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NEW COMPARISON PRINCIPLE WITH RAZUMIKHIN CONDITION FOR IMPULSIVE INFINITE DELAY DIFFERENTIAL SYSTEMS

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Abstract. In this paper, we will develop a comparison principle with Razumikhin condition relative to stability theory of impulsive functional differential system with infinite delay in terms of two different measures.

1. **Introduction.** It is known that the theory of impulsive functional differential systems provides a natural framework for mathematical model of many real-world phenomena, such as project, communication, biology, economy and so on. There has been a significant development in recent years[1-6]. Since the interval $(-\infty, t_0]$ is not compact so there should be some difficulty in the study of impulsive differential systems with infinite delay. Though, must authors have done more[1,2,3,5],most of which are about existence and stable conclusions with Lyapunov direct method. As is known, impulsive functional differential systems with state-dependent impulses as an extension of systems with fixed impulses have more application.[9] However, acted by "delay" and "impulse", it is more difficult to investigate this kind of systems. Up to now, the results about these impulsive functional differential systems are really few, and most of them are under the assumption that every solution meets each surface exactly once.

Therefore, it is necessary to study this system with so called "pulse phenomena".[10] In this paper, we will give some conclusions by method of comparison composed with Razumikhin technique,which improve some earlier results.

2. **Definitions and marks.** Let R^n be the n-dimensional Euclidean space,$R = (-\infty, +\infty), R^+ = [0, +\infty)$. For all $x \in R^n$, let $\|x\|$ be the norm in Euclidean space.

Consider the following impulsive functional differential system

$$\begin{cases} x'(t) &= f(t, x_t), & t \neq \tau_k(x(t^-)), (2.1) \\ x(t) &= x(t^-) + I_k(x(t^-)), & t = \tau_k(x(t^-)), (2.2) \\ x_{t_0} &= \varphi_0, & t_0 \geq t^*, \qquad (2.3) \end{cases} \qquad (I)$$

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where $x_t = x(t+s), t \geq t^*, s \in (-\infty, 0], \tau_k(x) < \tau_{k+1}(x), k = 1, 2, \cdots$,and $\lim_{k\to+\infty} \tau_k(x) = +\infty$.

For $\forall I \subset R$,definite $PC[I, R^n] = \{x(t) : I \to R^n, x(t)$ is continuous except at points $t = \tau_k(x(t^-))(k = 1, 2, \cdots)$, where $x(t)$ is right continuous $\}$.Let $PC(0) = PC[(-\infty, 0], R^n] = PC$.For all $\varphi \in PC$, define $\|\varphi\| = \sup_{s\in(-\infty,0]} \|\varphi(s)\|$.

Definition 2.1. A function $x(t)$ is said to be a solution of (2.1)$\sim$(2.2)with given initial function $\varphi_0 \in PC$ at $t = t_0$, if there exist $\alpha > 0$, such that

(i)(2.1)$\sim$(2.3)are satisfied;

(ii)$x(t)$ is continuous at points $t \in [t_0, \alpha)\backslash\{t = \tau_k(x(t^-))\}_{k=1}^{\infty}$;

(iii)$x(t)$ is right continuous at points $\{t = \tau_k(x(t^-))\}_{k=1}^{\infty}$.

We assume that the system (I) satisfies certain conditions to guarantee the global existence and uniqueness of solutions[6],and every solution can beat each impulsive surface several times at most in turn,$t_0 \neq \tau_k(x(t_0)), k = 1, 2, \cdots$.

Definition 2.2. Definite the following kind of functions:

$K = \{a(s) \in C[R^+,\ R^+] :\ a(0) = 0$ and $a(s)$ is strictly increasing in $s\}$,

$CK = \{a \in C[R^{2+},\ R^+] :\ a(t, w) \in K$ for each $t \in R^+, \}$,

$\Sigma = \{Q(u) \in C^1[R, R^+], Q(0) = 0, Q(u)$is strictly increase in $u\}$,

$\Gamma = \{h \in C[R \times R^n,\ R^+] :\ \inf_{(t,x)} h(t, x) = 0,\ (t, x) \in R \times R^n\}$,

$S_k = \{(t, x) \in R \times R^n, t = \tau_k(x(t^-)), k = 1, 2, ...\}$,

$G_k = \{(t, x(t)) \in R \times R^n, t$ belongs to the left closed and right open interval conducted by the kth and $(k+1)$th impulsive time which produced by $x(t)\}$,

$G = \bigcup_{k=1}^{\infty} G_k$.

Definition 2.3. $V : R \times R^n \to R^+$, is said to belong to the class of ν^0,if
$(i)V(t, x)$ is continuous in G_k and locally Lipschitzian in x for every t;
(ii) For all $(t_k, x) \in S_k$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y)$ exists, $\ k = 1, 2, \cdots$.

For any function $V \in \nu^0$, we define the function

$$D^+V(t, x(t)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x(t) + hf(t, x_t)) - V(t, x(t))],$$

where $x(t) = x(t, t_0, \varphi_0)$ is any solution of the system (I).

Definition 2.4 Let $h^0, h \in \Gamma, \varphi_0 \in PC$, set $h_0(t, \varphi_0) = \sup_{s\leq 0}\{h^0(t+s, \varphi_0(s))\}$. The impulsive functional differential system (I) is said to be
(i) $(h_0, h)-$stable, if for $\forall\ \epsilon > 0,\ \forall t_0 \geq t^*$, there exists $\delta = \delta(t_0, \epsilon) > 0$, such that $h_0(t_0, \varphi_0) < \delta$ implies $h(t, x(t)) < \epsilon,\ t \geq t_0$;
(ii) $(h_0, h)-$uniformly stable, if the δ in (i) is independent of t_0.

3. **Main result.**

Lemma 3.1. [7] *Assume that*

(i) $m(t) \in C[R, R]$,and $D^+m(t) \leq g(t, m(t))$,while $m(s) \leq m(t), s \leq t$,where $g \in C[R \times R, R^+]$,$D^+m(t)$ is any Dini derivative of $m(t)$;

(ii) Let $u(t) = u(t, t_0, u_0)$ is the maximal solution on $[t_0, \infty)$ of the system (II)

$$\begin{cases} u^{'}(t) &= g(t, u), (3.1) \\ u(t_0) &= u_0 \geq 0, (3.2) \end{cases} \qquad (II)$$

then $m(s) \leq u_0, s \leq t$ implies $m(t) \leq u(t), t \geq t_0$.

Theorem 3.2. *Assume that*

(i)$g \in C[R \times R, R^+], u(t) = u(t, t_0, u_0)$is the maximal solution of the system (II) on $[t_0, \infty)$;

(ii)$V \in \nu^0$,and if $x(t)$ is the solution of the system (I),we have

$$D^+V(t, x(t)) \leq g(t, V(t, x(t))), (t, x) \in G, \tag{3.3}$$

while $V(t, x(t)) \leq V(s, x(s)), s \leq t$;

(iii)$V(t, x(t)) = V(t, x(t^-) + I_k(x(t^-))) \leq V(t, x(t^-)), (t, x) \in S_k, k = 1, 2, \cdots$, then $\sup\limits_{s \leq 0} V(t_0 + s, \varphi_0(s)) \leq u_0$ implies $V(t, x(t)) \leq u(t), t \geq t_0$.

Proof. If $x(t)$ is the solution of the system (I)since $t_0 \neq \tau_k(x(t_0)), k = 1, 2 \cdots$, there must be $k_0, k_1, k_0 \leq k_1$, such that $\tau_{k_0}(x(t_0)) < t_0 < \tau_{k_1}(x(t_0))$. Let $t_i = \tau_{k_i}(x(t_i^-))\tau_{k_i}$,$\tau_{k_j}$can be sameand assume $t_i < t_{i+1}$.Let $m(t) = V(t, x(t))$.

Consider the interval $[t_0, t_1)$. For $\forall s \leq t_0$,$m(s) = V(s, x(s)) \leq \sup\limits_{s \leq 0} V(t_0 + s, \varphi_0) \leq u_0$, in view of Lemma 3.1, it is easy to know that $m(t) \leq u(t), t \in [t_0, t_1)$.

At present , $m(t_1) \leq V(t_1, x(t_1^-)) \leq u(t_1, t_0, u_0) \doteq u_1$.

Consider the interval $[t_1, t_2)$, since for $\forall s \leq t_1, m(s) \leq u_1$, again, in view of Lemma 3.1, we have $m(t) \leq u_1(t, t_1, u_1)$, where $u_1(t, t_1, u_1)$ is the maximal solution of the system (II) with initial point (t_1, u_1) on $[t_1, t_2)$. At this time, $m(t_2) \leq V(t_2, x(t_2^-)) \leq u_1(t_2, t_1, u(t_1)) \doteq u_2$.

Repeating, we can get $m(t) \leq u_i(t, t_i, u_i), t \in [t_i, t_{i+1})$,$i = 1, 2, \cdots$, where $u_i(t, t_i, u_i)$is the maximal solution of the system (II) with initial point (t_i, u_i) on $[t_i, t_{i+1})$, and $u_i = u_{i-1}(t_i, t_{i-1}, u_{i-1})$.

$$\text{Let } r(t) = \begin{cases} u(t), & t \in [t_0, t_1) \\ u_1(t, t_1, u_1), & t \in [t_1, t_2) \\ \quad\cdots\cdots & \\ u_i(t, t_{i-1}, u_{i-1}), & t \in [t_{i-1}, t_i), \quad i = 2, 3, 4 \cdots \end{cases}$$

then, $r(t)$ is the solution of the system $(II)$with initial point (t_0, u_0), and $m(t) \leq r(t)$, since $r(t) \leq u(t)$, then $m(t) \leq u(t), t \geq t_0$, the proof is completed. □

Definition 3.3. Let $Q_0, Q \in \Sigma$, the system (II) is said to be $(Q_0, Q)-$stable, if for $\forall \epsilon > 0$, there exists $\delta = \delta(t_0, \epsilon)$, such that $Q_0(u_0) < \delta_0$ implies $Q(u(t)) < \epsilon, t \geq t_0$, where $u(t)$ is any solution of the system (II).

Theorem 3.4. *Assume that*

(i) the condition $(i), (ii), (iii)$of Theorem 3.1 hold;

(ii) $Q_0, Q \in \Sigma, h^0, h \in \Gamma$, and there exist $\rho_0 > 0, \varphi \in K$ such that

$$h(t, x) \leq \varphi(h^0(t, x)), (t, x) \in S(h^0, \rho_0);$$

(iii) there exist $a, b \in K, \rho > 0$ such that

$$b(h(t, x)) \leq Q(V(t, x)), (t, x) \in S(h, \rho),$$

$$Q_0(V(t, x)) \leq a(h^0(t, x));$$

(iv) for the solution $x(t)$ of the system (I), $(t, x(t^-) + I_k(x(t^-))) \in S(h, \rho)$ while $(t, x(t^-)) \in S(h, \rho) \bigcap S_k$,

then , the (Q_0, Q)-stability of the system (II) implies the responding $(h_0, h)-$stability of the system (I).

Proof. We only prove the $(h_0, h)-$uniform stability of the system (I). If $x(t)$ is the solution of the system (I)since $t_0 \neq \tau_k(x(t_0)), k = 1, 2 \cdots$, there must be $k_0, k_1, k_0 \leq k_1$, such that $\tau_{k_0}(x(t_0)) < t_0 < \tau_{k_1}(x(t_0))$. Let $t_i = \tau_{k_i}(x(t_i^-))\tau_{k_i}$,$\tau_{k_j}$can be sameand assume $t_i < t_{i+1}$ and $m(t) = V(t, x(t))$.

For $\forall\epsilon > 0, (\epsilon < \rho), \forall t_0 \geq t^*$, let the system (II) is $(Q_0, Q)-$uniformly stable, then for above $\epsilon > 0, t_0$, there exists $\delta_1 = \delta_1(\epsilon) > 0$, such that $Q_0(u_0) < \delta_1$ implies

$$Q(u(t)) < b(\epsilon), t \geq t_0, \tag{3.4}$$

where $u(t)$ is any solution of the system (II) with given initial point (t_0, u_0).

Choose $\delta < \min\{\rho_0, \varphi^{-1}(\epsilon), a^{-1}(\delta_1)\}$, then if $h_0(t_0, \varphi_0) < \delta$we have

$$h(t_0, x_0) \leq \varphi(h^0(t_0, x_0)) = \varphi(h^0(t_0, \varphi_0(0))) \leq \varphi(h_0(t_0, \varphi_0)) < \varphi(\delta) < \epsilon.$$

Now we will prove for above $\epsilon, \delta, t_0,\ h_0(t_0, \varphi_0) < \delta$ implies

$$h(t, x(t)) < \epsilon, t \geq t_0, \tag{3.5}$$

where $x(t)$ is the solution of the system (I).

If (3.5) is not holdthen there must exist $i \geq 0, t^0 > t_0, t^0 \in [t_i, t_{i+1})$, such that $h(t^0, x(t^0)) \geq \epsilon$ and $h(t, x(t)) < \epsilon, t \in [t_0, t_i)$.

If $t_i < t^0 < t_{i+1}$,then $h(t^0, x(t^0)) = \epsilon < \rho$, so $h(t, x(t)) < \rho, t \in [t_0, t^0]$;

If $t^0 = t_i$, then $h(t_i^-, x(t_i^-)) < \epsilon < \rho$, from (iv) we have $h(t, x(t)) < \rho$, so $h(t, x(t)) < \rho, t \in [t_0, t^0]$.

Considering the interval $[t_0, t^0]$, we choose $u_0 = \sup\limits_{s\in(-\infty,0]} V(t_0 + s, \varphi_0(s))$, by Theorem 3.2 we have

$$V(t, x(t)) \leq r(t), t \in [t_0, t^0],$$

where $r(t)$ is the maximal solution of the system (II) on $[t_0, \infty)$.

Now,

$$\begin{aligned} Q_0(u_0) &= Q_0(\sup_{s\in(-\infty,0]} V(t_0 + s, \varphi_0(s))) \leq \sup_{s\in(-\infty,0]} a(h^0(t_0 + s, \varphi_0(s))) \\ &\leq a(h_0(t_0, \varphi_0)) \leq a(\delta) < \delta_1, \end{aligned}$$

then by (3.4) we get $Q(r(t, t_0, u_0)) < b(\epsilon), t \geq t_0$. Especially,$Q(r(t^0, t_0, u_0)) < b(\epsilon)$, But $b(\epsilon) \leq b(h(t^0, x(t^0))) \leq Q(V(t^0, x(t^0))) \leq Q(r(t^0, t_0, u_0)) < b(\epsilon)$, which is impossible. The proof is completed. ☐

4. **Example.** To verify our theorems, we consider the following system:

$$\begin{cases} x_1{}' &= e^{-t}x_1 + x_2 sint - (x_1^3 + x_1x_2^2)e^t + \int_{-\infty}^t e^{-2t+s}x_1(s)ds, \quad t \neq \tau_k(x(t^-)) \\ x_2{}' &= x_1 sint + e^{-t}x_2 - (x_1^2x_2 + x_2^3)e^t + \int_{-\infty}^t e^{-2t+s}x_2(s)ds, \quad t \neq \tau_k(x(t^-)) \\ \Delta x_1 &= -\dfrac{1}{2}x_1 + \dfrac{1}{2}x_2, \quad t = \tau_k(x(t^-)), \\ \Delta x_2 &= \dfrac{1}{2}x_1 - \dfrac{1}{2}x_2, \quad t = \tau_k(x(t^-)), \\ x_{t_0} &= \varphi(s), \end{cases} \tag{I$'$}$$

where $\tau_1(x) < \tau_2(x) < \cdots < \tau_k(x) < \cdots, \lim\limits_{k\to\infty} \tau_k(x) = \infty$.

Let $V(t, x(t)) = \dfrac{1}{2}(x_1 + x_2)^2, h^0 = h = (x_1 + x_2)^2, Q(s) = Q_0(s) = 2s$.

Then, $Q_0(V) = Q(V) = (x_1 + x_2)^2 = h^0 = h$,

$$\begin{array}{rcl} D^+V(t,x(t)) & = & (x_1(t)+x_2(t))(x_1'(t)+x_2'(t) \\ & \leq & (x_1(t)+x_2(t))^2(e^{-t}+sint) \\ & & +(x_1+x_2)(t)\int_{-\infty}^{t} e^{-2t+s}(x_1(s)+x_2(s)ds \end{array}$$

When $V(t,x(t)) \geq V(s,x(s)), s \leq t$, we have

$$\frac{1}{2}(x_1(t)+x_2(t))^2 \geq \frac{1}{2}(x_1(s)+x_2(s))^2,$$

that is

$$|x_1(t)+x_2(t)| \geq |x_1(s)+x_2(s)|,$$

at this time ,we have

$$\begin{array}{rcl} D^+V(t,x(t)) & \leq & (x_1(t)+x_2(t))^2(e^{-t}+sint)+(x_1(t)+x_2(t))^2\int_{-\infty}^{t} e^{-2t+s}ds \\ & = & (x_1(t)+x_2(t))^2(e^{-t}+sint)+(x_1(t)+x_2(t))^2e^{-2t}\int_{-\infty}^{t} e^{s}ds \\ & = & (x_1(t)+x_2(t))^2(2e^{-t}+sint) \\ & = & 2V(t,x(t))(e^{-t}+sint) \end{array}$$

We choose the following comparison system

$$\begin{cases} u'(t) = 2u(t)(e^{-t}+sint) \\ u(t_0) = \sup\limits_{s\leq 0} V(t_0+s,\varphi(s)) \end{cases} \qquad (II)'$$

The solution of $(II)'$ is

$$u(t) = u(t_0)exp(2e^{-t} - 2cost + 2e^{-t_0} + 2cost_0),$$

it is easy to know that $u(t)$ is (Q_0, Q)-stable. When $t = \tau_k(x(t^-))$,we have

$$V(t,x(t)) = V(t,x(t^-)),$$

by Theorem 3.4,the system$(I)'$ is (h_0, h)-stable.

REFERENCES

[1] Zhiguo Luo and Jianhua Shen, *Stability results for impulsive functional differential equations with infinite delays*, J.Com.and Appl.Math., **131** (2001), 55–64.

[2] Zhiguo Luo and Jianhua Shen, *Stability of functional differential equations with infinite delays*, Appl.Math.J.Chinese Univ.Ser.B, **20** (2005), 142–150.

[3] Zhiguo Luo and Jianhua Shen, *Stability and boundedness results for impulsive functional differential equations with infinite delays*, Nonlinear Analysis, **46** (2001), 475–493.

[4] Shaoping Liu, *Stability of impulsive delay differential equations*, Mathematical Application, **18** (2005), 28–32.

[5] Yu Zhang and Jitao Sun, *Stability of impulsive infinite delay differential equations* , Applied Mathematics Letters, **19** (2006), 1100–1106.

[6] Xilin Fu and Baoqiang Yan, *The global solution of impulsive retarded functional differential equations*, International Journal of Applied Mathematics, **3** (2000), 389–398.

[7] X.Liu andD.Xu, *Uniform Asymptotic Stability of Abstract Functional Differential Equations*, J.Math.Anal.Appl., **216** (1997), 626–643.

[8] V.Lakshmikantham and D.D.Bainov,PS.Simeonov, “Theory of Impulsive Differential Equations,” 1^{nd} edition, World Scientific,Singapore, 1989.

[9] Lin Wang and Xilin Fu, *A new comparison principle for impulsive differential systems with variable impulsive perturbations and stability theory* , Com.and Math.with Appl., **54** (2007), 730–736.

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