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A REMARK ON BLOW-UP AT SPACE INFINITY

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Abstract. In this note we discuss blow-up at space infinity for quasilinear parabolic equation $u_t = \Delta u^m + u^p$. It is known that if initial data is not a constant and takes its maximum at space infinity in a certain sense, the solution blows up only at space infinity at minimal blow-up time. We show that if $m \geq 1$ and a solution blows up at minimal blow-up time, then it blows up completely at the blow-up time.

1. **Introduction.** Let us consider the Cauchy problem for quasilinear parabolic equations

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1}$$

where $p > 1$, $m > 0$ are constants and initial data $u_0 \not\equiv 0$ is a nonnegative bounded continuous function in $\mathbf{R}^N$.

We begin with the definition of weak solution.

Definition 1.1 (*weak solution*)**.** Let G be a domain in $\mathbf{R}^N$. A function $u = u(x,t)$ defined on $\bar{G} \times [0,T)$ is said to be a weak solution of (1) in $G \times ([0,T)$ if:
(i) $u(x,t) \geq 0$ in $\bar{G} \times [0,T)$ and $u \in BC(\bar{G} \times [0,\tau])$ for each $0 < \tau < T$;
(ii) For any bounded domain $\Omega \subset G$ with smooth boundary $\partial\Omega$, $0 < \tau < T$ and nonnegative $\eta \in C^{1,0}(\bar{\Omega} \times [0,T)) \cap C^{2,1}(\Omega \times [0,T))$ which vanishes on $\partial\Omega$,

$$\int_\Omega u(x,\tau)\eta(x,\tau)dx - \int_\Omega u(x,0)\eta(x,0)dx = \int_0^\tau \int_\Omega \{u\partial_t\eta - u^m\Delta\eta + u^p\eta\}dxdt - \int_0^\tau \int_{\partial\Omega} u^m \partial_\nu \eta dSdt, \tag{2}$$

where ν denotes the outer unit normal to the boundary.

A supersolution [or subsolution] is similarly defined with the equality of (2) replaced by $\geq$ [or $\leq$].

Problem (1) admits a unique weak solution u in $\mathbf{R}^N \times [0,T)$ for some $T \in (0,\infty]$ and it can be prolonged in time as long as it remains bounded. ([17, 15, 5, 3, 6, 4]). We set

$$T(u_0) = \sup\{T \in [0,\infty) | \sup_{0<t<T} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} < \infty\}.$$

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If $T(u_0) = \infty$, then the solution is said to be a global (weak) solution problem (1), and if $T(u_0) < \infty$, then the solution is said to blow up at the time $T(u_0)$. In this case, it is readily seen that

$$\lim_{t \nearrow T(u_0)} \|u(\cdot, t)\|_\infty = \infty.$$

The blow-up set of solution u is defined as the set of all the points $a \in \mathbf{R}^N$ for which u is not locally bounded at $(a, T(u_0))$.

The simplest example of blowing up solution is a flat one solving ordinary differential equation

$$\frac{dv}{dt} = v^p, \quad t > 0, \quad v(0) = M, \tag{3}$$

where M is a positive constant. Henceforth we denote the solution of (3) by v_M. A simple computation reveals that

$$v_M(t) = \kappa(T_M - t)^{-1/(p-1)} \ \text{ with } \ \kappa = (p-1)^{-1/(p-1)} \text{ and } T_M = \frac{1}{(p-1)M^{p-1}}.$$

A solution u of problem (1) is said to blow up at space infinity if there exists a sequence $\{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T(u_0))$ such that

$$|x_n| \to \infty, \ t_n \nearrow T(u_0) \ \text{ and } \ u(x_n, t_n) \to \infty \text{ as } n \to \infty. \tag{4}$$

Blow-up at space infinity was initially discussed in [14] for one-dimensional semilinear problems (see also [8] for related results) and then in [9, 10] for semilinear equations $u_t = \Delta u + f(u)$ in general space dimension. As is readily seen from this definition, if u blows up at minimal blow-up time, it blows up at space infinity in some directions. To be more precise, a direction $\psi \in S^{N-1}$, where S^{N-1} is the $(N-1)$-dimensional unit sphere, is said to be a blow-up direction if there exists a sequence $\{(x_n, t_n)\} \subset \mathbf{R}^N \times (0, T(u_0))$ satisfying (4) and $x_n/|x_n| \to \psi$ as $n \to \infty$. We note that this notion was first introduced in [10] and that the blow-up directions are characterized there by initial data.

The results of [9, 10] were generalized to quasilinear equations including the equation of (1) with $m \geq 1$ in [19] and with $m \leq 1$ in [18] as a typical equation. Moreover, in these articles, the authors introduced the notion of minimal blow-up time and showed necessary and sufficient conditions on initial data for a solution of (1) to have a minimal blow-up time. We would like to emphasize that [19, 18] are the only articles which give a characterization on initial data for a solution to have a minimal blow-up time in all the articles we have introduced (See also [11]).

We shall recall the notion of minimal blow-up time and some properties of the solutions with minimal blow-up times. Hereafter we select $M = \|u_0\|_\infty$. A solution u of problem (1) with initial data u_0 is said to blow up at *minimal blow-up time* or *the least (possible) blow-up time* if $T(u_0) = T_M$.

In order to state the characterization result obtained in [19, 18], let us introduce the following condition on initial data. A direction $\psi \in S^{N-1}$ is said to be a *direction of mean convergence* of u_0 (to M) if there exists a sequence $\{x_n\} \subset \mathbf{R}^N$ such that

$$|x_n| \to \infty, \ \frac{x_n}{|x_n|} \to \psi \text{ and } u_0(x + x_n) \to \|u_0\|_\infty \ \text{ a.e. in } \mathbf{R}^N \text{ as } n \to \infty. \tag{5}$$

Henceforth we denote by $B_R(a)$ the open ball in $\mathbf{R}^N$ with radius $R > 0$ centered at $a \in \mathbf{R}^N$ and write $B_R = B_R(0)$ for simplicity.

Proposition 1. *Suppose that $\psi \in S^{N-1}$ is a direction of mean convergence of u_0. Then the solution u of problem (1) blows up at minimal blow-up time and ψ is a blow-up direction of u. Moreover, if $m \geq 1$, then u satisfies the condition that*

$$\lim_{n\to\infty} \sup_{x\in B_R(x_n)} |u(x,t) - v_M(t)| = 0 \tag{6}$$

uniformly on each compact sets of $\{0 < t < T_M\}$ for each $R > 0$, where $\{x_n\}$ is a sequence satisfying the condition (5).

Proposition 2. *Assume that $p > \max\{m, 1\}$. Let u be a solution of problem (1) that has a minimal blow-up time. Then the following hold:*

(i) *If the initial data u_0 is not a constant, then the solution u blows up only at space infinity, that is, the blow-up set is empty.*

(ii) *A direction is a blow-up direction of u if and only if it is a direction of mean convergence of u_0.*

Moreover, if $m \geq 1$, then for any direction of mean convergence $\psi \in S^{N-1}$ and any $R > 0$, the solution u satisfies (6).

Proposition 3. *Assume that $p > \max\{m, 1\}$. Then a solution u of problem (1) blows up at minimal blow-up time if and only if the initial data u_0 has at least one direction of mean convergence.*

We would like to consider the behavior of the solutions beyond the blow-up times. Weak solutions, of course, do not make sense after blow-up times, but there is a notion of solution which enables us to discuss the possibility of extending solutions of (1) beyond blow-up times ([2, 7, 21]).

Following [7, 21], we shall define the notion of generalized solution. Let u_n be a unique solution to the problem

$$\begin{cases} (u_n)_t = \Delta u_n^m + f_n(u_n), & \text{in } \mathbf{R}^N \times (0,\infty), \\ u_n(x,0) = u_0(x), & \text{in } \mathbf{R}^N, \end{cases} \tag{7}$$

$n = 1, 2, \ldots$, where $f_n(u) := \min\{n, u^p\}$. Note that for each n, the problem (7) has a global weak solution by virtue of globally Lipschitz continuity of the term f_n.

Definition 1.2 (*proper solution, complete blow-up*)**.** Given initial data u_0, we call the limit function $u(x,t) := \lim_{n\to\infty} u_n(x,t) \in [0,\infty]$ a *proper solution* of problem (1). If $u \equiv \infty$ in $\mathbf{R}^N \times (T(u_0),\infty)$, we say that the solution u blows up completely at $t = T(u_0)$ and otherwise we say that incomplete blow-up occurs at $t = T(u_0)$. Following [21], we set

$$t_c(u_0) = \inf\{T \in [0,\infty] \mid u(x,t) \equiv \infty \text{ for each } (x,t) \in \mathbf{R}^N \times (T,\infty)\}$$

and call it *complete blow-up time*.

Clearly, any proper solution agrees with weak solution with the same data in $\mathbf{R}^N \times (0, T(u_0))$ and $T(u_0) \leq t_c(u_0)$.

Galaktionov and Vazquez [7] defined proper solution in the general framework using order-preserving semigroups acting on ordered topological space in order to deal with quasilinear equations not necessarily endowed with semilinear structures. Our definition of proper solution agrees with theirs, once we apply their general results to a suitable function space. Moreover, they applied the general results to the space of nonnegative, measurable functions and proved that when $p + m \leq 2$, incomplete blow-up always occurs and when $2 - m < p \leq m(N+2)/[N-2]_+$, every blowing up solution blows up completely at the blow-up time. We recall some related results for the Cauchy or the Cauchy-Dirichlet problem in a bounded convex domain Ω. For a semilinear heat equation

$$u_t = \Delta u + u^p, \quad p > 1, \tag{8}$$

Baras and Cohen [2] proved that all nonnegative blow-up solutions of the Cauchy-Dirichlet problem for blow up completely at the blow-up times if $p < p_s := (N+2)/[N-2]_+$ or $\Delta u_0 + u_0^p \geq 0$ in the distributional sense. Suzuki [21] generalized their results to the quasilinear equation $u_t = \Delta u^m + u^p$. He also obtained similar results for the corresponding Cauchy problem in $\mathbf{R}^N$.

Recently, M. Shimojō [20] studied the Cauchy problem for semilinear heat equations (8), which corresponds to the case of $m = 1$ in the equation of (1). He gave a sufficient (but not necessary) condition on initial data for a solution to have a minimal blow-up time in our notation and proved that blow-up occurs only at space infinity for initial data satisfying the condition. He not only proved that the blow-up sets of such solutions are empty but also discussed blow-up profiles at the blow-up time and, moreover, showed that such blow-up is always complete at the time.

Our aim is to show that for any initial data that provides a solution of problem (1) with minimal blow-up time, the blow-up is surely complete, that is, if a solution u of problem (1) has a minimal blow-up time T_M, it blows up completely at the blow-up time. In other words, it cannot be prolonged beyond the time as a proper solution that takes finite values. We are now in the position to state our main result.

Theorem 1.3. *Assume that $p \geq m > 1$ or $m = 1$. Suppose that a solution u of problem (1) blows up at minimal blow-up time T_M. Then u blows up completely at the time. Namely, the complete blow-up time $t_c(u_0)$ coincides with T_M.*

Remark 1. We have assumed the condition of $p > m$ in Propositions 2 and 3, but the assumption is weaken so as to include the case $p = m > 1$ in this theorem.

2. **Proof of Theorem.** Let $\tau > 0, R > 0$ and set $\epsilon = e^{-R}$. Let $w_R(x,t)$ be the solution to the problem

$$\begin{cases} w_t = \Delta w^m + w^p, & \text{in } B_R \times (\tau, T), \\ w(x,t) = 0, & \text{on } \partial B_R \times (\tau, T), \\ w(x,\tau) = w_{0,R}(x), & \text{in } B_R, \end{cases} \tag{9}$$

where $w_{0,R}$ is a smooth, bounded, radial nonincreasing function on B_R satisfying

$$w_{0,R}(x) = \begin{cases} v_M(\tau) - \epsilon, & x \in B_{(1-\epsilon)R}, \\ 0, & x \in \partial B_R, \end{cases}$$

$w_{0,R} \leq v_M(\tau) - \epsilon$ in B_R and $\int_{B_R} w_{0,R}(x)\psi(x)dx \geq v_M(\tau) - 2\epsilon$. Denote by $T(R)$ the blow-up time of w_R, that is,

$$T(R) \equiv \sup\{T > 0; \sup_{0<t<T} \|w_R(\cdot,t)\|_{L^\infty(B_R)} < \infty\}. \tag{10}$$

Lemma 2.1. *Assume the same hypotheses as in Theorem 1.3 and take a sequence $\{x_n\} \subset \boldsymbol{R}^N$ satisfying condition (5). Let u be a proper solution with initial data u_0. Then, for any $\mu > 0$, there exist sufficiently large $R > 0$ and $n_0 \in \mathbf{N}$ such that*

$$u(x,t) = \infty \quad \text{in } B_R(x_n) \times (T_M + \mu, \infty) \tag{11}$$

provided $n \geq n_0$.

Proof. For any $R > 0$, we have that

$$\lim_{n\to\infty} \sup_{x\in B_R(x_n)} |u(x,t) - v_M(t)| = 0$$

uniformly on each compact sets of $\{0 < t < T_M\}$ by Proposition A. Then there exists an $n_0 \in \mathbf{N}$ such that

$$w_{0,R}(x - x_n) \le v_M(\tau) - \epsilon < u(x,\tau) \quad \text{for } x \in B_R(x_n) \quad \text{provided } n \ge n_0.$$

Hence we deduce that

$$w_R(x - x_n, t) \le u(x,t) \quad \text{for } (x,t) \in B_R(x_n) \times (\tau, T_M) \quad \text{provided } n \ge n_0$$

by the comparison theorem.

We use the well-known eigenfunction method developed in [13, 12, 21]. Let λ_R be the principal eigenvalue of $-\Delta$ in B_R with homogeneous Dirichlet boundary condition and let $\psi_R(x) > 0$ be the corresponding eigenfunction normalized as $\int_{B_R} \psi_R(x)dx = 1$. We set

$$J_R(t) = \int_{B_R} w_R(x,t)\psi_R(x)dx, \quad t > 0,$$

and define a nondecreasing convex function Γ_R as follows: When $p > m$,

$$\Gamma_R(\xi) \equiv \begin{cases} \xi^p - \lambda_R \xi^m, & \textit{if} \quad \left(\dfrac{\lambda_R m}{p}\right)^{1/(p-m)} \le \xi, \\ -\left(\dfrac{\lambda_R m}{p}\right)^{m/(p-m)} \dfrac{\lambda_R(p-m)}{p}, & \textit{if} \quad 0 \le \xi \le \left(\dfrac{\lambda_R m}{p}\right)^{1/(p-m)} \end{cases}$$

and when $p = m > 1$, $\Gamma_R(\xi) = (1 - \lambda_R)\xi^p$ for any $\xi \ge 0$. Here we may assume that $\lambda_R < 1$, taking R sufficiently large. For $\xi > 0$, we set

$$\tilde{t}_R(\xi) = \int_\xi^\infty \frac{ds}{\Gamma_R(s)}.$$

We then make use of [21, Proposition 3.8], to obtain

$$w_R(x,t) = \infty \quad \text{in } B_R(0) \times (\tilde{t}_R(J_R(\tau)), \infty).$$

Here we used the convexity of the function Γ_R. We have then obtained

$$w_R(x - x_n, t) = \infty \quad \text{in } B_R(x_n) \times (\tilde{t}_R(J_R(\tau)), \infty).$$

The conclusion (11) then follows at once if we show that

$$\tilde{t}_R(J_R(\tau)) \le T_{v_M(\tau)} + \mu \tag{12}$$

for some $R > 0$, since $T_M = \tau + T_{v_M(\tau)}$. Note that, if $R > 0$ is sufficiently large, $\Gamma_R(\xi) = \xi^p - \lambda_R\xi^m$ for $\xi \ge M - 2\epsilon = M - 2e^{-R}$. Since $J_R(\tau) \ge M - 2e^{-R}$,

$$\tilde{t}_R(J_R(\tau)) = \int_{J_R(\tau)}^\infty \frac{d\xi}{\xi^p - \lambda_R\xi^m} \to T_{v_M(\tau)} \quad \text{as } R \to \infty.$$

We thus obtain (12). □

Proof of Theorem. Applying Lemma 2.1 and [21, Proposition 3.5], we have

$$u(x,t) = \infty \quad \text{in } \mathbf{R}^N \times (T_M + \mu, \infty),$$

which implies $t_c(u_0) \le T_M + \mu$. Since μ is arbitrary, we conclude $t_c(u_0) = T_M$, which completes the proof. □

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