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RANDOM ATTRACTORS FOR WAVE EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. The asymptotic behavior of stochastic wave equations on $\mathbb{R}^n$ is studied. The existence of a random attractor for the corresponding random dynamical system in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is established, where the nonlinearity has an arbitrary growth order for $n \le 2$ and is subcritical for $n = 3$.

1. **Introduction.** In this paper, we investigate the asymptotic behavior of solutions of the stochastic wave equation defined on $\mathbb{R}^n$ with $n \le 3$:

$$u_{tt} + \alpha u_t - \Delta u + \lambda u + f(x, u) = g(x) + h(x)\frac{dw}{dt}, \tag{1}$$

with the initial conditions

$$u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x), \tag{2}$$

where $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, α and λ are positive numbers, $g \in L^2(\mathbb{R}^n)$ and $h \in H^1(\mathbb{R}^n)$ are given, f is a nonlinear function satisfying certain growth and dissipative conditions, and w is an independent two-sided real-valued Wiener process on a probability space which will be specified later.

The concept of random attractor for random dynamical systems was introduced in [5, 7]. The existence of such attractors was studied in [4, 5, 7] for PDEs defined on bounded domains, and in [3, 9] for PDEs defined on unbounded domains. Notice that Sobolev embeddings are not compact for unbounded domains. This introduces a major obstacle for proving the asymptotic compactness of solutions. In this paper, we will use a splitting technique to solve the problem. We first show that the solutions are uniformly small outside a bounded domain as in [8]. Then we decompose the solutions in a bounded domain in terms of eigenfunctions of negative Laplacian. To ensure the desired estimates on solutions, we require the nonlinearity f be subcritical when $n = 3$. More precisely, the estimates in Lemma 3.3 are valid only for subcritical nonlinearity, and hence the approach of this paper does not apply to the critical or supercritical case. For critical wave equations on $\mathbb{R}^3$, the existence of random attractors can be proved by combining the energy equation method of [1] and the uniform estimates on the tails of solutions. The reader is referred to [10] for more details in this case.

In this paper, we denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norm of $L^p(\mathbb{R}^n)$ is written as $\| \cdot \|_p$ for $1 \le p \le \infty$.

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2. **Random Dynamical Systems.** In this section, we define a continuous random dynamical system for the stochastic wave equation (1). To this end, we introduce a transformation $z = u_t + \delta u$ where δ is a small positive number to be determined later. Then problem (1)-(2) is equivalent to the system

$$\frac{du}{dt} + \delta u = z, \tag{3}$$

$$\frac{dz}{dt} + (\alpha - \delta)z + (\lambda + \delta^2 - \alpha\delta)u - \Delta u + f(x,u) = g(x) + h(x)\frac{dw}{dt}, \tag{4}$$

with the initial conditions

$$u(x,\tau) = u_0(x), \quad z(x,\tau) = z_0(x), \tag{5}$$

where $z_0(x) = u_1(x) + \delta u_0(x)$, $x \in \mathbb{R}^n$, $t > \tau$ with $\tau \in \mathbb{R}$, α and λ are positive numbers, $g \in L^2(\mathbb{R}^n)$ and $h \in H^1(\mathbb{R}^n)$ are given, and w is an independent two-sided real-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, P)$ with path $\omega(\cdot)$ in $C(\mathbb{R},\mathbb{R})$ satisfying $\omega(0) = 0$. As usual, a family of measure preserving shift operators on $(\Omega, \mathcal{F}, P)$ is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

$(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system. Denote by $F(x,u) = \int_0^u f(x,s)ds$ for $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. Throughout this paper, we assume that the nonlinear function f satisfies the following conditions, for every $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$,

$$|f(x,u)| \le c_1|u|^r + \phi_1(x), \quad \phi_1 \in L^2(\mathbb{R}^n), \tag{6}$$

$$f(x,u)u - c_2F(x,u) \ge \phi_2(x), \quad \phi_2 \in L^1(\mathbb{R}^n), \tag{7}$$

$$F(x,u) \ge c_3|u|^{r+1} - \phi_3, \quad \phi_3 \in L^1(\mathbb{R}^n), \tag{8}$$

$$|f_u(x,u)| \le c_4|u|^{r-1} + \phi_4, \quad \phi_4 \in H^1(\mathbb{R}^n), \tag{9}$$

where $1 \le r < \infty$ for $n = 1, 2$ and $1 \le r < 3$ for $n = 3$. Notice that (6) and (7) imply

$$F(x,u) \le c(|u|^2 + |u|^{r+1} + \phi_1^2 + \phi_2). \tag{10}$$

As a typical model, the nonlinearity $f(x,u) = |u|^{r-1}u$ arising from relativistic quantum mechanics satisfies all conditions (6)-(9).

For our purpose, we need to convert the stochastic system (3)-(4) with a random term into a deterministic one with a random parameter. Set $v(t,\tau,\omega) = z(t,\tau,\omega) - h\omega(t)$. Then it follows from (3)-(5) that

$$\frac{du}{dt} + \delta u - v = h\omega(t), \tag{11}$$

$$\frac{dv}{dt} + (\alpha - \delta)v + (\lambda + \delta^2 - \alpha\delta)u - \Delta u + f(x,u) = g + (\delta - \alpha)h\omega(t), \tag{12}$$

with the initial conditions

$$u(x,\tau) = u_0(x), \quad v(x,\tau) = v_0(x), \tag{13}$$

where $v_0(x) = z_0(x) - h\omega(\tau)$.

By a standard method as in [6], it can be proved that problem (11)-(13) under assumptions (6)-(9) is well-posed in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, that is, for P-a.e. $\omega \in \Omega$, for every $\tau \in \mathbb{R}$ and $(u_0, v_0) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, problem (11)-(13) has a unique solution $(u(t,\tau,\omega), v(t,\tau,\omega)) \in C([\tau,\infty), H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ with $(u(\tau,\tau,\omega), v(\tau,\tau,\omega)) = (u_0, v_0)$. Further, the solution is continuous with respect to initial data in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We now define a mapping Φ: $\mathbb{R}^+ \times \Omega \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by

$$\Phi(t, \omega, (u_0, z_0)) = (u(t, 0, \omega), z(t, 0, \omega)) = (u(t, 0, \omega), v(t, 0, \omega) + h\omega(t)), \tag{14}$$

for every $(t, \omega, (u_0, z_0)) \in \mathbb{R}^+ \times \Omega \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Then Φ is a continuous random dynamical system over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Notice that Φ satisfies the identity, for all $\omega \in \Omega$ and $t \geq 0$,

$$\Phi(t, \theta_{-t}\omega, (u_0, z_0)) = (u(t, 0, \theta_{-t}\omega), z(t, 0, \theta_{-t}\omega)) = (u(0, -t, \omega), z(0, -t, \omega)). \tag{15}$$

Let $B = \{B(\omega)\}_{\omega\in\Omega}$ be a random subset of $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. B is said to be tempered if for every positive number β,

$$e^{\beta\tau} d(B(\theta_\tau\omega)) \to 0 \quad \text{as } \tau \to -\infty,$$

where $d(B(\theta_\tau\omega)) = \sup\limits_{(u,z)\in B(\theta_\tau\omega)} \|(u, z)\|_{H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$. Denote by $\mathcal{D}$ the collection of all tempered random subsets of $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. In this paper, we will show that Φ has a random attractor which attracts every random set in $\mathcal{D}$.

3. Uniform Estimates.

Let $\delta > 0$ be small enough such that $\alpha - \delta > 0$ and $\lambda + \delta^2 - \alpha\delta > 0$. Set

$$\kappa = \frac{1}{2} \min\{\alpha - \delta, \delta, \delta c_2\}, \tag{16}$$

where c_2 is the positive constant in (7).

Lemma 3.1. *Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (6)-(9) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for P-a.e. $\omega \in \Omega$, there is $T = T(B, \omega) < 0$ such that for all $\tau \leq T$, the solution $(u(t, \tau, \omega), v(t, \tau, \omega))$ of problem (11)-(13) with $(u_0, v_0) \in B(\theta_\tau\omega)$ satisfies*

$$\|u(0, \tau, \omega)\|^2_{H^1(\mathbb{R}^n)} + \|v(0, \tau, \omega)\|^2 \leq r_1(\omega), \tag{17}$$

and

$$\int_\tau^0 e^{\kappa\xi} \left(\|u(\xi, \tau, \omega)\|^2_{H^1(\mathbb{R}^n)} + \|v(\xi, \tau, \omega)\|^2 \right) d\xi \leq r_1(\omega), \tag{18}$$

where $r_1(\omega)$ is a positive random function with

$$e^{\beta s} r_1(\theta_s\omega) \to 0 \quad \text{as} \quad s \to -\infty, \quad \forall\, \beta > 0. \tag{19}$$

Proof. Taking the inner product of (12) with v in $L^2(\mathbb{R}^n)$, we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + (\alpha - \delta)\|v\|^2 + (\lambda + \delta^2 - \alpha\delta)(u, v) - (\Delta u, v) + (f(x, u), v) = (g, v) + (\delta - \alpha)(h, v)\omega(t). \tag{20}$$

Solving v from (11) and then substituting v into the third, fourth and fifth terms on the left-hand side of (20), after some computations we find that

$$\begin{aligned} &\frac{d}{dt}\left(\|v\|^2 + (\lambda + \delta^2 - \alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n} F(x, u)dx \right) \\ &+2(\alpha - \delta)\|v\|^2 + 2\delta(\lambda + \delta^2 - \alpha\delta)\|u\|^2 + 2\delta\|\nabla u\|^2 + 2\delta(f(x, u), u) \\ &= 2(\lambda + \delta^2 - \alpha\delta)(h, u)\omega(t) + 2(\nabla u, \nabla h)\omega(t) + 2(f(x, u), h)\omega(t) \\ &\quad + 2(g, v) + 2(\delta - \alpha)(h, v)\omega(t). \end{aligned} \tag{21}$$

We first deal with the nonlinear terms in (21). By (7) we have

$$(f(x, u), u) \geq c_2 \int_{\mathbb{R}^n} F(x, u)dx + \int_{\mathbb{R}^n} \phi_2(x)dx. \tag{22}$$

Using (6) and (8), we obtain

$$2(f(x,u),h)\omega(t) \le 2\|\phi_1\|\|h\||\omega(t)| + c\left(\int_{\mathbb{R}^n}|u|^{r+1}\right)^{\frac{r}{r+1}}\|h\|_{r+1}|\omega(t)|$$

$$\le 2\|\phi_1\|\|h\||\omega(t)| + c\left(\int_{\mathbb{R}^n}(F(x,u)+\phi_3)\right)^{\frac{r}{r+1}}\|h\|_{r+1}|\omega(t)|$$

$$\le 2\|\phi_1\|\|h\||\omega(t)| + \delta c_2\int_{\mathbb{R}^n}F(x,u)dx + \delta c_2\int_{\mathbb{R}^n}\phi_3(x)dx + c\|h\|_{H^1}^{r+1}|\omega(t)|^{r+1}. \quad (23)$$

Then applying Young's or Holder's inequality to other terms on the right-hand side of (21) and using (22)-(23), we find that

$$\frac{d}{dt}\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx\right)$$
$$+(\alpha-\delta)\|v\|^2 + \delta(\lambda+\delta^2-\alpha\delta)\|u\|^2 + \delta\|\nabla u\|^2 + \delta c_2\int_{\mathbb{R}^n}F(x,u)dx$$
$$\le c\left(1+|\omega(t)|^2+|\omega(t)|^{r+1}\right). \quad (24)$$

By (8) and (16) we have

$$\delta c_2\int_{\mathbb{R}^n}F(x,u)dx \ge 2\kappa\int_{\mathbb{R}^n}F(x,u)dx + (2\kappa-\delta c_2)\int_{\mathbb{R}^n}\phi_3(x)dx.$$

Therefore, it follows from (24) that

$$\frac{d}{dt}\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx\right)$$
$$+\kappa\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx\right)$$
$$+\kappa\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2\right) \le c\left(1+|\omega(t)|^2+|\omega(t)|^{r+1}\right). \quad (25)$$

Integrating (25) on $(\tau, 0)$, we get

$$\|v(0,\tau,\omega)\|^2 + (\lambda+\delta^2-\alpha\delta)\|u(0,\tau,\omega)\|^2 + \|\nabla u(0,\tau,\omega)\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx$$
$$+\kappa\int_\tau^0 e^{\kappa\xi}\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2\right)d\xi$$
$$\le e^{\kappa\tau}\left(\|v_0\|^2 + (\lambda+\delta^2-\alpha\delta)\|u_0\|^2 + \|\nabla u_0\|^2 + 2\int_{\mathbb{R}^n}F(x,u_0)dx\right)$$
$$+c\int_\tau^0 e^{\kappa\xi}\left(1+|\omega(\xi)|^2+|\omega(\xi)|^{r+1}\right)d\xi, \quad (26)$$

from which we can obtain Lemma 3.1 by some computations. □

By an argument similar to (25), we also have the inequality

$$\frac{d}{dt}\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx\right)$$
$$+\frac{1}{3}\kappa\left(\|v\|^2 + (\lambda+\delta^2-\alpha\delta)\|u\|^2 + \|\nabla u\|^2 + 2\int_{\mathbb{R}^n}F(x,u)dx\right)$$
$$\le c\left(1+|\omega(t)|^2+|\omega(t)|^{r+1}\right),$$

and hence we can prove that there exists $T = T(B, \omega) < 0$ such that for all $\tau \le T$ and $\tau \le \xi \le 0$,

$$\|v(\xi, \tau, \omega)\|^2 + (\lambda + \delta^2 - \alpha\delta)\|u(\xi, \tau, \omega)\|^2 + \|\nabla u(\xi, \tau, \omega)\|^2$$
$$\le c e^{-\frac{\kappa}{3}\xi} \int_{-\infty}^{0} e^{\frac{\kappa}{3}s} \left(1 + |\omega(s)|^2 + |\omega(s)|^{r+1}\right) ds. \quad (27)$$

We point out that (27) is useful for deriving uniform estimates of solutions later.

Next, we derive uniform estimates on the tails of solutions when x and t approach infinity. Given $k \ge 1$, denote by $Q_k = \{x \in \mathbb{R}^n \colon |x| < k\}$ and $\mathbb{R}^n \backslash Q_k$ the complement of Q_k.

Lemma 3.2. *Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (6)-(9) hold. Let $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $T = T(B, \omega, \epsilon) < 0$ and $k_0 = k_0(\omega, \epsilon) > 0$ such that for all $\tau \le T$, the solution $(u(t, \tau, \omega), v(t, \tau, \omega))$ of problem (11)-(13) with $(u_0, v_0) \in B(\theta_\tau \omega)$ satisfies*

$$\int_{\mathbb{R}^n \backslash Q_k} \left(|u(0, \tau, \omega)|^2 + |\nabla u(0, \tau, \omega)|^2 + |v(0, \tau, \omega)|^2\right) dx \le \epsilon. \quad (28)$$

Proof. Take a smooth function ρ such that $0 \le \rho \le 1$ for all $s \in \mathbb{R}$ and

$$\rho(s) = \begin{cases} 0, & \text{if} \quad |s| < 1, \\ 1, & \text{if} \quad |s| > 2. \end{cases} \quad (29)$$

Then there is a positive constant c such that $|\rho'(s)| \le c$ for all $s \in \mathbb{R}$.

Taking the inner product of (12) with $\rho\left(\frac{|x|^2}{k^2}\right) v$ in $L^2(\mathbb{R}^n)$, and then substituting v into the resulting identity via (11), after some manipulations we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(|v|^2 + (\lambda + \delta^2 - \alpha\delta)|u|^2 + |\nabla u|^2 + 2F(x, u)\right) dx$$
$$+ \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(2(\alpha - \delta)|v|^2 + 2\delta(\lambda + \delta^2 - \alpha\delta)|u|^2 + 2\delta|\nabla u|^2 + 2\delta f(x, u)u\right) dx$$
$$= 2(\lambda + \delta^2 - \alpha\delta) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) hu\omega(t) dx - 4 \int_{\mathbb{R}^n} \rho'\left(\frac{|x|^2}{k^2}\right) v \nabla u \frac{x}{k^2} dx$$
$$+ 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(x, u) h\omega(t) dx + 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \nabla u \nabla h \omega(t) dx$$
$$+ 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(gv + (\delta - \alpha) hv\omega(t)\right) dx. \quad (30)$$

By (7) we have

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(x, u) u dx \ge c_2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) F(x, u) dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \phi_2(x) dx. \quad (31)$$

By (6) and (8) as in (23), we also have

$$2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(x, u) h\omega(t) dx \le \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\phi_1|^2 dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |h|^2 |\omega(t)|^2 dx$$
$$+ \delta c_2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \left(F(x, u) + \phi_3(x)\right) dx + c \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |h|^{r+1} |\omega(t)|^{r+1} dx. \quad (32)$$

Notice that by (29) we have

$$\int_{\mathbb{R}^n} |\rho'\left(\frac{|x|^2}{k^2}\right) v\nabla u \frac{x}{k^2}|dx \le \int_{k\le|x|\le\sqrt{2}k} |\rho'||v||\nabla u|\frac{|x|}{k^2}dx \le \frac{c}{k}\left(\|\nabla u\|^2+\|v\|^2\right). \tag{33}$$

Now applying Young's or Holder's inequality to other terms on the right-hand side of (30) and using (31)-(33), we get that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+(\lambda+\delta^2-\alpha\delta)|u|^2+|\nabla u|^2+2F(x,u)\right)dx\\&+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left((\alpha-\delta)|v|^2+\delta(\lambda+\delta^2-\alpha\delta)|u|^2+\delta|\nabla u|^2+\delta c_2F(x,u)\right)dx\\&\le\frac{c}{k}(\|\nabla u\|^2+\|v\|^2)+c|\omega(t)|^2\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|h|^2+|\nabla h|^2)dx\\&+c\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)(|\phi_1|^2+|\phi_2|+|\phi_3|+|g|^2+|\omega(t)|^{r+1}|h|^{r+1})dx.\end{aligned}\tag{34}$$

Since $\phi_1, g\in L^2(\mathbb{R}^n)$, $\phi_2,\phi_3\in L^1(\mathbb{R}^n)$, $h\in H^1(\mathbb{R}^n)$, and $\rho\left(\frac{|x|^2}{k^2}\right)=0$ for $|x|\le k$, we find that there exists $k_1=k_1(\epsilon)\ge 1$ such that for all $k\ge k_1$, the last two terms on the right-hand side of (34) are bounded by $c\epsilon(1+|\omega(t)|^2+|\omega(t)|^{r+1})$. Therefore, we have that for all $k\ge k_1$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+(\lambda+\delta^2-\alpha\delta)|u|^2+|\nabla u|^2+2F(x,u)\right)dx\\&+\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left((\alpha-\delta)|v|^2+\delta(\lambda+\delta^2-\alpha\delta)|u|^2+\delta|\nabla u|^2+\delta c_2F(x,u)\right)dx\\&\le\frac{c}{k}(\|\nabla u\|^2+\|v\|^2)+c\epsilon(1+|\omega(t)|^2+|\omega(t)|^{r+1}).\end{aligned}\tag{35}$$

By (8), (16) and (35) we find that for all $k\ge k_1$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+(\lambda+\delta^2-\alpha\delta)|u|^2+|\nabla u|^2+2F(x,u)\right)dx\\&+\kappa\int_{\mathbb{R}^n}\rho\left(\frac{|x|^2}{k^2}\right)\left(|v|^2+(\lambda+\delta^2-\alpha\delta)|u|^2+|\nabla u|^2+2F(x,u)\right)dx\\&\le\frac{c}{k}(\|\nabla u\|^2+\|v\|^2)+c\epsilon(1+|\omega(t)|^2+|\omega(t)|^{r+1}).\end{aligned}\tag{36}$$

Integrating (36) on $(\tau,0)$, by Lemma 3.1 we find that, for all $k\ge k_1$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\left(|v(0,\tau,\omega)|^2+(\lambda+\delta^2-\alpha\delta)|u(0,\tau,\omega)|^2+|\nabla u(0,\tau,\omega)|^2+2F(x,u)\right)\\&\le e^{\kappa\tau}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\left(|v_0|^2+(\lambda+\delta^2-\alpha\delta)|u_0|^2+|\nabla u_0|^2+2F(x,u_0)\right)dx\\&+\frac{c}{k}\int_\tau^0 e^{k\xi}(\|\nabla u(\xi)\|^2+\|v(\xi)\|^2)d\xi+c\epsilon\int_\tau^0 e^{\kappa\xi}(|\omega(\xi)|^2+|\omega(\xi)|^{r+1})d\xi+c\epsilon\\&\le e^{\kappa\tau}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})\left(|v_0|^2+(\lambda+\delta^2-\alpha\delta)|u_0|^2+|\nabla u_0|^2+2F(x,u_0)\right)dx\\&+\frac{c}{k}r_1(\omega)+c\epsilon\int_{-\infty}^0 e^{\kappa\xi}(|\omega(\xi)|^2+|\omega(\xi)|^{r+1})d\xi+c\epsilon.\end{aligned}\tag{37}$$

Since $(u_0, v_0) \in B(\theta_\tau \omega)$, we find that the first term on the right-hand side of (37) goes to zero as $\tau \to -\infty$. Hence, there exist $T = T(B, \omega, \epsilon) < 0$ and $k_2(\epsilon) \geq k_1(\epsilon)$ such that for all $\tau \leq T$ and $k \geq k_2$,

$$\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) \left(|v(0,\tau,\omega)|^2 + (\lambda + \delta^2 - \alpha\delta)|u(0,\tau,\omega)|^2 + |\nabla u|^2 + 2F(x,u)\right) \leq c\epsilon r(\omega),$$

where $r(\omega) = 1 + r_1(\omega) + \int_{-\infty}^0 e^{\kappa\xi}(|\omega(\xi)|^2 + |\omega(\xi)|^{r+1}) d\xi$. This and (8) imply the lemma. □

We now denote by $\psi = 1 - \rho$ with ρ given by (29). Fix $k \geq 1$ and let

$$\tilde{u}(x,t,\tau,\omega) = \psi(\frac{|x|^2}{k^2}) u(x,t,\tau,\omega) \quad \text{and} \quad \tilde{v}(x,t,\tau,\omega) = \psi(\frac{|x|^2}{k^2}) v(x,t,\tau,\omega). \tag{38}$$

Then $\tilde{u}(\cdot,t,\tau,\omega), \tilde{v}(\cdot,t,\tau,\omega) \in H_0^1(Q_{2k})$. Multiplying (11) and (12) by ψ and using (38) we find that

$$\tilde{u}_t + \delta\tilde{u} - \tilde{v} = \psi h\omega(t), \tag{39}$$

$$\tilde{v}_t + (\alpha - \delta)\tilde{v} + (\lambda + \delta^2 - \alpha\delta)\tilde{u} - \Delta\tilde{u} + \psi f(x,u) = \psi g + (\delta - \alpha)\psi h\omega(t) - u\Delta\psi - 2\nabla\psi\nabla u. \tag{40}$$

Consider the eigenvalue problem:

$$-\Delta\tilde{u} = \lambda\tilde{u} \quad \text{in} \quad Q_{2k}, \quad \text{with} \quad \tilde{u}|_{\partial Q_{2k}} = 0.$$

Then the problem has a family of eigenfunctions $\{e_j\}_{j=1}^\infty$ with corresponding eigenvalues $\{\lambda_j\}_{j=1}^\infty$ such that $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(Q_{2k})$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ as $j \to \infty$. Given n, let $X_n = \text{span}\{e_1, \cdots, e_n\}$ and $P_n : L^2(Q_{2k}) \to X_n$ be the projection operator.

Lemma 3.3. *Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (6)-(9) hold. Let $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then for every $\epsilon > 0$ and P-a.e. $\omega \in \Omega$, there exist $K = K(\omega,\epsilon) > 0$, $T = T(B,\omega,\epsilon) < 0$ and $N = N(\omega,\epsilon) > 0$ such that for all $k \geq K$, $\tau \leq T$ and $n \geq N$,*

$$\|(I - P_n)\tilde{u}(\cdot,0,\tau,\omega)\|_{H_0^1(Q_{2k})} + \|(I - P_n)\tilde{v}(\cdot,0,\tau,\omega)\|_{L^2(Q_{2k})} \leq \epsilon.$$

Proof. Let $\tilde{u}_{n,1} = P_n\tilde{u}$, $\tilde{u}_{n,2} = \tilde{u} - \tilde{u}_{n,1}$, $\tilde{v}_{n,1} = P_n\tilde{v}$ and $\tilde{v}_{n,2} = \tilde{v} - \tilde{v}_{n,1}$. Then applying $I - P_n$ to (39) we get

$$\tilde{v}_{n,2} = \frac{d}{dt}\tilde{u}_{n,2} + \delta\tilde{u}_{n,2} - (I - P_n)(\psi h\omega(t)). \tag{41}$$

Similarly, applying $I - P_n$ to (40) and taking the inner product of the resulting equation with $\tilde{v}_{n,2}$ in $L^2(Q_{2k})$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde{v}_{n,2}\|^2 + (\alpha - \delta)\|\tilde{v}_{n,2}\|^2 + (\lambda + \delta^2 - \alpha\delta)(\tilde{v}_{n,2}, \tilde{u}_{n,2}) - (\Delta\tilde{u}, \tilde{v}_{n,2}) + (\psi f(x,u), \tilde{v}_{n,2})$$
$$= (\psi g + (\delta - \alpha)\psi h\omega(t) - u\Delta\psi - 2\nabla\psi\nabla u, \tilde{v}_{n,2}).$$

Substituting (41) into the last three terms on the left-hand side of the above we get

$$\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{v}_{n,2}\|^2 + (\lambda + \delta^2 - \alpha\delta)\|\tilde{u}_{n,2}\|^2 + \|\nabla\tilde{u}_{n,2}\|^2 + 2(\psi f(x,u), \tilde{u}_{n,2})\right) + 2\delta\|\nabla\tilde{u}_{n,2}\|^2 \\
&\quad + 2(\alpha - \delta)\|\tilde{v}_{n,2}\|^2 + 2\delta(\lambda + \delta^2 - \alpha\delta)\|\tilde{u}_{n,2}\|^2 = 2(\lambda + \delta^2 - \alpha\delta)(\psi h\omega, \tilde{u}_{n,2}) \\
&\quad + 2(\nabla\tilde{u}_{n,2}, \nabla(\psi h\omega)) + 2(\psi f_u(x,u)u_t, \tilde{u}_{n,2}) + 2(\psi f(x,u), (I - P_n)(\psi h\omega)) \\
&\quad - 2\delta(\psi f(x,u), \tilde{u}_{n,2}) + 2(\psi g + (\delta - \alpha)\psi h\omega - u\Delta\psi - 2\nabla\psi\nabla u, \tilde{v}_{n,2}).
\end{aligned} \tag{42}$$

We need to estimate every term on the right-hand side of (42). For the nonlinear terms, by (9) we have

$$
\begin{aligned}
|2(\psi f_u(x,u)u_t, \tilde{u}_{n,2})| &\le c\|\phi_4\|_6\|u_t\|\|\tilde{u}_{n,2}\|_3 + c\|u_t\|\|u\|_6^{r-1}\|\tilde{u}_{n,2}\|_{\frac{6}{4-r}} \\
\le c\|\phi_4\|_{H^1}\|u_t\|\|\tilde{u}_{n,2}\|^{\frac12}\|\nabla\tilde{u}_{n,2}\|^{\frac12} &+ c\|u_t\|\|u\|_{H^1}^{r-1}\|\tilde{u}_{n,2}\|^{\frac{3-r}{2}}\|\nabla\tilde{u}_{n,2}\|^{\frac{r-1}{2}} \\
&\le c\lambda_{n+1}^{-\frac14}\|u_t\|\|\nabla\tilde{u}_{n,2}\| + c\lambda_{n+1}^{\frac{r-3}{4}}\|u_t\|\|u\|_{H^1}^{r-1}\|\nabla\tilde{u}_{n,2}\| \\
&\le \frac14\delta\|\nabla\tilde{u}_{n,2}\|^2 + c\lambda_{n+1}^{-\frac12}\|u_t\|^2 + c\lambda_{n+1}^{\frac{r-3}{2}}\|u_t\|^2\|u\|_{H^1}^{2r-2}. \qquad (43)
\end{aligned}
$$

By (6) we also have

$$
|2(\psi f(x,u),(I-P_n)(\psi h\omega))| \le c\|(I-P_n)(\psi h\omega)\| + c\|u\|_{H^1}^r\|(I-P_n)(\psi h\omega)\|. \quad (44)
$$

Applying Young's inequality to other terms on the right-hand side of (42), by (16) and (43)-(44), after detailed computations we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{v}_{n,2}\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}\|^2 + \|\nabla\tilde{u}_{n,2}\|^2 + 2(\psi f(x,u),\tilde{u}_{n,2})\right) \\
&+2\kappa\left(\|\tilde{v}_{n,2}\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}\|^2 + \|\nabla\tilde{u}_{n,2}\|^2 + 2(\psi f(x,u),\tilde{u}_{n,2})\right) \\
&\le c\|(I-P_n)(\psi h)\|^2|\omega(t)|^2 + \frac{c}{k^2}\|h\|^2|\omega(t)|^2 + c\|(I-P_n)(\psi\nabla h)\|^2|\omega(t)|^2 + c\lambda_{n+1}^{-\frac12}\|u_t\|^2 \\
&+c\lambda_{n+1}^{\frac{r-3}{2}}\|u_t\|^2\|u\|_{H^1}^{2r-2} + c\|(I-P_n)(\psi h)\||\omega(t)| + c\|(I-P_n)(\psi h)\|\|u\|_{H^1}^r|\omega(t)| \\
&+ c\lambda_{n+1}^{-1}(1+\|u\|_{H^1}^r) + c\|(I-P_n)g\|^2 + \frac{c}{k^2}\|\nabla u\|^2 + \frac{c}{k^4}\|u\|^2. \qquad (45)
\end{aligned}
$$

Since $1 \le r < 3$ and $\lambda_n \to \infty$, there are $N_1 = N_1(\epsilon)$ and $k_1 = k_1(\epsilon)$ such that for all $n \ge N_1$ and $k \ge k_1$, the right-hand side of (45) is bounded by

$$
\begin{aligned}
&c\epsilon + c\epsilon|\omega(t)|^2 + c\epsilon\|u_t\|^2 + c\epsilon\|u_t\|^2\|u\|_{H^1}^{2r-2} + c\epsilon\|u\|_{H^1}^{2r} \\
&\le c\epsilon\left(1+|\omega(t)|^2+\|u_t\|^6+\|u\|_{H^1}^6\right),
\end{aligned}
$$

which along with (45) implies that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\tilde{v}_{n,2}\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}\|^2 + \|\nabla\tilde{u}_{n,2}\|^2 + 2(\psi f(x,u),\tilde{u}_{n,2})\right) \\
&+2\kappa\left(\|\tilde{v}_{n,2}\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}\|^2 + \|\nabla\tilde{u}_{n,2}\|^2 + 2(\psi f(x,u),\tilde{u}_{n,2})\right) \\
&\le c\epsilon\left(1+|\omega(t)|^2+\|u_t\|^6+\|u\|_{H^1}^6\right).
\end{aligned}
$$

Integrating the above on $(0,\tau)$ we get, for all $n \ge N_1$ and $k \ge k_1$,

$$
\begin{aligned}
&\|\tilde{v}_{n,2}(0,\tau,\omega)\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}(0,\tau,\omega)\|^2 + \|\nabla\tilde{u}_{n,2}(0,\tau,\omega)\|^2 + 2(\psi f(x,u),\tilde{u}_{n,2}) \\
&\le e^{2\kappa\tau}\left(\|\tilde{v}_{n,2}(\tau,\tau,\omega)\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}(\tau,\tau,\omega)\|^2\right) \\
&+e^{2\kappa\tau}\left(\|\nabla\tilde{u}_{n,2}(\tau,\tau,\omega)\|^2 + 2(\psi f(x,u_0),\tilde{u}_{n,2}(\tau,\tau,\omega))\right) \\
&+c\epsilon\int_\tau^0 e^{2\kappa\xi}\left(1+|\omega(\xi)|^2+\|u_t(\xi,\tau,u_0)\|^6+\|u(\xi,\tau,u_0)\|_{H^1}^6\right)d\xi \\
&\le ce^{2\kappa\tau}\left(1+\|v_0\|^2+(\lambda+\delta^2-\alpha\delta)+\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^{r+1}\right) \\
&+ c\epsilon\int_{-\infty}^0 e^{2\kappa\xi}\left(1+|\omega(\xi)|^2+\|u_t(\xi,\tau,u_0)\|^6+\|u(\xi,\tau,u_0)\|_{H^1}^6\right)d\xi. \qquad (46)
\end{aligned}
$$

By (11) and (27) we obtain that

$$
\|u_t(\xi,\tau,\omega)\|^6 \le c(\|u(\xi,\tau,\omega)\|^6 + \|v(\xi,\tau,\omega)\|^6 + |\omega(\xi)|^6) \le ce^{-\kappa\xi}r^3(\omega) + c|\omega(\xi)|^6,
$$

and

$$
\|u(\xi,\tau,\omega)\|_{H^1}^6 \le ce^{-\kappa\xi}r^3(\omega),
$$

where $r(\omega) = \int_{-\infty}^0 e^{\frac{k}{3}s}\left(1 + |\omega(s)|^2 + |\omega(s)|^6\right) ds$. Hence, it follows from (46) that

$$\|\tilde{v}_{n,2}(0,\tau,\omega)\|^2 + (\lambda+\delta^2-\alpha\delta)\|\tilde{u}_{n,2}(0,\tau,\omega)\|^2 + \|\nabla\tilde{u}_{n,2}(0,\tau,\omega)\|^2 + 2(\psi f(x,u),\tilde{u}_{n,2})$$
$$\le ce^{2\kappa\tau}\left(1 + \|v_0\|^2 + (\lambda+\delta^2-\alpha\delta) + \|u_0\|^2_{H^1} + \|u_0\|^{r+1}_{H^1}\right)$$
$$+ c\epsilon r^3(\omega) + c\epsilon\int_{-\infty}^0 e^{2\kappa\xi}\left(1 + |\omega(\xi)|^2 + |\omega(\xi)|^6\right) d\xi. \quad (47)$$

Since $(u_0, v_0) \in B(\theta_\tau\omega)$, the the first term on the right-hand side of (47) approaches zero as $\tau \to -\infty$, which along with (6), (17) and (47) concludes the proof. □

4. **Random Attractors.** In this section, we prove the existence of a random attractor for the wave equation defined on $\mathbb{R}^n$. Notice that $z(t,\tau,\omega) = v(t,\tau,\omega) + h(\omega(t)$. Therefore it follows from Lemma 3.1 that, given $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, for P-a.e. $\omega \in \Omega$, there is $T = T(B,\omega) > 0$ such that for all $t \ge T$,

$$\|\Phi(t,\theta_{-t}\omega,(u_0,z_0))\|^2_{H^1\times L^2} = \|u(0,-t,\omega)\|^2_{H^1} + \|z(0,\tau,\omega)\|^2 \le r_1(\omega), \quad (48)$$

where $r_(\omega)$ is the random function in (17) satisfying (19). Denote by

$$E(\omega) = \{(u,z) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : \|u\|^2_{H^1} + \|z\|^2 \le r_1(\omega)\}. \quad (49)$$

Then (48) indicates that $E = \{E(\omega)\}_{\omega\in\Omega}$ is a closed random absorbing set for Φ in $\mathcal{D}$. Next, we show that Φ is pullback asymptotically compact in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

Lemma 4.1. *Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (6)-(9) hold. Then the random dynamical system Φ is $\mathcal{D}$-pullback asymptotically compact in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$; that is, for P-a.e. $\omega \in \Omega$, the sequence $\{\Phi(t_m,\theta_{-t_m}\omega,(u_{0,m},z_{0,m}))\}$ has a convergent subsequence in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ provided $t_m \to \infty$ and $(u_{0,m},z_{0,m}) \in B(\theta_{-t_m}\omega)$ with $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.*

Proof. Since $t_m \to \infty$, by Lemma 3.1 we find that, for P-a.e. $\omega \in \Omega$, there is $M_1 = M_1(B,\omega) > 0$ such that for all $m \ge M_1$,

$$\|u(0,-t_m,\omega)\|^2_{H^1(\mathbb{R}^n)} + \|v(0,-t_m,\omega)\|^2 \le r_1(\omega). \quad (50)$$

On the other hand, given $\epsilon > 0$, by Lemma 3.2 there are $M_2 = M_2(B,\omega,\epsilon) > 0$ and $k_0 = k_0(\omega,\epsilon) > 0$ such that for all $m \ge M_2$,

$$\int_{\mathbb{R}^n\setminus Q_{k_0}} \left(|u(0,-t_m,\omega)|^2 + |\nabla u(0,-t_m,\omega)|^2 + |v(0,-t_m,\omega)|^2\right) dx \le \epsilon. \quad (51)$$

Denote by

$$\tilde{u}(x,t,\tau,\omega) = \psi\left(\frac{|x|^2}{k^2}\right)u(x,t,\tau,\omega) \quad\text{and}\quad \tilde{v}(x,t,\tau,\omega) = \psi\left(\frac{|x|^2}{k^2}\right)v(x,t,\tau,\omega). \quad (52)$$

Then it follows from Lemma 3.3 that there are $k_1 = k_1(\omega,\epsilon) \ge k_0$, $M_3 = M_3(B,\omega,\epsilon)$ and $N = N(\omega,\epsilon)$ such that for all $m \ge M_3$,

$$\|(I - P_N)\tilde{u}(0,-t_m,\omega)\|_{H^1(Q_{2k_1})} + \|(I - P_N)\tilde{v}(0,-t_m,\omega)\|_{L^2(Q_{2k_1})} \le \epsilon. \quad (53)$$

By (50) and (52) we find that $\{P_N(\tilde{u}(0,-t_m,\omega),\tilde{v}(0,-t_m,\omega))\}$ is bounded in the finite-dimensional space $P_N(H^1(Q_{2k_1}) \times L^2(Q_{2k_1}))$, which along with (53) shows that $\{(\tilde{u}(0,-t_m,\omega),\tilde{v}(0,-t_m,\omega))\}$ is precompact in $H^1_0(Q_{2k_1}) \times L^2(Q_{2k_1})$. Since $\psi(\frac{|x|^2}{k_1^2}) = 1$ for $|x| \le k_1$, we get by (52) that $\{(u(0,-t_m,\omega),v(0,-t_m,\omega))\}$ is precompact in $H^1(Q_{k_1}) \times L^2(Q_{k_1})$, which together with (51) implies the precompactness of this sequence in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. □

As an immediate consequence of (48) and Lemma 4.1, the existence of a random attractor for Φ follows from a standard result in [2, 3, 5, 7, 9].

Theorem 4.2. *Assume that $g \in L^2(\mathbb{R}^n)$, $h \in H^1(\mathbb{R}^n)$ and (6)-(9) hold. Then the random dynamical system Φ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.*

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