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EQUIVALENCE BETWEEN OBSERVABILITY AND STABILIZATION FOR A CLASS OF SECOND ORDER SEMILINEAR EVOLUTION EQUATIONS

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Abstract. We consider an abstract second order semilinear evolution equation with a bounded dissipation. We establish an equivalence between the stabilization of this system and the observability of the corresponding undamped system. Our technique of proof relies on an appropriate decomposition of the solution, and the energy method. Our result generalizes an earlier one by Haraux [5] who studied the same type of problem for linear systems. Some applications of our result are provided, and the paper ends with a few open problems.

1. **Introduction and statements of main results.** Let H be a Hilbert space, and let A be an unbounded coercive operator on H with $A = A^*$. Also let B be a bounded nonnegative self-adjoint operator on H. Denote $(.,.)$, the scalar product on H, and $|.|$, the corresponding norm on H. Set $V = D(A^{\frac{1}{2}})$, and for every $v \in V$, set $||v|| = |A^{\frac{1}{2}} v|$. About two decades ago, Haraux [5] considered the following abstract second order evolution equations

$$y_{tt} + Ay + By_t = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

and

$$\varphi_{tt} + A\varphi = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

He proved the following result:
Theorem 0. *The following assertions are equivalent:*
i) *There exist $T_0 > 0$, and $C > 0$ such that every solution of (1.2) satisfies:*

$$|\varphi_t(0)|^2 + ||\varphi(0)||^2 \leq C \int_0^{T_0} |B^{\frac{1}{2}} \varphi_t(t)|^2 \, dt. \tag{1.3}$$

ii) *There exist $M > 0$, and $\lambda > 0$ such that every solution of (1.1) satisfies:*

$$|y_t(t)|^2 + ||y(t)||^2 \leq M e^{-\lambda t} \left(|y_t(0)|^2 + ||y(0)||^2 \right), \quad \forall t \geq 0. \tag{1.4}$$

Now let $f : H \longrightarrow H$ be a globally Lipschitz operator, namely,

$$\exists L \geq 0 : \quad |f(u) - f(v)| \leq L|u - v|, \quad \forall u, v \in H. \tag{1.5}$$

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Consider the following semilinear equations:

$$y_{tt} + Ay + f(y) + By_t = 0, \quad t \in \mathbb{R}, \tag{1.6}$$

and

$$\varphi_{tt} + A\varphi + f(\varphi) = 0, \quad t \in \mathbb{R}. \tag{1.7}$$

Our purpose in this note is to show that Theorem 0 remains valid when the linear operator A is perturbed by a globally Lipschitz nonlinearity satisfying

$$f(0) = 0, \quad (f(v), v) \geq 0, \quad \forall v \in H. \tag{1.8}$$

Let λ_0 be the best positive constant such that:

$$||v||^2 = |A^{\frac{1}{2}} v|^2 \geq \lambda_0^2 |v|^2, \quad \forall v \in V. \tag{1.9}$$

More precisely, we will prove the following:
Theorem 1. *The following assertions are equivalent:*
i) *There exist $T_0 > 0$, and $C > 0$ such that every solution of (1.7) satisfies:*

$$|\varphi_t(0)|^2 + ||\varphi(0)||^2 \leq C \int_0^{T_0} |B^{\frac{1}{2}} \varphi_t(t)|^2 \, dt. \tag{1.10}$$

ii) *There exist $M > 0$, and $\lambda > 0$ such that for every $\{y^0, y^1\} \in V \times H$, the solution of (1.6) with $y(0) = y^0$ and $y_t(0) = y^1$ satisfies:*

$$|y_t(t)|^2 + ||y(t)||^2 \leq M e^{-\lambda t} \left(|y^1|^2 + ||y^0||^2 \right), \quad \forall t \geq 0. \tag{1.11}$$

Proof of Theorem 1. First, we prove (1.10)⇒(1.11). To this end, let $\{y^0, y^1\} \in V \times H$, and let $F : H \longrightarrow \mathbb{R}$ be the nonnegative Fréchet differentiable function with $F(0) = 0$, and whose Fréchet differential is f. Consider the solution φ of (1.7) with $\varphi(0) = y^0$, and $\varphi_t(0) = y^1$. Set:

$$E_\varphi(t) = \frac{1}{2} \left\{ |\varphi_t(t)|^2 + ||\varphi(t)||^2 \right\} + F(\varphi(t)). \tag{1.12}$$

The function E_φ denotes the natural energy associated with (1.7); it is a constant function of the time variable, and for each t, one has the equivalence $E_\varphi(t) \simeq |\varphi_t(t)|^2 + ||\varphi(t)||^2$, thanks to (1.5) and (1.9).

Let y be the solution of (1.6) with $y(0) = y^0$ and $y_t(0) = y^1$, and set $\psi = y - \varphi$. Then ψ satisfies:

$$\begin{aligned} &\psi_{tt} + A\psi = f(\varphi) - f(y) - By_t \\ &\psi(0) = 0 = \psi_t(0). \end{aligned} \tag{1.13}$$

Now, set

$$E_\psi(t) = \frac{1}{2} \left\{ |\psi_t(t)|^2 + ||\psi(t)||^2 \right\}. \tag{1.14}$$

We have:

$$E_\psi(t) = \int_0^t (f(\varphi) - f(y), \psi_t) \, ds - \int_0^t (By_t, \psi_t) \, ds. \tag{1.15}$$

Thanks to (1.5), the boundedness of B, and Cauchy-Schwarz inequality, one easily derives from (1.15) that for every $0 \leq t \leq T_0$:

$$\begin{aligned} E_\psi(t) &\leq L \int_0^t |\psi(s)||\psi_t(s)|\, ds + M_0^{\frac{1}{2}} \int_0^t |B^{\frac{1}{2}} y_t(s)||\psi_t(s)|\, ds \\ &\leq \frac{L}{\lambda_0} \int_0^t ||\psi(s)|||\psi_t(s)|\, ds + \frac{M_0}{2} \int_0^t |B^{\frac{1}{2}} y_t(s)|^2\, ds + \frac{1}{2} \int_0^t |\psi_t(s)|^2\, ds \\ &\leq (1 + (L/\lambda_0)) \int_0^t E_\psi(s)\, ds + \frac{M_0}{2} \int_0^{T_0} |B^{\frac{1}{2}} y_t(s)|^2\, ds, \end{aligned} \tag{1.16}$$

where M_0 is the norm in $L(H)$ of the bounded operator B.
Applying Gronwall lemma to (1.16), we derive:

$$E_\psi(t) \leq M_0 e^{\frac{(L+\lambda_0)T_0}{\lambda_0}} \int_0^{T_0} |B^{\frac{1}{2}} y_t|^2\, dt. \tag{1.17}$$

Combining (1.10), (1.17), and using the fact that B is bounded, one easily gets:

$$\begin{aligned} E_y(0) = E_\varphi(0) &\leq C \int_0^{T_0} |B^{\frac{1}{2}} \varphi_t|^2\, dt \\ &\leq 2C \int_0^{T_0} |B^{\frac{1}{2}} y_t|^2\, dt + 2C \int_0^{T_0} |B^{\frac{1}{2}} \psi_t|^2\, dt \\ &\leq K_0 \int_0^{T_0} |B^{\frac{1}{2}} y_t|^2\, dt, \end{aligned} \tag{1.18}$$

for some positive constant K_0 independent of y^0 and y^1, and where the energy E_y associated with (1.6) is given by:

$$E_y(t) = \frac{1}{2} \left\{ |y_t(t)|^2 + ||y(t)||^2 \right\} + F(y(t)). \tag{1.19}$$

It is worth noting that E_y is a nonincreasing function of the time variable as

$$E_y'(t) = -(By_t, y_t), \quad \forall t \geq 0. \tag{1.20}$$

We then derive from (1.18) and (1.20):

$$E_y(T_0) \leq \gamma E_y(0), \tag{1.21}$$

with $\gamma = K_0/(K_0 + 1)$. Thanks to the semigroup property, (1.11) follows with $M = 1/\gamma$, and $\lambda = (\log M)/T_0$.
We now turn to (1.11)⇒(1.10). Let φ be an arbitrary solution of (1.7). Set $\varphi = y + \theta$, where y solves (1.6), and θ solves:

$$\begin{aligned} &\theta_{tt} + A\theta + B\theta_t = -f(\varphi) + f(y) + B\varphi_t \\ &\theta(0) = 0 = \theta_t(0). \end{aligned} \tag{1.22}$$

Thanks to (1.11) and (1.20), for large enough T_1, one has:

$$E_y(T_1) = E_y(0) - \int_0^{T_1} (By_t, y_t)\, dt \leq \frac{1}{2} E_y(0) = \frac{1}{2} E_\varphi(0), \tag{1.23}$$

from which one derives:

$$E_\varphi(0) \le 2\int_0^{T_1} (By_t, y_t)\,dt = 2\int_0^{T_1} \{(B\varphi_t, \varphi_t) + (B\theta_t, \theta_t)\}\,dt \\ - 4\int_0^{T_1} (B\varphi_t, \theta_t)\,dt, \quad \forall t \ge 0. \tag{1.24}$$

It follows from (1.24), and the Cauchy-Schwarz inequality:

$$2|(Bu, v)| \le (Bu, u) + (Bv, v), \quad \forall u, v \in H,$$

that:

$$E_\varphi(0) \le 4\int_0^{T_1} (B\varphi_t, \varphi_t)\,dt + 4\int_0^{T_1} (B\theta_t, \theta_t)\,dt \quad \forall t \ge 0 \tag{1.25}$$

For System (1.22), if we set:

$$E_\theta(t) = \frac{1}{2}\left\{|\theta_t(t)|^2 + ||\theta(t)||^2\right\}. \tag{1.26}$$

We have

$$E_\theta(t) + \int_0^t (B\theta_t, \theta_t)\,ds = \int_0^t (f(y) - f(\varphi), \theta_t)\,ds + \int_0^t (B\varphi_t, \theta_t)\,ds. \tag{1.27}$$

Thanks to (1.5), and Cauchy-Schwarz inequality, one easily derives from (1.27) that for every $0 \le t \le T_1$,

$$\begin{aligned} &E_\theta(t) + \frac{1}{2}\int_0^t (B\theta_t, \theta_t)\,ds \\ &\le L\int_0^t |\theta(s)||\theta_t(s)|\,ds + \frac{1}{2}\int_0^t (B\varphi_t(s), \varphi_t(s))\,ds \\ &\le \frac{L}{\lambda_0}\int_0^t ||\theta(s)|||\theta_t(s)|\,ds + \frac{1}{2}\int_0^{T_1} (B\varphi_t(s), \varphi_t(s))\,ds \\ &\le \frac{L}{\lambda_0}\int_0^t E_\theta(s)\,ds + \frac{1}{2}\int_0^{T_1} (B\varphi_t(s), \varphi_t(s))\,ds. \end{aligned} \tag{1.28}$$

Applying Gronwall lemma to (1.28), we derive

$$E_\theta(t) \le e^{(L/\lambda_0)t}\int_0^{T_1} (B\varphi_t(s), \varphi_t(s))\,ds. \tag{1.29}$$

The combination of (1.28), and (1.29) yields:

$$\int_0^{T_1} (B\theta_t, \theta_t)\,dt \le K_1\int_0^{T_1} (B\varphi_t, \varphi_t)\,dt, \tag{1.30}$$

for some positive constant K_1 that depends on L, T_1, and λ_0 only. Reporting (1.30) in (1.25), one gets the claimed estimate, which completes the proof of Theorem 1. □

2. Applications. We are going to discuss some applications of our result in this section.

2.1 **A Hyperbolic equation in a bounded domain.** Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with boundary of class C^2. Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a globally Lipschitz function with

$$h(0) = 0, \quad sh(s) \geq 0, \quad \forall s \in \mathbb{R}. \tag{2.1}$$

Let $a \in L^\infty(\Omega)$ be a nonnegative function satisfying

$$\exists a_0 > 0 : a(x) \geq a_0, \quad \forall x \in \omega, \tag{2.2}$$

where ω is a neighboorhood of Γ_0, that is to say, the intersection of Ω and a neighborhood of Γ_0, where Γ_0 is a suitable portion of the boundary that will be defined later. Throughout the paper ∂_i stands for $\partial/\partial x_i$, and we use the Einstein summation convention on repeated indices. Consider the damped hyperbolic equation

$$\begin{cases} y_{tt} - \partial_i(b_{ij}(x)\partial_j y) + p(x)y + h(y) + ay_t = 0 \text{ in } \Omega \times (0, \infty) \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0, \infty) \\ y(0) = y^0 \text{ in } \Omega \\ y_t(0) = y^1 \text{ in } \Omega, \end{cases} \tag{2.3}$$

where p is a nonnegative function in $L^\infty(\Omega)$, and the coefficients $(b_{ij})_{i,j}$ satisfy:

$$b_{ij} \in C^1(\bar{\Omega}); \quad b_{ij} = b_{ji}, \quad \forall i,\ j = 1, 2, ..., N, \tag{2.4}$$

and

$$\exists b_0 > 0 : b_{ij}(x)z_i z_j \geq b_0 z_i z_i, \quad \forall (x, z) \in \bar{\Omega} \times \mathbb{R}^N. \tag{2.5}$$

If $\{y^0, y^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, then System (2.2) is well-posed in the space $H_0^1(\Omega) \times L^2(\Omega)$; thus System (2.2) has a unique weak solution $y \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$. This result is well-known (e.g. [9, 11]). Now introduce the energy given by

$$E(t) = \frac{1}{2}\int_\Omega \{|y_t(x,t)|^2 + b_{ij}(x)\partial_j y(x,t)\partial_i y(x,t) + p(x)|y(x,t)|^2\}dx + \int_\Omega F(y(x,t))dx, \tag{2.6}$$

where $F(s) = \int_0^s h(r)dr$.

It is also well-known that the energy E decays exponentially to zero as the time $t \to \infty$, even for some nonlinear dissipations (cf. e.g. [1, 9, 13, 17, 18, 19]). The method of proof in those earlier works relies on the multiplier techniques applied to the damped equation; a compactness-uniqueness argument was also used in [9, 13, 19] in order to get rid of lower order terms that naturally occur in the estimates. In [17, 18], using a Carleman estimate (e.g. [3, 15]) it was possible to prove a constructive exponential decay result without resorting to the compactness-uniqueness argument. The fact that we employed the damped equation directly in [17, 18] required that we use an H^{-1}-type Carleman estimate (cf. [3, 6, 15]). We note that an L^2-type Carleman estimate could not be used in [17, 18], due to the method employed, and the nonlinearity involved. The approach developed above enables us to prove the exponential decay of the energy using an observability estimate for the associated conservative semilinear system. For the sequel, we will need the following additional notations: Based on [3, 10], we suppose that there exists a function $d \in C^2(\bar{\Omega})$ satisfying for some $m_0 \geq 4$:

$$\begin{aligned} &\text{i)}\ \ (2b_{il}(b_{kj}d_{x_k})_{x_l} - b_{ij,x_l}b_{kl}d_{x_k})\, z_i z_j \geq m_0 b_{ij} z_i z_j, \quad \forall (x,z) \in \bar{\Omega} \times \mathbb{R}^N. \\ &\text{ii)}\ \ \min\left\{|\nabla d(x)|; x \in \bar{\Omega}\right\} > 0. \\ &\text{iii)}\ \ \frac{1}{4} b_{ij}(x) d_{x_i}(x) d_{x_j}(x) \geq R_1^2 \geq R_0^2 > 0, \quad \forall x \in \bar{\Omega}, \end{aligned} \tag{2.7}$$

where $R_0 = \min\left\{\sqrt{d(x)}; x \in \bar{\Omega}\right\}$, and $R_1 = \max\left\{\sqrt{d(x)}; x \in \bar{\Omega}\right\}$. We now define Γ_0; let ν be the unit normal pointing into the exterior of Ω, and set

$$\Gamma_0 = \left\{x \in \partial\Omega; b_{ij}\nu_i d_{x_j}(x) > 0\right\}. \tag{2.8}$$

It is easy to check that if $b_{ij} = \gamma\delta_{ij}$, setting $d(x) = |x - x_0|^2$ for any $x_0 \in \mathbb{R}^N \setminus \bar{\Omega}$, then d, or its scaled version [3, 10], satisfies (2.7); in this case,

$$\Gamma_0 = \{x \in \partial\Omega; (x - x_0) \cdot \nu > 0\}, \tag{2.9}$$

which is the usual portion of the boundary that arises in the framework of the multiplier method (e.g. [7, 8, 12, 16].

Now we are going to use Theorem 1 to prove the exponential decay of the energy E *via* the observability of the solution of the conservative system associated with (2.3). To this end, set $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $Au = -\partial_i(b_{ij}(x)\partial_j u) + p(x)u$, $Bu = a(x)u$, $f(u)(x) = h(u(x))$. If φ solves (2.3) with the function $a \equiv 0$. Then it can be shown that φ satisfies (1.10) with $T_0 = 2\inf\{R_1; d(\cdot) \text{ satisfies } (2.7)\}$, from which the sought exponential decay follows. One may prove this claim by noticing that $\Phi = \varphi_t$ solves a hyperbolic problem with a bounded potential that depends on both the space and time variables, then use the observability estimate provided by [4, Theorem 2.2]. □

2.2 **A Hyperbolic equation in the whole space.** We now consider System (2.3) with $\Omega = \mathbb{R}^N$; the boundary condition must be dropped. Now the coefficients b_{ij} are further assumed to be bounded, and to satisfy (2.7) locally only. We also assume now that $\omega = \{x \in \mathbb{R}^N; |x| > L\}$ for some $L > 0$, and that the function p further satisfies:

$$\exists b_0 > 0 : p(x) \geq b_0, \quad a.e.\ x \in \omega. \tag{2.10}$$

Hypothesis (2.10) is useful in order to prove the exponential decay of the energy when Ω is unbounded (e.g. [2, 20]); when (2.10) is dropped, only polynomial decay of the energy is known to hold for a smaller class of initial data (e.g. [14]).

If φ denotes the solution of the conservative semilinear system (2.3) on the whole space, and we set as above, $\Phi = \varphi_t$, then $\tilde{\varphi} = r\xi\Phi$, where r is the cut-off function given in [3, (2.33)], and $\xi \in C_0^\infty(\Omega)$ with $0 \leq \xi \leq 1$, $\xi = 1$ in ω, and $\xi = 0$ for $|x| > 2L$, satisfies the equation:

$$\begin{aligned} &\tilde{\varphi}_{tt} - \partial_i(b_{ij}(x)\partial_j\tilde{\varphi}) + p(x)\tilde{\varphi} + h'(y)\tilde{\varphi} = \xi(r''\Phi + 2r'\Phi_t) \\ &- r(2(\partial_i\Phi)(b_{ij}(x)\partial_j\xi) + \Phi(\partial_i(b_{ij}(x)\partial_j\xi)) \text{ in } B_L \times (0,\infty) \end{aligned} \tag{2.11}$$

where $B_L = \mathbb{R}^N \setminus \omega$. Using equation (2.11) and Theorem 2.4 of [3], one derives the observability estimate (1.10) for φ, from which the exponential decay of the energy follows. □

2.3 **Case of a locally Lipschitz nonlinearity.** We now assume $f : V \longrightarrow H$ to be a continuous function satisfying for some positive constants r and L:

$$\exists L \geq 0 : \quad |f(u) - f(v)| \leq L|u - v|(1 + ||u||^r + ||v||^r), \quad \forall u, v \in V. \tag{2.12}$$

and

$$f(0) = 0, \quad (f(v), v) \geq 0, \quad \forall v \in V. \tag{2.13}$$

In this case, the nonnegative function F is now assumed to be defined from V to $\mathbb{R}$, with Fréchet differential f.

The technique developed to prove Theorem 1 may be used to prove the following result:

Theorem 2. *The following assertions are equivalent:*
j) *Let* $(\varphi^0, \varphi^1) \in V \times H$. *There exist positive constants* $T_0 = T_0(E_\varphi(0))$ *and* $C = C(E_\varphi(0))$, *such that every solution of (1.7), with* $\varphi(0) = \varphi^0$ *and* $\varphi'(0) = \varphi^1$, *satisfies:*

$$|\varphi_t(0)|^2 + ||\varphi(0)||^2 \leq C \int_0^{T_0} |B^{\frac{1}{2}} \varphi_t(t)|^2 \, dt. \tag{2.14}$$

jj) *Let* $\{y^0, y^1\} \in V \times H$. *There exist positive constants* $M = M(E_y(0))$, *and* $\lambda = \lambda(E_y(0))$ *such that every solution of (1.6), with* $y(0) = y^0$ *and* $y_t(0) = y^1$, *satisfies:*

$$|y_t(t)|^2 + ||y(t)||^2 \leq M e^{-\lambda t} \left(|y^1|^2 + ||y^0||^2\right), \quad \forall t \geq 0. \tag{2.15}$$

As one can see, neither the observability estimate nor the stabilization result in Theorem 2 is uniform in the energy space. Therefore, it is fairly reasonable to wonder whether there exists a class of superlinear nonlinearities for which Theorem 2 holds uniformly in the energy space.

2.4 **Final remarks and open problems.** It should be noted that the method that we have developed can also be used to prove nonuniform observability estimates for some semilinear hyperbolic problems with superlinear nonlinearities (the nonlinearity h verifying (2.1)), from which new unique continuation results follow. We also note that we are able to apply the method to the hyperbolic equations above because we now have in the literature observability estimates for the underlying linear equations with a potential. The method is quite flexible and could also apply to semilinear Euler-Bernoulli or the elasticity equations once the analogue of the aforementioned observability estimates become available. More precisely, consider, for instance, the Euler-Bernoulli equation:

$$\begin{cases} \varphi_{tt} + \Delta^2 \varphi + V(x,t)\varphi = 0 \text{ in } \Omega \times (0, \infty) \\ \varphi = \dfrac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Gamma \times (0, \infty), \end{cases} \tag{2.16}$$

where the potential V is, say, a bounded measurable function.

To the best of the author's knowledge, the observability estimate: Given $T > 0$, there exists $C > 0$: $||\varphi(0)||^2_{L^2(\Omega)} + ||\varphi_t(0)||^2_{H^{-2}(\Omega)} \leq C \int_0^T \int_\omega |\varphi(x,t)|^2 \, dxdt$, is an open problem.

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