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EXISTENCE OF SOLUTIONS TO SINGULAR INTEGRAL EQUATIONS

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Abstract. We consider the system of integral equations

$$u_i(t) = \int_0^T g_i(t,s)[a_i(s,u_1(s),u_2(s),\cdots,u_n(s)) \\ + b_i(s,u_1(s),u_2(s),\cdots,u_n(s))]ds, \quad t\in[0,T],\ 1\le i\le n,$$

where $T>0$ is fixed and the nonlinearities $a_i(t,u_1,u_2,\cdots,u_n)$ can be *singular* at $t=0$ and $u_j=0$ where $j\in\{1,2,\cdots,n\}$. Criteria are established for the existence of *fixed-sign solutions* $(u_1^*,u_2^*,\cdots,u_n^*)$ to the above system, i.e., $\theta_i u_i^*(t)\ge 0$ for $t\in[0,T]$ and $1\le i\le n$, where $\theta_i\in\{1,-1\}$ is fixed. We also include an example to illustrate the usefulness of the results obtained.

1. **Introduction.**

In this paper we shall consider the system of integral equations

$$u_i(t) = \int_0^T g_i(t,s)[a_i(s,u_1(s),u_2(s),\cdots,u_n(s)) + b_i(s,u_1(s),u_2(s),\cdots,u_n(s))]ds, \\ t\in[0,T],\ 1\le i\le n \tag{1.1}$$

where $T>0$ is fixed. The nonlinearities $a_i(t,u_1,u_2,\cdots,u_n)$ can be *singular* at $t=0$ and $u_j=0$ where $j\in\{1,2,\cdots,n\}$.

Throughout, let $u=(u_1,u_2,\cdots,u_n)$. We are interested in establishing the existence of solutions u of the system (1.1) in $(C[0,T])^n$. Moreover, we are concerned with *fixed-sign* solutions u, by which we mean $\theta_i u_i(t)\ge 0$ for all $t\in[0,T]$ and $1\le i\le n$, where $\theta_i\in\{1,-1\}$ is fixed. Note that *positive* solution is a special case of fixed-sign solution when $\theta_i=1$ for $1\le i\le n$.

There are only a handful of papers in the literature (see [1, 8, 11, 12, 14, 16, 17, 18, 19] and the references therein) that tackle particular cases of (1.1) when $n=1$, $\theta_1=1$, and the nonlinearity has the special form $a(t,y)=y^{-r}$, $r>0$. As an example, in the axisymmetric stagnation flow (i.e. Homann flow [1]), the Navier-Stokes equation can be reduced to

$$y(t) = \int_t^1 \frac{(1-s)\left(\frac{1}{2}+\frac{3}{2}s\right)}{y(s)}\,ds + (1-t)\int_0^t \frac{s}{y(s)}\,ds,\ 0<t<1, \tag{1.2}$$

with $y(t)>0$, $t\in[0,1)$ and $y(1)=0$. Other examples can be found in problems arising from communications [17], as well as in boundary layer theory [19].

In the literature, the conditions placed on the kernel g are not natural. A new approach is thus employed in this paper to present new results for (1.1). In particular, new "lower type inequalities" on the solutions are presented. Our results extend, improve and complement the existing theory in the literature [1, 2, 6, 7, 9, 12, 18]. We have generalized the problems to (i) *systems*, (ii) *general* form of nonlinearities

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$a_i,\ 1 \le i \le n$ that can be singular in *both* independent and dependent variables, (iii) existence of *fixed-sign* solutions, which include *positive* solutions as special case. Other related work on *systems* of integral equations and singular integral equations/boundary value problems can be found in [3, 4, 5, 6, 7, 10, 13, 15, 20, 21, 22]. Note that the technique employed in singular integral equations [6, 7] is entirely different from the present work.

2. **Existence Results.** Let the Banach space $B = (C[0,T])^n$ be equipped with the norm

$$\|u\| = \max_{1\le i\le n} \sup_{t\in[0,T]} |u_i(t)| = \max_{1\le i\le n} |u_i|_0$$

where we let $|u_i|_0 = \sup_{t\in[0,T]} |u_i(t)|,\ 1 \le i \le n$. Our main tool is the following theorem.

Theorem 2.1. *Consider the system*

$$u_i(t) = c_i(t) + \int_0^T g_i(t,s)f_i(s,u(s))ds,\ t \in [0,T],\ 1 \le i \le n. \tag{2.1}$$

Let $1 \le p \le \infty$ be an integer and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume the following conditions hold for each $1 \le i \le n$:

$$c_i \in C[0,T]; \tag{2.2}$$

$$\begin{cases} f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R} \text{ is a } L^q\text{-Carathéodory function, i.e.,} \\ \text{(i) the map } u \mapsto f_i(t,u) \text{ is continuous for almost all } t \in [0,T], \\ \text{(ii) the map } t \mapsto f_i(t,u) \text{ is measurable for all } u \in \mathbb{R}^n, \\ \text{(iii) for any } r > 0, \text{ there exists } \mu_{r,i} \in L^q[0,T] \text{ such that} \\ \|u\| \le r \text{ implies } |f_i(t,u)| \le \mu_{r,i}(t) \text{ for almost all } t \in [0,T]; \end{cases} \tag{2.3}$$

$$g_i^t(s) = g_i(t,s) \in L^p[0,T] \text{ for each } t \in [0,T] \tag{2.4}$$

and

$$\text{the map } t \mapsto g_i^t \text{ is continuous from } [0,T] \text{ to } L^p[0,T]. \tag{2.5}$$

In addition, suppose there is a constant $M > 0$, independent of λ, with $\|u\| \ne M$ for any solution $u \in (C[0,T])^n$ to

$$u_i(t) = c_i(t) + \lambda \int_0^T g_i(t,s)f_i(s,u(s))ds,\ t \in [0,T],\ 1 \le i \le n \tag{2.6$_\lambda$}$$

for each $\lambda \in (0,1)$. Then, (2.1) has at least one solution in $(C[0,T])^n$.

Proof. The proof employs a similar technique as the proof of Theorem 4.2.2 in [18]. To outline briefly, we define the operator A by

$$Au(t) = (A_1u(t), A_2u(t), \cdots, A_nu(t)),\ t \in [0,T]$$

where

$$A_iu(t) = c_i(t) + \int_0^T g_i(t,s)f_i(s,u(s))ds,\ t \in [0,T],\ 1 \le i \le n.$$

Clearly, the system (2.1) is equivalent to $u = Au$, and $(2.6)_\lambda$ is the same as $u = \lambda Au + (1-\lambda)c$ where $c = (c_1, c_2, \cdots, c_n)$. We can show that $A : (C[0,T])^n \to (C[0,T])^n$ is continuous and also completely continuous. Then, the Nonlinear Alternative (Theorem 1.2.1, [18]) is applied with $\tilde{N} = A$, $U = \{u \in (C[0,T])^n \mid \|u\| < M\}$, $C = E = (C[0,T])^n$ and $p^* = c$ to obtain the conclusion of the theorem. □

Let $\theta_i \in \{-1, 1\},\ 1 \le i \le n$ be fixed. For each $1 \le j \le n$, define

$$[0,\infty)_j = \begin{cases} [0,\infty), & \theta_j = 1 \\ (-\infty, 0], & \theta_j = -1 \end{cases}$$

($(0,\infty)_j$ is similarly defined.) We shall apply Theorem 2.1 to obtain an existence result for (1.1).

Theorem 2.2. *Let $\theta_i \in \{-1, 1\},\ 1 \le i \le n$ be fixed and let the following conditions be satisfied for each $1 \le i \le n$:*

$$\begin{cases} \theta_i a_i(t,u) > 0 \text{ and is continuous for } (t,u) \in (0,T] \times \prod_{j=1}^{n} (0,\infty)_j, \\ \theta_i b_i(t,u) \ge 0 \text{ and is continuous for } (t,u) \in [0,T] \times \prod_{j=1}^{n} [0,\infty)_j; \end{cases} \tag{2.7}$$

$$\begin{cases} \theta_i a_i \text{ is 'nonincreasing' in } u, \text{ i.e., if } \theta_j u_j \ge \theta_j v_j \text{ for some } j \in \{1,2,\cdots,n\}, \text{ then} \\ \theta_i a_i(t,u_1,\cdots,u_j,\cdots,u_n) \le \theta_i a_i(t,u_1,\cdots,v_j,\cdots,u_n),\ t \in (0,T]; \end{cases} \tag{2.8}$$

$$\begin{cases} \text{there exist nonnegative } r_i \text{ and } \gamma_i \text{ such that} \\ r_i \in C(0,T],\ \gamma_i \in C(0,\infty),\ \gamma_i > 0 \text{ is nonincreasing, and} \\ \theta_i a_i(t,u) \ge r_i(t)\gamma_i(|u_i|),\ (t,u) \in (0,T] \times \prod_{j=1}^{n} (0,\infty)_j; \end{cases} \tag{2.9}$$

$$\begin{cases} \text{there exist nonnegative } d_i \text{ and } h_{ij},\ 1 \le j \le n \text{ such that} \\ d_i \in C(0,T],\ h_{ij} \in C(0,\infty),\ h_{ij} \text{ is nondecreasing, and} \\ \dfrac{b_i(t,u)}{a_i(t,u)} \le d_i(t)h_{i1}(|u_1|)h_{i2}(|u_2|)\cdots h_{in}(|u_n|),\ (t,u) \in (0,T] \times \prod_{j=1}^{n} (0,\infty)_j; \end{cases} \tag{2.10}$$

$$g_i^t(s) = g_i(t,s) \in L^1[0,T] \text{ for each } t \in [0,T]; \tag{2.11}$$

$$\text{the map } t \mapsto g_i^t \text{ is continuous from } [0,T] \text{ to } L^1[0,T]; \tag{2.12}$$

$$g_i(t,s) \ge 0 \text{ for each } t \in [0,T],\ \text{a.e. } s \in [0,T]; \tag{2.13}$$

$$\text{for } t_1, t_2 \in (0,T) \text{ with } t_1 < t_2, \text{ we have } g_i(t_1,s) \ge g_i(t_2,s),\ \text{a.e. } s \in [0,T]; \tag{2.14}$$

$$\begin{cases} \int_0^T g_i(0,s)\theta_i a_i(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s))ds < \infty \text{ where} \\ \beta_i(s) = G_i^{-1}\left(\int_s^T g_i(s,x)r_i(x)dx\right) \text{ for } s \in [0,T] \text{ and } G_i(z) = \dfrac{z}{\gamma_i(z)} \text{ for } z > 0; \end{cases} \tag{2.15}$$

$$\begin{cases} \text{there exists } \rho_i \in C[0,T] \text{ such that for } t,x \in [0,T] \text{ with } t < x, \text{ we have} \\ \int_0^T [g_i(t,s) - g_i(x,s)]\theta_i a_i(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s))ds \le |\rho_i(x) - \rho_i(t)| \end{cases} \tag{2.16}$$

and

$$\begin{cases} \text{if } z > 0 \text{ satisfies } z \le K + L\left\{1 + \max_{1\le i\le n}\left[\sup_{t\in[0,T]} d_i(t)\right]\left[\prod_{j=1}^{n} h_{ij}(z)\right]\right\} \\ \text{for some constants } K, L \ge 0, \text{ then there exists a constant } M \\ \text{(which may depend on } K \text{ and } L) \text{ such that } z \le M. \end{cases} \tag{2.17}$$

Then, (1.1) has a fixed-sign solution $u \in (C[0,T])^n$ with $\theta_i u_i(t) \ge \beta_i(t)$ for $t \in [0,T]$ and $1 \le i \le n$ (β_i is defined in (2.15)).

Proof. Let $N = \{1, 2, \cdots\}$ and $k = (k_1, k_2, \cdots, k_n) \in N^n$. We shall first show that the nonsingular system

$$u_i(t) = \frac{\theta_i}{k_i} + \int_0^T g_i(t,s)\,[a_i^*(s,u(s)) + b_i(s,u(s))]\,ds,\ t \in [0,T],\ 1 \le i \le n \quad (2.18)^k$$

has a solution for each $k \in N^n$, where

$$a_i^*(t, u_1, \cdots, u_n) = a_i(t, v_1, \cdots, v_n),\ t \in (0,T]$$

with

$$v_j = \begin{cases} \dfrac{\theta_j}{k_j}, & \theta_j u_j \le \dfrac{1}{k_j} \\ u_j, & \theta_j u_j \ge \dfrac{1}{k_j}. \end{cases}$$

Let $k \in N^n$ be fixed. In order to apply Theorem 2.1, we need to consider the family of problems

$$u_i(t) = \frac{\theta_i}{k_i} + \lambda \int_0^T g_i(t,s)\,[a_i^*(s,u(s)) + b_i(s,u(s))]\,ds,\ t \in [0,T],\ 1 \le i \le n \quad (2.19)_\lambda^k$$

where $\lambda \in (0,1)$. Let $u \in (C[0,T])^n$ be any solution of $(2.19)_\lambda^k$. Then, for each $1 \le i \le n$, $\theta_i u_i(t) \ge \frac{1}{k_i} > 0$ for $t \in [0,T]$, and so $a_i^*(t,u) = a_i(t,u)$. Using (2.8) and (2.10), we get for $t \in [0,T]$ and $1 \le i \le n$,

$$\begin{aligned}
|u_i(t)| \ &= \ \theta_i u_i(t) \\
&= \frac{1}{k_i} + \lambda \int_0^T g_i(t,s)\,[\theta_i a_i^*(s,u(s)) + \theta_i b_i(s,u(s))]\,ds \\
&= \frac{1}{k_i} + \lambda \int_0^T g_i(t,s)\theta_i a_i(s,u(s)) \left[1 + \frac{b_i(s,u(s))}{a_i(s,u(s))}\right] ds \\
&\le 1 + \int_0^T g_i(t,s)\theta_i a_i\left(s, \frac{\theta_1}{k_1}, \cdots, \frac{\theta_n}{k_n}\right) [1 + d_i(s) h_{i1}(|u_1(s)|) \cdots h_{in}(|u_n(s)|)]\,ds \\
&\le 1 + \max_{1 \le i \le n} \left[\sup_{t \in [0,T]} \int_0^T g_i(t,s)\theta_i a_i\left(s, \frac{\theta_1}{k_1}, \cdots, \frac{\theta_n}{k_n}\right) ds\right] \\
&\qquad \times \left\{1 + \max_{1 \le i \le n} \left[\sup_{s \in [0,T]} d_i(s)\right] \prod_{j=1}^n h_{ij}(\|u\|)\right\} \ \equiv \ A.
\end{aligned}$$

Thus, we have $\|u\| \le A$ and so by (2.17) there exists a constant M_k with $\|u\| \le M_k$. Hence, Theorem 2.1 guarantees that $(2.18)^k$ has a solution $u^k \in (C[0,T])^n$ with $\theta_i u_i^k(t) \ge \frac{1}{k_i}$ for $t \in [0,T]$ and $1 \le i \le n$. Consequently, $a_i^*(t,u^k) = a_i(t,u^k)$ and u^k is a solution of the system

$$u_i(t) = \frac{\theta_i}{k_i} + \int_0^T g_i(t,s)[a_i(s,u(s)) + b_i(s,u(s))]ds,\ t \in [0,T],\ 1 \le i \le n. \quad (2.20)$$

Moreover, noting (2.14) we see that $\theta_i u_i^k$ is nonincreasing on $(0,T)$.

Next, we shall obtain a solution to (1.1) as a limit of solutions of $(2.18)^k$ as $k_i \to \infty$, $1 \le i \le n$. The Arzéla-Ascoli theorem will be used. To begin, we shall show that

$$\{u^k\}_{k \in N^n} \text{ is a bounded, equicontinuous family on } [0,T]. \quad (2.21)$$

We need a lower bound for $\theta_i u_i^k(t),\ t \in [0,T],\ 1 \le i \le n$. Using (2.9) and the fact that $\theta_i u_i^k = |u_i^k|$ is nonincreasing on $(0,T)$, we find

$$\begin{aligned} |u_i^k(t)| &= \theta_i u_i^k(t) \\ &= \frac{1}{k_i} + \int_0^T g_i(t,s)[\theta_i a_i(s,u^k(s)) + \theta_i b_i(s,u^k(s))]ds \\ &\ge \int_t^T g_i(t,s)\theta_i a_i(s,u^k(s))ds \\ &\ge \int_t^T g_i(t,s) r_i(s)\gamma_i(|u_i^k(s)|)ds \\ &\ge \gamma_i(|u_i^k(t)|)\int_t^T g_i(t,s)r_i(s)ds \end{aligned}$$

or equivalently

$$G_i(|u_i^k(t)|) = \frac{|u_i^k(t)|}{\gamma_i(|u_i^k(t)|)} \ge \int_t^T g_i(t,s)r_i(s)ds.$$

Since G_i is an increasing function (γ_i is nonincreasing), the above yields

$$\theta_i u_i^k(t) = |u_i^k(t)| \ge G_i^{-1}\left(\int_t^T g_i(t,s)r_i(s)ds\right) = \beta_i(t), \qquad t \in [0,T],\ 1 \le i \le n \tag{2.22}$$

for each $k \in N^n$.

We shall now show that $\{u^k\}_{k\in N^n}$ is a bounded family on $[0,T]$. Fix $k \in N^n$. Note that $|u_i^k|_0 = \sup_{t\in[0,T]}|u_i^k(t)| = \theta_i u_i^k(0),\ 1 \le i \le n$. Applying (2.22), (2.8) and (2.10), we get for $1 \le i \le n$,

$$\begin{aligned} &|u_i^k|_0 = \theta_i u_i^k(0) \\ &= \frac{1}{k_i} + \int_0^T g_i(0,s)\theta_i a_i(s,u^k(s))\left[1 + \frac{b_i(s,u^k(s))}{a_i(s,u^k(s))}\right]ds \\ &\le 1 + \int_0^T g_i(0,s)\theta_i a_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right) \\ &\qquad \times \left[1 + d_i(s)h_{i1}(|u_1^k(s)|)\cdots h_{in}(|u_n^k(s)|)\right]ds \\ &\le 1 + \max_{1\le i\le n}\left[\int_0^T g_i(0,s)\theta_i a_i\left(s,\theta_1\beta_1(s),\cdots,\theta_n\beta_n(s)\right)ds\right] \\ &\qquad \times \left\{1 + \max_{1\le i\le n}\left[\sup_{s\in[0,T]} d_i(s)\right]\prod_{j=1}^n h_{ij}(\|u^k\|)\right\} \ \equiv\ A_0. \end{aligned}$$

It follows that $\|u^k\| \le A_0$ and so noting (2.17) there exists a constant M (independent of k) with $\|u^k\| \le M$. Thus, $\{u^k\}_{k\in N^n}$ is bounded.

Next, we shall show that $\{u^k\}_{k\in N^n}$ is equicontinuous. Fixed $k \in N^n$. For $t, x \in [0,T]$ with $t < x$, using an earlier technique we obtain for each $1 \le i \le n$,

$$\begin{aligned}|u_i^k(t) - u_i^k(x)| &= \theta_i u_i^k(t) - \theta_i u_i^k(x)\\ &\le \left\{1 + \left[\sup_{s\in[0,T]} d_i(s)\right] \prod_{j=1}^n h_{ij}(M)\right\}\\ &\quad \times \int_0^T [g_i(t,s) - g_i(x,s]\theta_i a_i\left(s, \theta_1\beta_1(s), \cdots, \theta_n\beta_n(s)\right) ds\\ &\le \left\{1 + \left[\sup_{s\in[0,T]} d_i(s)\right] \prod_{j=1}^n h_{ij}(M)\right\} |\rho_i(x) - \rho_i(t)|\end{aligned}$$

where we have also used (2.16) in the last inequality. This shows that $\{u^k\}_{k\in N^n}$ is an equicontinuous family on $[0,T]$.

Now, the Arzéla-Ascoli theorem guarantees the existence of a subsequence N^* of N, and a function $u^* \in (C[0,T])^n$ with u^k converging uniformly on $[0,T]$ to u^* as $k_i \to \infty,\ 1 \le i \le n$ through N^*. Further,

$$\beta_i(t) \le \theta_i u_i^*(t) \le M,\ t \in [0,T],\ 1 \le i \le n. \tag{2.23}$$

Finally, to see that u^* is indeed a solution of (1.1), fix $t \in [0,T]$. Then, from (2.20) we have for each $1 \le i \le n$,

$$u_i^k(t) = \frac{\theta_i}{k_i} + \int_0^T g_i(t,s)[a_i(s,u^k(s)) + b_i(s,u^k(s))]ds.$$

Let $k_i \to \infty$ through N^*, and use the Lebesgue dominated convergence theorem with (2.15), to obtain for each $1 \le i \le n$,

$$u_i^*(t) = \int_0^T g_i(t,s)[a_i(s,u^*(s)) + b_i(s,u^*(s))]ds.$$

This argument holds for each $t \in [0,T]$, therefore u^* is indeed a solution of (1.1). □

Remark 2.1. If $b_i \equiv 0$, then we can pick $d_i = 0$ in (2.10), and trivially (2.17) is satisfied with $M = K + L$.

There is also an analogue of Theorem 2.2 if (2.14) is changed to

$$\text{for } t_1, t_2 \in (0,T) \text{ with } t_1 < t_2, \text{ we have } g_i(t_1,s) \le g_i(t_2,s),\ a.e.\ s \in [0,T]. \tag{2.14$'$}$$

We state the result as follows.

Theorem 2.3. *Let $\theta_i \in \{-1,1\}$, $1 \le i \le n$ be fixed and let the following conditions be satisfied for each $1 \le i \le n$: (2.7)–(2.13),* (2.14)′, *(2.17),*

$$\left\{\begin{aligned}&\int_0^T g_i(T,s)\theta_i a_i(s,\theta_1\beta_1^*(s),\cdots,\theta_n\beta_n^*(s))ds < \infty \text{ where}\\ &\beta_i^*(s) = G_i^{-1}\left(\int_0^s g_i(s,x)r_i(x)dx\right) \text{ for } s \in [0,T] \text{ and } G_i(z) = \frac{z}{\gamma_i(z)} \text{ for } z > 0\end{aligned}\right. \tag{2.15$'$}$$

and

$$\left\{\begin{aligned}&\text{there exists } \rho_i \in C[0,T] \text{ such that for } t, x \in [0,T] \text{ with } t < x, \text{ we have}\\ &\int_0^T [g_i(x,s) - g_i(t,s)]\theta_i a_i(s,\theta_1\beta_1^*(s),\cdots,\theta_n\beta_n^*(s))ds \le |\rho_i(x) - \rho_i(t)|.\end{aligned}\right. \tag{2.16$'$}$$

Then, (1.1) has a fixed-sign solution $u \in (C[0,T])^n$ *with* $\theta_i u_i(t) \ge \beta_i^*(t)$ *for* $t \in [0,T]$ *and* $1 \le i \le n$ *(*β_i^* *is defined in* (2.15)$'$*).*

Proof. The proof is similar to that of Theorem 2.2, except that here for each $1 \le i \le n$, $\theta_i u_i^k$ is *nondecreasing* on $(0,T)$ and so $|u_i^k|_0 = \theta_i u_i^k(T)$.

To obtain a lower bound for $\theta_i u_i^k(t),\ t \in [0,T],\ 1 \le i \le n$, we have

$$\begin{aligned} |u_i^k(t)| \ = \ \theta_i u_i^k(t) \ &= \ \frac{1}{k_i} + \int_0^T g_i(t,s)[\theta_i a_i(s,u^k(s)) + \theta_i b_i(s,u^k(s))]ds \\ &\ge \ \int_0^t g_i(t,s)\theta_i a_i(s,u^k(s))ds \\ &\ge \ \int_0^t g_i(t,s) r_i(s)\gamma_i(|u_i^k(s)|)ds \\ &\ge \ \gamma_i(|u_i^k(t)|)\int_0^t g_i(t,s)r_i(s)ds \end{aligned}$$

which gives

$$\theta_i u_i^k(t) = |u_i^k(t)| \ge G_i^{-1}\left(\int_0^t g_i(t,s)r_i(s)ds\right) = \beta_i^*(t),\ t \in [0,T],\ 1 \le i \le n \quad (2.22)'$$

for each $k \in N^n$.

Moreover, to see that $\{u^k\}_{k\in N^n}$ is a bounded family on $[0,T]$, we find for $1 \le i \le n$,

$$\begin{aligned} &|u_i^k|_0 = \theta_i u_i^k(T) \\ &= \frac{1}{k_i} + \int_0^T g_i(T,s)\theta_i a_i(s,u^k(s))\left[1 + \frac{b_i(s,u^k(s))}{a_i(s,u^k(s))}\right]ds \\ &\le 1 + \int_0^T g_i(T,s)\theta_i a_i\left(s,\theta_1\beta_1^*(s),\cdots,\theta_n\beta_n^*(s)\right) \\ &\qquad \times \left[1 + d_i(s)h_{i1}(|u_1^k(s)|)\cdots h_{in}(|u_n^k(s)|)\right]ds \\ &\le 1 + \max_{1\le i\le n}\left[\int_0^T g_i(T,s)\theta_i a_i\left(s,\theta_1\beta_1^*(s),\cdots,\theta_n\beta_n^*(s)\right)ds\right] \\ &\qquad \times \left\{1 + \max_{1\le i\le n}\left[\sup_{s\in[0,T]} d_i(s)\right]\prod_{j=1}^n h_{ij}(\|u^k\|)\right\} \ \equiv \ A_1. \end{aligned}$$

Hence, $\|u^k\| \le A_1$ and so (2.17) guarantees a constant M (independent of k) such that $\|u^k\| \le M$. The rest of the proof follows that of Theorem 2.2. □

3. **Example.** We shall now illustrate our results by an example which is related to one-dimensional Homann flow represented by the singular integral equation (1.2). Consider the system of singular integral equations

$$\left\{ \begin{aligned} u_1(t) &= \int_t^1 (1-s)\left(\frac{1}{2}+\frac{3}{2}\ s\right)\left[(u_1(s))^{-1} + (u_2(s))^{-\frac{1}{3}}\right]ds \\ &\qquad + (1-t)\int_0^t s\left[(u_1(s))^{-1} + (u_2(s))^{-\frac{1}{3}}\right]ds, \quad t \in [0,1] \\ u_2(t) &= \int_t^1 (1-s)\left(\frac{1}{2}+\frac{3}{2}\ s\right)\left[(u_1(s))^{-\frac{1}{4}} + (u_2(s))^{-1}\right]ds \\ &\qquad + (1-t)\int_0^t s\left[(u_1(s))^{-\frac{1}{4}} + (u_2(s))^{-1}\right]ds, \quad t \in [0,1]. \end{aligned} \right. \quad (3.1)$$

Here, (3.1) is of the form (1.1) with

$$g_1(t,s)=g_2(t,s)\equiv g(t,s)=\begin{cases}(1-s)\left(\frac{1}{2}+\frac{3}{2}\,s\right), & s>t\\ (1-t)s, & s<t\end{cases}$$

$$a_1(t,u_1,u_2)=u_1^{-1}+u_2^{-\frac{1}{3}},\quad a_2(t,u_1,u_2)=u_1^{-\frac{1}{4}}+u_2^{-1}\quad \text{and}\quad b_1=b_2=0.$$

Suppose we are interested in *positive solutions* of (3.1), i.e., when $\theta_1=\theta_2=1$. From Remark 2.1, conditions (2.10) and (2.17) are trivial. Moreover, conditions (2.7), (2.8) and (2.11)–(2.13) are clearly met. To satisfy condition (2.9), we shall pick

$$r_1=r_2=1 \qquad \text{and} \qquad \gamma_1(z)=\gamma_2(z)=\frac{1}{z}.$$

It is also easy to see that condition (2.14) holds, since for $t_1,t_2\in(0,1)$ with $t_1<t_2$, we have

$$\begin{aligned}&g(t_1,s)-g(t_2,s)\\ &=\begin{cases}s(t_2-t_1)\ge 0, & 0<s<t_1\\ (1-s)\left(\frac{1}{2}+\frac{3}{2}\,s\right)-(1-t_2)s & \\ \qquad \ge (1-s)\left(\frac{1}{2}+\frac{3}{2}\,s\right)-(1-s)s=\frac{1}{2}(1-s^2)\ge 0, & t_1<s<t_2\\ 0, & t_2<s<1.\end{cases}\end{aligned}$$

To check condition (2.15), we note that $G_1(z)=G_2(z)=z^2$ and so $G_1^{-1}(z)=G_2^{-1}(z)=\sqrt{z}$. Hence, we have

$$\begin{aligned}\beta_1(t)=\beta_2(t)=\sqrt{\int_t^1 g(t,x)dx}&=\sqrt{\int_t^1(1-x)\left(\frac{1}{2}+\frac{3}{2}\,x\right)dx}\\ &=(1-t)\sqrt{\frac{1+t}{2}},\ t\in[0,1].\end{aligned} \tag{3.2}$$

Clearly,

$$\beta_1(t),\beta_2(t)\ge\frac{1-t}{\sqrt{2}},\ t\in[0,1]. \tag{3.3}$$

Using (3.3), we get

$$\begin{aligned}\int_0^1 g(0,s)a_1(s,\beta_1(s),\beta_2(s))ds&\le\int_0^1(1-s)\left(\frac{1}{2}+\frac{3}{2}\,s\right)\left[\frac{\sqrt{2}}{1-s}+2^{\frac{1}{6}}(1-s)^{-\frac{1}{3}}\right]ds\\ &=\int_0^1\left[\frac{\sqrt{2}}{2}(1+3s)+2^{-\frac{5}{6}}(1-s)^{\frac{2}{3}}(1+3s)\right]ds\ <\ \infty\end{aligned}$$

and

$$\begin{aligned}\int_0^1 g(0,s)a_2(s,\beta_1(s),\beta_2(s))ds&\le\int_0^1(1-s)\left(\frac{1}{2}+\frac{3}{2}\,s\right)\left[2^{\frac{1}{8}}(1-s)^{-\frac{1}{4}}+\frac{\sqrt{2}}{1-s}\right]ds\\ &=\int_0^1\left[\frac{\sqrt{2}}{2}(1+3s)+2^{-\frac{7}{8}}(1-s)^{\frac{3}{4}}(1+3s)\right]ds\ <\ \infty.\end{aligned}$$

Thus, (2.15) holds.

Finally, we shall check condition (2.16). Let $t, x \in [0,1]$ with $t < x$. Using (3.3) again, we find

$$\begin{aligned}
&\int_0^1 [g(t,s) - g(x,s)] a_1(s, \beta_1(s), \beta_2(s)) ds \\
&= (x-t) \int_0^t s a_1(s, \beta_1(s), \beta_2(s)) ds \\
&\qquad + \int_t^x \left[(1-s) \left(\frac{1}{2} + \frac{3}{2}\, s \right) - (1-x)s \right] a_1(s, \beta_1(s), \beta_2(s)) ds \\
&\le \int_t^x \int_0^z s a_1(s, \beta_1(s), \beta_2(s)) ds dz - \int_t^x s(x-s) a_1(s, \beta_1(s), \beta_2(s)) ds \\
&\qquad + \int_t^x \left[(1-s) \left(\frac{1}{2} + \frac{3}{2}\, s \right) - (1-x)s \right] a_1(s, \beta_1(s), \beta_2(s)) ds \\
&= \int_t^x \int_0^z s a_1(s, \beta_1(s), \beta_2(s)) ds dz - \frac{1}{2} \int_t^x (1-s)^2 a_1(s, \beta_1(s), \beta_2(s)) ds \\
&\le \int_t^x \int_0^z s a_1(s, \beta_1(s), \beta_2(s)) ds dz \\
&\le \int_t^x \int_0^z s \left[\frac{\sqrt{2}}{1-s} + 2^{\frac{1}{6}} (1-s)^{-\frac{1}{3}} \right] ds dz \\
&= \int_t^x \left\{ \sqrt{2}[-z - \ln(1-z)] + 2^{\frac{1}{6}} \left[\frac{9}{10} - \frac{3}{10}(1-z)^{\frac{2}{3}}(2z+3) \right] \right\} dz \\
&= \rho_1(x) - \rho_1(t)
\end{aligned}$$

where

$$\rho_1(z) = \sqrt{2}(1-z)\ln(1-z) + \frac{9}{40}(2)^{\frac{1}{6}}(1-z)^{\frac{5}{3}}(z+3) - \frac{\sqrt{2}}{2}\, z^2 + z \left[\frac{9}{10}(2)^{\frac{1}{6}} + \sqrt{2} \right].$$

Note that $\rho_1 \in C[0,1]$ since

$$\lim_{z \to 1} (1-z)\ln(1-z) = \lim_{z \to 1} \frac{\ln(1-z)}{(1-z)^{-1}} = \lim_{z \to 1} -(1-z) = 0.$$

Using a similar technique, we get

$$\begin{aligned}
&\int_0^1 [g(t,s) - g(x,s)] a_2(s, \beta_1(s), \beta_2(s)) ds \\
&\le \int_t^x \int_0^z s a_2(s, \beta_1(s), \beta_2(s)) ds dz \\
&\le \int_t^x \int_0^z s \left[2^{\frac{1}{8}} (1-s)^{-\frac{1}{4}} + \frac{\sqrt{2}}{1-s} \right] ds dz \\
&= \int_t^x \left\{ \sqrt{2}[-z - \ln(1-z)] + 2^{\frac{1}{8}} \left[\frac{16}{21} - \frac{4}{21}(1-z)^{\frac{3}{4}}(3z+4) \right] \right\} dz \\
&= \rho_2(x) - \rho_2(t)
\end{aligned}$$

where

$$\rho_2(z) = \sqrt{2}(1-z)\ln(1-z) + \frac{16}{231}(2)^{\frac{1}{8}}(1-z)^{\frac{7}{4}}(3z+8) - \frac{\sqrt{2}}{2}\, z^2 + z \left[\frac{16}{21}(2)^{\frac{1}{8}} + \sqrt{2} \right].$$

Note also that $\rho_2 \in C[0,1]$. Thus, (2.16) holds.

All the conditions of Theorem 2.2 are satisfied. Hence, the system (3.1) has a *positive solution* $u \in (C[0,1])^2$ with

$$u_i(t) \geq \beta_i(t) = (1-t)\sqrt{\frac{1+t}{2}},\ t \in [0,1],\ i = 1,2.$$

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