to this problem is $w = u$. Then

$$(1-\epsilon)v(x) \le u(x) \le (1+\epsilon)v(x)$$

holds in $B_{R-\delta}(x_0)$, therefore it is true in $B_R(x_0)$. Letting $\epsilon \to 0$, we arrive at $u = v$. □

3. **Examples.** In this section, we construct three examples to illustrate the relationship between blow-up rates of large solutions and decay rates of weight functions. Here we only give solutions in one dimension and the corresponding solutions in higher dimension can be easily converted. The corresponding singular boundary value problem (3) in one dimension is

$$\begin{cases} \psi'' + \frac{N-1}{r}\psi' &= -\lambda\psi + b(r)\psi f(\psi) \text{ in } (0,R), \\ \lim_{r\to R}\psi(r) &= \infty, \\ \psi'(0) &= 0, \end{cases} \tag{17}$$

where N is the dimension and R is the radius of ball domain. In the following examples, b and f satisfy the assumptions $\mathcal{B}$ and $\mathcal{F}$ respectively by direct computation.

Example 3.1. Let $N = 3$, $\lambda = 0$, $R = 1$, $f(\psi) = \psi + \psi^2$ and $b(r) = \mu(r)6(r-1)^2$, where $\mu(r) = \left(2\,(r-1)^2 + r^2\right)^{-2}\left(3\,(r-1)^2 + r^2\right)^{-1}$.
Then $p = 3$, $F = \lim_{\psi\to\infty} f(\psi)/\psi^{p-1} = 1$, $\beta = \frac{1}{p-1} = 1/2$, and

$$C_0 = \lim_{r\to 1} \frac{((b^*(r))')^2}{b^*(r)(b^*(r))''} = \frac{4}{3}, K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}} = \frac{\sqrt{2}}{2}.$$

Because $\mu(r) = 1 - 6\,(r-1) + 14\,(r-1)^2 - 93\,(r-1)^4 + 258\,(r-1)^5 + O\left((r-1)^6\right)$, the weight function $b(r)$ has a polynomial decay rate $6(r-1)^2$ as $r \to 1$. Then $(b^*(r))^{-\beta}$ has a polynomial blow-up rate $(\frac{1}{2}(r-1)^4)^{-1/2} = \sqrt{2}(r-1)^{-2}$ as $r \to 1$. By theorem 1.1, the blow-up rate of the unique solution $\psi(r)$ is same as the blow-up rate of $KF^{-\beta}(b^*(r))^{-\beta} \sim (r-1)^{-2}$ as $r \to 1$.

On the other hand, we know that $\psi(r) = 2 + r^2(r-1)^{-2}$ is the exact solution of (17). In fact, $\psi(r)$ is a smooth function in $(0,1)$, $\psi'(r) = \frac{2r}{(r-1)^2} - \frac{2r^2}{(r-1)^3}$,

$$\psi''(r) = 2\,(r-1)^{-2} - 8\,\frac{r}{(r-1)^3} + 6\,\frac{r^2}{(r-1)^4},$$

$\psi'' + \frac{N-1}{r}\psi' = 6\,(r-1)^{-4}$, and $\psi f(\psi) = (\mu(r))^{-1}\,(r-1)^{-6}$. So $\lim_{r\to 1}\psi(r) = \infty$ and $\psi'(0) = 0$. The exact blow-up rate of the solution is $(r-1)^{-2}$ as $r \to 1$. The graph of the weight function $b(r)$ and the graph of the exact solution $\psi(r)$ are shown in Figure 1.

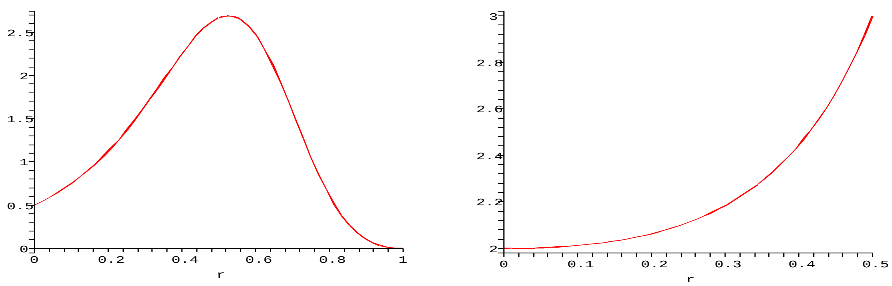

FIGURE 1. Left: the graph of $b(r) = \mu(r)6(r-1)^2$ (Polynomial Decay); Right: the graph of the exact solution $\psi(r) = 2 + r^2(r-1)^{-2}$.

Example 3.2. Let $N = 3$, $R = 1$, $f(\psi) = \psi^2$ and $b(r) = \mu(r)\frac{1}{4}e^{-(1-r)^{-2}}$, where

$$\mu(r) = \frac{24\frac{a^*}{a} + 12\frac{A}{a} + 3r^2\frac{A^2}{a^* a} - 2r^2}{(2\sqrt{a^*} + r^2)^3},$$

and $a = e^{-(1-r)^{-2}}$, $A = \int_r^R a(s)ds$, $a^* = \int_r^R A(s)ds$.
Then $p = 3$, $F = \lim_{\psi\to\infty} f(\psi)/\psi^{p-1} = 1$, $\beta = \frac{1}{p-1} = 1/2$, and

$$C_0 = \lim_{r\to 1}\frac{((b^*(r))')^2}{b^*(r)(b^*(r))''} = 1, K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}} = \frac{1}{2}.$$

Because $\lim_{r\to R}\mu(r) = 1$, the weight function $b(r)$ has an exponential decay rate $\frac{1}{4}e^{-(1-r)^{-2}}$ as $r \to 1$, which could be not approximated by a polynomial. Then $(b^*(r))^{-\beta}$ has an exponential blow-up rate $\left(\int_r^1\int_t^1 \frac{1}{4}e^{-(1-s)^{-2}}dsdt\right)^{-1/2} = 2\left(\int_r^1\int_t^1 e^{-(1-s)^{-2}}dsdt\right)^{-1/2}$ as $r \to 1$. By theorem 1.1, the blow-up rate of the unique solution $\psi(r)$ is same as the blow-up rate of

$$\psi(r) \sim KF^{-\beta}(b^*(r))^{-\beta} \sim \left(\int_r^1\int_t^1 e^{-(1-s)^{-2}}dsdt\right)^{-1/2} \quad \text{as } r \to 1.$$

On the other hand, the exact solution is

$$\psi(r) = 2 + r^2\left(\int_r^1\int_t^1 e^{-(1-s)^{-2}}dsdt\right)^{-1/2},$$

which has the exact blow-up rate as we found by theorem 1.1. The graph of the weight function $b(r)$ and the graph of the exact solution $\psi(r)$ are shown in Figure 2.

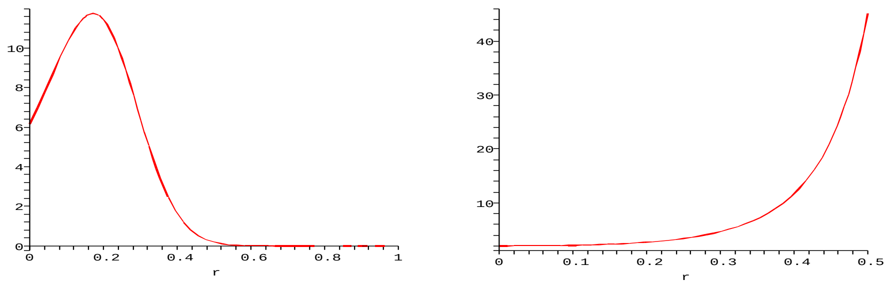

FIGURE 2. Left: the graph of $b(r) = \mu(r)\frac{1}{4}e^{-(1-r)^{-2}}$ (Exponential Decay); Right: the graph of the exact solution $\psi(r) = 2 + r^2\left(\int_r^1\int_t^1 e^{-(1-s)^{-2}}dsdt\right)^{-1/2}$.

Example 3.3. Let $N = 3$, $R = 1$, $f(\psi) = \psi^2$ and

$$b(r) = \mu(r)\,(1-r)\left(2 + 3\,(1-r)\sin\left((1-r)^{-1}\right)\right)$$

where

$$\mu(r) = \frac{24\frac{a^*}{a} + 12\frac{A}{a} + 3r^2\frac{A^2}{a^*a} - 2r^2}{4(2\sqrt{a^*} + r^2)^3},$$

and $a = (1-r)\left(2 + 3\,(1-r)\sin\left((1-r)^{-1}\right)\right)$, $A = \int_r^R a(s)ds$, $a^* = \int_r^R A(s)ds$. Then $p = 3$, $F = \lim_{\psi\to\infty} f(\psi)/\psi^{p-1} = 1$, $\beta = \frac{1}{p-1} = 1/2$, and

$$C_0 = \lim_{r\to 1}\frac{((b^*(r))')^2}{b^*(r)(b^*(r))''} = \frac{3}{2}, K = [\beta((\beta+1)C_0 - 1)]^{\frac{1}{p-1}} = \frac{\sqrt{10}}{4}.$$

Because $\lim_{r\to R}\mu(r) = \frac{5}{8}$ and $\sin\left((1-r)^{-1}\right)$ is not monotone, $b(r)$ is not a monotone decreasing function as $r \to 1$. The decay rate of $b(r)$ is $\frac{5}{4}(1-r)$ as $r \to 1$ and the blow-up rate of $(b^*(r))^{-\beta}$ is $(\frac{5}{24}(1-r)^3)^{-1/2}$ as $r \to 1$. By theorem 1.1, the blow-up rate of the unique solution $\psi(r)$ is same as the blow-up rate of

$$\psi(r) \sim KF^{-\beta}(b^*(r))^{-\beta} \sim \sqrt{3}(1-r)^{-3/2} \text{ as } r \to 1.$$

On the other hand, the exact solution is

$$\psi(r) = 2 + r^2\left(\int_r^1\int_t^1 (1-s)\left(2 + 3\,(1-s)\sin\left((1-s)^{-1}\right)\right)dsdt\right)^{-1/2}$$

which has the exact blow-up rate as we found by theorem 1.1. The graph of the weight function $b(r)$ and the graph of the exact solution $\psi(r)$ are shown in Figure 3.

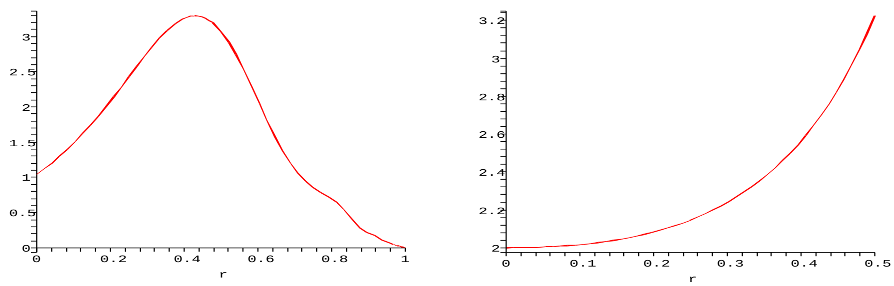

FIGURE 3. Left: the graph of $b(r) = \mu(r)\,(1-r)\left(2+3\,(1-r)\sin\left((1-r)^{-1}\right)\right)$ for $r \in [0,1]$ (Not Monotone); Right: the graph of the exact solution $\psi(r)$.

Acknowledgments. The author thanks sincerely the anonymous referees for their careful reading of the manuscript and the very helpful suggestions made for its improvement. We have shortened the proof of theorem 1.1 by eliminating some technical details and providing more explicit expressions on the examples in section 3.

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Received June 2008; revised July 2009.

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