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NONCOMMUTATIVE AKNS SYSTEMS AND MULTISOLITON SOLUTIONS TO THE MATRIX SINE-GORDON EQUATION

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Abstract. The main result is a very general solution formula for the noncommutative AKNS system, extending work by Bauhardt and Pöppe. As an application, we construct for the matrix sine-Gordon equation N-soliton solutions analogous to the multisoliton solutions for the KdV equation due to Goncharenko.

1. **Introduction.** In his pioneering work Marchenko [15] discovered how to apply operator theory to the study of integrable systems. The principal idea is to first study noncommutative (nc) equations and their solutions and to return then to the scalar setting via projection techniques. Later a brilliant observation of B. Carl [4], see also [8], made it possible to exploit the full power of advanced Banach space theory, and gave solution formulas, which are general enough to contain, at least for the KdV equation, all solutions accessible to the usual Inverse Scattering Method (see [7]). Moreover it turned out that these solution formulas are well suited for the study of complicated solution families, leading to quite definitive results on multipole solutions and countable superpositions of solitons (see [8], [20] for a survey and references). On the other hand, nc integrable systems for their own sake have become an active subject in the past decade. We refer to [10], [11], [13], [16], [17], [18] to select only a few papers related to our approach.

At first the implementation of the method for the classical soliton equations was done in a case-by-case way, the degree of complexity (and the number of ad-hoc choices) depending greatly on the equation under consideration. In view of many known results about integrable equations coming in families one might expect that new insight would be gained from a universal approach. Here we study the celebrated AKNS system [1], [22] (see also [2], [3]), a family of integro-differential systems which depends on a functional parameter and comprises the KdV, mKdV, sine-Gordon (SG) and Nonlinear Schrödinger (NLS) equations as particular reductions. It was observed by Bauhardt and Pöppe [6] that the AKNS system and its soliton-like solutions have very natural nc counterparts. The main goal of the present paper is to give a rigorous proof of an extended version of this result (we have to supply details at places where [6] argues formally, in order to justify our later applications). The extension aims at obtaining solution formulas with the

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right number of operator parameters and is crucial in obtaining the full family of soliton solutions (see Remark 2).

In Section 6 we explain the transition from the ncAKNS system to the ncSG and ncNLS equations (Finding nc solutions of these equations is a far from routine matter). This will also give us the opportunity to indicate some classical problems which can be attacked using our methods. For a thorough presentation, we have to refer to the author's habilitation thesis [20], see also [19], [21]. For interesting alternative approaches to solution formulas we refer to [5], [14]. In Section 7 we derive N-soliton solutions for the matrix SG equation as a quite particular case of our solution formulas. The family we obtain exactly corresponds to the multisoliton solutions of the matrix KdV equation constructed by Goncharenko [12].

2. **From the commutative to the ncAKNS system.** For the sake of motivation we will, following [1], [3], recall the basic facts on the scalar AKNS system. For given non-trivial polynomials f, $g \in \mathbb{C}[x]$ the AKNS system reads

$$g\left(T_{r,q} \right) \begin{pmatrix} r_t \\ q_t \end{pmatrix} = f\left(T_{r,q} \right) \begin{pmatrix} r \\ -q \end{pmatrix}. \tag{1}$$

It is an integro-differential system in two unknown functions $r(x,t)$, $q(x,t)$. Here $T_{r,q}$ denotes the (r,q)-dependent operator

$$T_{r,q} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_x - 2r\big(\int_{-\infty}^x qu \, d\xi + \int_{-\infty}^x rv \, d\xi \big) \\ -v_x + 2q\big(\int_{-\infty}^x qu \, d\xi + \int_{-\infty}^x rv \, d\xi \big) \end{pmatrix}. \tag{2}$$

Interpreting $\left(T_{r,q}\right)^n$ as n-fold iteration, we obtain operators $f\left(T_{r,q}\right)$, $g\left(T_{r,q}\right)$ acting on pairs of functions and arrive at (1) by inserting $(r,-q)^T$, $(r_t, q_t)^T$.

The right nc interpretation of the AKNS system is due to work of Bauhardt and Pöppe [6], who introduced the system

$$g\left(\mathcal{T}_{R,Q} \right) \begin{pmatrix} R_t \\ Q_t \end{pmatrix} = f\left(\mathcal{T}_{R,Q} \right) \begin{pmatrix} R \\ -Q \end{pmatrix}, \quad \text{where}$$

$$\mathcal{T}_{R,Q} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_x - \Big(R \int_{-\infty}^x (QU + VR) d\xi + \int_{-\infty}^x (UQ + RV) d\xi \, R \Big) \\ -V_x + \Big(Q \int_{-\infty}^x (UQ + RV) d\xi + \int_{-\infty}^x (QU + VR) d\xi \, Q \Big) \end{pmatrix}.$$

In [6] this system is formulated only for endomorphisms. However, it can be extended to operator-valued functions $R(x,t)$, $Q(x,t)$ taking values in $\mathcal{L}(F,E)$, $\mathcal{L}(E,F)$, where E, F are Banach spaces.

Note that the ncAKNS system with $Q(x,t)$, $R(x,t)$ mapping between different Banach spaces almost necessarily leads to the right nc interpretation. However, the most important advantage of this generalization is that our solution depends on *two* independent operator parameters, whereas one operator parameter would not be sufficient to obtain the full solution variety, confer Section 6.

As in the representation of the AKNS system as a compatibility condition of a corresponding linear system (see [1], [3], [22]), a more compact formulation can be obtained by looking at $G = E \oplus F$, the involution

$$J = \begin{pmatrix} I_E & 0 \\ 0 & -I_F \end{pmatrix} \in \mathcal{L}(G),$$

and interpreting the above operator-functions appropriately as $\mathcal{L}(G)$-valued operator-functions

$$S(x,t) = \begin{pmatrix} 0 & R(x,t) \\ Q(x,t) & 0 \end{pmatrix}, \qquad W(x,t) = \begin{pmatrix} 0 & U(x,t) \\ V(x,t) & 0 \end{pmatrix}.$$

With these conventions, the ncAKNS system can be rewritten in the form (3), (4) used in the present work.

3. **Solving the ncAKNS system.** Fix polynomials f, g. Let G be a Banach space and $J \in \mathcal{L}(G)$ an involution. We say that an $\mathcal{L}(G)$-valued function $S = S(x,t)$ is a solution of the ncAKNS system, if

$$g(\mathcal{R})(S_t) \; = \; f(\mathcal{R})(JS), \tag{3}$$

$$\text{where } \mathcal{R}(W) = J\left(W_x - \left\{S, \int_{-\infty}^{x} \{W,S\} d\xi \right\}\right), \tag{4}$$

and $\{W, S\}$ denotes the anticommutator of W and S. Note that $\mathcal{R}$ depends on S, making (3) truly nonlinear.

Because (3) depends on improper integrals, appropriate global conditions must be imposed in order that it is well defined. For $t_1 < t_2$ we may look at the space $\mathcal{F}_n(t_1, t_2)$ of functions $W(x,t)$ defined on $\mathbb{R} \times (t_1, t_2)$ satisfying the following properties:

1. $W(x,t)$ is $\mathcal{C}^1$ in t, and both $W(x,t)$ and $W_t(x,t)$ are $\mathcal{C}^n$ in x.
2. For each integer j, $0 \le j \le n-1$, we have convergence $|x| \, \|W^{(j)}(x,t)\|$ and $|x| \, \|W_t^{(j)}(x,t)\| \to 0$ for $x \to -\infty$, locally uniformly in t.

It is easy to check that both sides of (3) are defined for $S(x,t) \in \mathcal{F}_n(t_1,t_2)$ if $n = \max(\deg f, \deg g)$. To keep assumptions at a manageable size, we shall not include explicit conditions in the theorems but somewhat loosely require that operator functions are sufficiently smooth and behave sufficiently well for $x \to -\infty$. Actually in all concrete solution formulas we will obtain, the operator functions will even belong to the one-sided Schwartz space of functions for which all x-derivatives decrease faster than any power $|x|^{-k}$ for $x \to -\infty$.

Theorem 3.1. *Let $C \in \mathcal{L}(G)$ be a constant operator, commuting with J, such that the spectrum of C does not intersect the polar set of $f_0 = f/g$. Assume that $N = N(x,t)$ is an $\mathcal{L}(G)$-valued operator function, anticommuting with J, and satisfying the base equations*

$$N_x = JC\,N, \qquad \text{and} \qquad N_t = Jf_0(C)\,N. \tag{5}$$

Assume furthermore that, on a strip $\mathbb{R} \times (t_1, t_2)$, N is sufficiently smooth and behaves sufficiently well for $x \to -\infty$, and that $(I + N)$ is invertible on $\mathbb{R} \times (t_1, t_2)$.

Then

$$S = \frac{1}{2}\left[\, J, (I+N)^{-1} C (I+N) \,\right]. \tag{6}$$

is a solution of the ncAKNS system (3) on $\mathbb{R} \times (t_1, t_2)$.

Remark 1. a) Note that Theorem 3.1 does not claim the global existence of the solutions (6). In fact, it is known that soliton-like solutions of the AKNS system may explode in finite time [1]. In [20] natural sufficient conditions for global existence are given for the solutions to reduced equations which are obtained by imposing $r = -\overline{q}$ or $r = -q$ (for appropriate functional parameters f and g).

b) The base equations can be solved explicitly via $N(x,t) = \exp\big(JCx + Jf_0(C)t\big)D$, $D \in \mathcal{L}(G)$ an arbitrary constant operator. This indicates that the method has the potential for semigroup generalizations [9].

4. **Preparations.** To begin with we introduce some auxiliary notations. With respect to the involution J, the diagonal part $W^{(d)}$ and the off-diagonal part $W^{(od)}$ of an operator W are defined by

$$W^{(d)} = \frac{1}{2}\{J, JW\} \qquad \text{and} \qquad W^{(od)} = \frac{1}{2}[J, JW]. \tag{7}$$

Equivalently, $[J,W] = 2JW^{(od)}$ and $\{J,W\} = 2JW^{(d)}$. As usual, $[J,W] = JW - WJ$ and $\{J,W\} = JW + WJ$ stand for the commutator and the anticommutator of J and W, respectively. The following observation is obvious.

Lemma 4.1. *Diagonal and off-diagonal part $W^{(d)}$, $W^{(od)}$ of an operator W are characterized as the unique operators $A^{(d)}$, $A^{(od)}$ satisfying $W = A^{(d)} + A^{(od)}$, $\{J, A^{(od)}\} = 0$ and $[J, A^{(d)}] = 0$.*

Finally we need, for $n \in \mathbb{N}_0$, the operator hierarchies

$$\begin{aligned} T_n &= J(I+N)^{-1}\, C^n\, (I+N), & (8)\\ \widehat{T}_n &= J(I+N)^{-1}\, C^n f_0(C)\, (I+N). & (9) \end{aligned}$$

Note that with these notations, (6) reads $S = \frac{1}{2}[J, JT_1] = T_1^{(od)}$.

Some crucial relations for T_n, $\widehat{T}_n$ are collected below. The first one is obvious.

Proposition 4.2. *The following identity holds both for $W_j = T_j$ and $W_j = \widehat{T}_j$:*

$$W_{n+m} = W_n J T_m = T_n J W_m.$$

For the diagonal and the off-diagonal parts of the operator-functions, it translates as follows.

Corollary 4.3. *(Rules for the evaluation of products)*

$$\begin{aligned} JW_{n+m}^{(d)} &= W_n^{(d)} T_m^{(d)} - W_n^{(od)} T_m^{(od)} & (10)\\ &= T_n^{(d)} W_m^{(d)} - T_n^{(od)} W_m^{(od)}, & (11)\\ JW_{n+m}^{(od)} &= W_n^{(d)} T_m^{(od)} - W_n^{(od)} T_m^{(d)} & (12)\\ &= T_n^{(d)} W_m^{(od)} - T_n^{(od)} W_m^{(d)}, & (13) \end{aligned}$$

where both choices $W_j = \widehat{T}_j$ and $W_j = T_j$ are admissible.

Proof. Starting from the first identity in Proposition 4.2,

$$\begin{aligned} W_{n+m} &= W_n J T_m \\ &= (W_n^{(od)} + W_n^{(d)}) J (T_m^{(od)} + T_m^{(d)}) \\ &= W_n^{(od)} J T_m^{(od)} + W_n^{(d)} J T_m^{(d)} + W_n^{(od)} J T_m^{(d)} + W_n^{(d)} J T_m^{(od)}. \end{aligned}$$

Now we check for the diagonal and off-diagonal parts of W_{n+m} using Lemma 4.1. To this end, we calculate

$$\begin{aligned} \big\{\, J, W_n^{(od)} J T_m^{(d)} \,\big\} &= J(W_n^{(od)} J) T_m^{(d)} + W_n^{(od)} (J T_m^{(d)}) J \\ &= J(-J W_n^{(od)}) T_m^{(d)} + W_n^{(od)} (T_m^{(d)} J) J \\ &= 0, \end{aligned}$$

where we have used $\{J, W_n^{(od)}\} = 0$ and $[J, T_m^{(d)}] = 0$. Analogously, we can check $\{J, W_n^{(d)} J T_m^{(od)}\} = 0$ and $[J, W_n^{(d)} J T_m^{(d)}] = [J, W_n^{(od)} J T_m^{(od)}] = 0$. Thus Lemma 4.1 implies

$$\begin{aligned} W_{n+m}^{(d)} &= W_n^{(d)} J T_m^{(d)} + W_n^{(od)} J T_m^{(od)} = J(W_n^{(d)} T_m^{(d)} - W_n^{(od)} T_m^{(od)}), \\ W_{n+m}^{(od)} &= W_n^{(od)} J T_m^{(d)} + W_n^{(d)} J T_m^{(od)} = J(-W_n^{(od)} T_m^{(d)} + W_n^{(d)} T_m^{(od)}), \end{aligned}$$

yielding (10), (12). The other two identities (11), (13) can be proved similarly starting from the second identity in Proposition 4.2. □

Proposition 4.4. *The following identities hold both for $W_j = T_j$ and $W_j = \widehat{T}_j$:*

$$\begin{aligned} W_{n,x} &= T_1 \, W_n^{(od)}, \\ W_{n,t} &= \widehat{T}_0 \, W_n^{(od)}. \end{aligned}$$

Proof. For illustration we will show the assertion for the x-derivative of $W_n = \widehat{T}_n$. We need the following auxiliary identity

$$\begin{aligned} (I+N)^{-1}[J,N] &= (I+N)^{-1} \left(J(I+N) - (I+N)J \right) \\ &= -J + (I+N)^{-1} J (I+N). \end{aligned} \tag{14}$$

Using the fact that C and J commute, we get

$$\begin{aligned} J\widehat{T}_{n,x} &= \left(-(I+N)^{-1} N_x (I+N)^{-1} \right) C^n f_0(C)(I+N) + (I+N)^{-1} C^n f_0(C) \, N_x \\ &= -(I+N)^{-1} JCN(I+N)^{-1} C^n f_0(C)(I+N) + (I+N)^{-1} C^n f_0(C) \, JCN \\ &= (I+N)^{-1} C \Big(-JN(I+N)^{-1} C^n f_0(C)(I+N) + C^n f_0(C) JN \Big) \\ &= JT_1 \, (I+N)^{-1} \Big(-JNJ\widehat{T}_n + C^n f_0(C) JN \Big) \\ &= JT_1 \, \Big(-(I+N)^{-1} JNJ\widehat{T}_n + (I+N)^{-1} C^n f_0(C) \, JN \Big) \\ &= JT_1 \, \Big(-(I+N)^{-1} JNJ\widehat{T}_n + J\widehat{T}_n (I+N)^{-1} \, JN \Big) \\ &= JT_1 \, \Big[J\widehat{T}_n, (I+N)^{-1} JN \Big]. \end{aligned}$$

Now we use the fact that J and N anticommute, which yields $JN = \frac{1}{2}[J, N]$, and the auxiliary identity (14) to obtain

$$\begin{aligned} \Big[J\widehat{T}_n, (I+N)^{-1} JN \Big] &= \frac{1}{2} \Big[J\widehat{T}_n, (I+N)^{-1}[J,N] \Big] \\ &= \frac{1}{2} \Big[J\widehat{T}_n, -J + (I+N)^{-1} J (I+N) \Big]. \end{aligned}$$

Since $[J\widehat{T}_n, (I+N)^{-1} J (I+N)] = (I+N)^{-1}[C^n f_0(C), J](I+N) = 0$ we end up with

$$\Big[J\widehat{T}_n, (I+N)^{-1} JN \Big] = \frac{1}{2} \, [J, J\widehat{T}_n].$$

As a result, $\widehat{T}_{n,x} = \frac{1}{2} T_1 [J, J\widehat{T}_n] = T_1 \, \widehat{T}_n^{(od)}$. This completes the proof. □

Again we translate the result to derivation rules for diagonal and off-diagonal parts. For example for the x-derivative of $\widehat{T}_n$ these follow immediately from $\widehat{T}_{n,x} = (T_1^{(d)} + T_1^{(od)}) \, \widehat{T}_n^{(od)}$ observing $\{J, T_1^{(d)} \widehat{T}_n^{(od)}\} = 0$ and $[J, T_1^{(od)} \widehat{T}_n^{(od)}] = 0$.

Corollary 4.5. *(Derivation rules with respect to x)*

$$W_{n,x}^{(d)} = T_1^{(od)}\ W_n^{(od)}, \tag{15}$$

$$W_{n,x}^{(od)} = T_1^{(d)}\ W_n^{(od)}, \tag{16}$$

where both choices $W_j = \widehat{T}_j$ *and* $W_j = T_j$ *are admissible.*

However, we only need one particular case for the t-derivatives.

Corollary 4.6. *(Derivation rule for (6) with respect to t)* $\quad T_{1,t}^{(od)} = \widehat{T}_0^{(d)}\ T_1^{(od)}$.

5. **Proof of Theorem 3.1.** For the proof we define the operator functions $S_n = T_n T_1^{(od)}$ and $\widehat{S}_n = \widehat{T}_n T_1^{(od)}$, where T_n, $\widehat{T}_n$ are given in (8), (9), respectively.

Step 1. To begin with we prove the recursion relation

$$S_{n+1}^{(od)} = JS_{n,x} - J\ \{\ S_n, JS_0\ \}\ . \tag{17}$$

Applying (15), (16) to $W_j = T_j$ we obtain

$$\begin{aligned} S_{n,x} &= \Big(\ (\ T_n^{(d)} + T_n^{(od)}\ )\ T_1^{(od)}\Big)_x \\ &= \Big(\ (\ T_1^{(d)} + T_1^{(od)}\ )\ T_n^{(od)}\Big) T_1^{(od)} + (\ T_n^{(d)} + T_n^{(od)}\ )(\ T_1^{(d)} T_1^{(od)}\ ) \\ &= \Big(\{\ T_1^{(d)}, T_n^{(od)}\ \} + T_1^{(od)} T_n^{(od)} + T_n^{(d)} T_1^{(d)}\ \Big)\ T_1^{(od)}. \end{aligned}$$

Using Corollary 4.3 with $W_j = T_j$ yields

$$\begin{aligned} T_n^{(d)} T_1^{(d)} &= JT_{n+1}^{(d)} + T_n^{(od)} T_1^{(od)}, \\ \Big\{T_1^{(d)}, T_n^{(od)}\Big\} &= \Big\{T_1^{(od)}, T_n^{(d)}\Big\}. \end{aligned}$$

The first identity is (10) slightly rewritten, the second is an immediate consequence of (12) applied twice but interchanging the role of the indices. This gives

$$\begin{aligned} S_{n,x} &= \Big(\{\ T_1^{(od)}, T_n^{(d)}\ \} + T_1^{(od)} T_n^{(od)} + (\ JT_{n+1}^{(d)} + T_n^{(od)} T_1^{(od)}\ )\Big)\ T_1^{(od)} \\ &= \Big(\ JT_{n+1}^{(d)} + \{\ T_1^{(od)}, T_n\ \}\Big)\ T_1^{(od)} \\ &= JT_{n+1}^{(d)} T_1^{(od)} + \{\ T_1^{(od)}, T_n T_1^{(od)}\ \}, \end{aligned}$$

where we have used $\{a, b\}a = \{a, ba\}$ for the last identity.

Next we take a closer look at both terms separately. Since $\{J, T_1^{(od)}\} = 0$ by Lemma 4.1, we observe

$$\begin{aligned} T_{n+1}^{(d)} T_1^{(od)} &= \frac{1}{2}\ (\ T_{n+1} + JT_{n+1}J\ )\ T_1^{(od)} = \frac{1}{2}\Big(T_{n+1} T_1^{(od)} + JT_{n+1} J T_1^{(od)}\Big) \\ &= \frac{1}{2}\Big(T_{n+1} T_1^{(od)} - JT_{n+1} T_1^{(od)} J\Big) = \Big(T_{n+1} T_1^{(od)}\Big)^{(od)} \\ &= S_{n+1}^{(od)}. \end{aligned}$$

Moreover, inserting $S_0 = T_0 T_1^{(od)} = JT_1^{(od)}$, we get

$$\{\ T_1^{(od)}, T_n T_1^{(od)}\ \} = \{\ JS_0, S_n\ \}\ .$$

To sum up, $S_{n,x} = JS_{n+1}^{(od)} + \{\ S_n, JS_0\ \}$, which was to be shown.

Step 2. Next we show that

$$S_n^{(od)} = \mathcal{R}^n(JS). \tag{18}$$

Splitting up the recursion relation (17) into its diagonal and off-diagonal parts using Lemma 4.1, we obtain

$$\begin{aligned} S_{n,x}^{(d)} &= \{\ S_n^{(od)}, JS_0\ \}, \\ S_{n,x}^{(od)} &= JS_{n+1}^{(od)} + \{\ S_n^{(d)}, JS_0\ \}. \end{aligned}$$

Inserting the expression for the diagonal part into that for the off-diagonal part then yields

$$S_{n,x}^{(od)} = JS_{n+1}^{(od)} + \left\{ \int_{-\infty}^{x} \{\ S_n^{(od)}, JS_0\ \}\, d\xi\ ,\ JS_0 \right\}.$$

With $JS_0 = T_1^{(od)} = S$, the latter becomes

$$S_{n+1}^{(od)} = J\left(S_{n,x}^{(od)} - \left\{ \int_{-\infty}^{x} \{\ S_n^{(od)}, S\ \}\, d\xi\ ,\ S \right\}\right) = \mathcal{R}\ (\ S_n^{(od)}\ ),$$

and, by induction,

$$S_n^{(od)} = \mathcal{R}^n\ (\ S_0^{(od)}\ ).$$

It remains to verify $S_0^{(od)} = JS$ which follows from $S_0 = JS$ and the fact that $S_0^{(od)} = (JS)^{(od)} = JS^{(od)} = JS$, the latter by the very definition of S.

Step 3. Similarly as in Step 1 we prove a corresponding recursion relation for the operator function $\widehat{S}_n$, namely

$$\widehat{S}_{n+1}^{(od)} = J\widehat{S}_{n,x} - J\ \{\ \widehat{S}_n, JS_0\ \}. \tag{19}$$

For the proof we essentially follow the line of arguments of the first step. Thus we focus on the part of the arguments relying on the rules for derivation and evaluation of products which are somehow more involved as before.

As in Step 1, applying the derivation rules (15), (16), now for $W_j = \widehat{T}_j$, yields in a straightforward way

$$\widehat{S}_{n,x} = \left(\{\ T_1^{(d)}, \widehat{T}_n^{(od)}\ \} + T_1^{(od)}\widehat{T}_n^{(od)} + \widehat{T}_n^{(d)}T_1^{(d)}\right) T_1^{(od)}.$$

Next, we infer from the rules for evaluation of products in Corollary 4.3, now used with $W_j = \widehat{T}_j$, that

$$\begin{aligned} \widehat{T}_n^{(d)}T_1^{(d)} &= J\widehat{T}_{n+1}^{(d)} + \widehat{T}_n^{(od)}T_1^{(od)}, \\ \left\{T_1^{(d)}, \widehat{T}_n^{(od)}\right\} &= \left\{T_1^{(od)}, \widehat{T}_n^{(d)}\right\}. \end{aligned}$$

Indeed the first identity is due to (10), and for the second one has to consider the difference of (12) and (13) with the right choice of indices. Inserting yields

$$\widehat{S}_{n,x} = J\widehat{T}_{n+1}^{(d)}T_1^{(od)} + \left\{\ T_1^{(od)}, \widehat{T}_n T_1^{(od)}\ \right\}.$$

To see that the latter expression coincides with $J\widehat{S}_{n+1}^{(od)} + \{\ JS_0, \widehat{S}_n\ \}$, we can carry over the corresponding arguments from Step 1 almost literally.

Step 4. We show that

$$\widehat{S}_n^{(od)} = \mathcal{R}^n(S_t). \tag{20}$$

Here we can follow closely the line of arguments from Step 2 from which we correspondingly arrive at

$$\widehat{S}_n^{(od)} = \mathcal{R}^n(\widehat{S}_0^{(od)}),$$

and it thus remains to verify $\widehat{S}_0^{(od)} = S_t$. To this end, we calculate using the identity $[a, bc] = \{a, b\}c$ for $\{a, c\} = 0$

$$\begin{aligned} \widehat{S}_0^{(od)} &= \frac{1}{2}\left[\, J,\; J\widehat{T}_0\; T_1^{(od)} \,\right] = \frac{1}{2}\left\{\, J, J\widehat{T}_0 \,\right\} T_1^{(od)} \\ &= \widehat{T}_0^{(d)}\; T_1^{(od)}. \end{aligned}$$

On the other hand we recall that $S = T_1^{(od)}$. By Corollary 4.6 we thus infer $S_t = T_{1,t}^{(od)} = \widehat{T}_0^{(d)}\; T_1^{(od)} = \widehat{S}_0^{(od)}$.

Step 5. Let the polynomials f, g be given by $f(z) = \sum_{n=0}^{N} a_n z^n$, $g(z) = \sum_{n=0}^{M} b_n z^n$. We claim that the following identity holds:

$$\sum_{n=0}^{N} a_n S_n^{(od)} = \sum_{n=0}^{M} b_n \widehat{S}_n^{(od)}. \tag{21}$$

This can be checked directly as follows:

$$\begin{aligned} \sum_{n=0}^{M} b_n \widehat{S}_n &= \sum_{n=0}^{M} b_n\; \widehat{T}_n T_1^{(od)} = \sum_{n=0}^{M} b_n \left(\, J(1+N)^{-1} f_0(C) C^n (1+N) \,\right) T_1^{(od)} \\ &= J(1+N)^{-1} f_0(C) \left(\sum_{n=0}^{M} b_n C^n \right) (1+N)\; T_1^{(od)} \\ &= J(1+N)^{-1} \left(\, f_0(C) g(C) \,\right) (1+N)\; T_1^{(od)} \\ &= J(1+N)^{-1}\; f(C)\; (1+N)\; T_1^{(od)} \\ &= J(1+N)^{-1} \left(\sum_{n=0}^{N} a_n C^n \right) (1+N)\; T_1^{(od)} \\ &= \sum_{n=0}^{N} a_n \left(\, J(1+N)^{-1} C^n (1+N) \,\right) T_1^{(od)} = \sum_{n=0}^{N} a_n\; T_n T_1^{(od)} \\ &= \sum_{n=0}^{N} a_n S_n. \end{aligned}$$

The relation for the off-diagonal parts follows directly.

Step 6. It only takes a few lines to conclude that

$$\begin{aligned} g(\mathcal{R})(S_t) &= \sum_{n=0}^{M} b_n \mathcal{R}^n(S_t) \stackrel{(20)}{=} \sum_{n=0}^{M} b_n \widehat{S}_n^{(od)} \\ &\stackrel{(21)}{=} \sum_{n=0}^{N} a_n S_n^{(od)} \stackrel{(18)}{=} \sum_{n=0}^{N} a_n \mathcal{R}^n(JS) = f(\mathcal{R})(JS). \qquad \square \end{aligned}$$

6. **Consequences for the ncNLS and ncSG equations.** As in the scalar case, prominent nc systems arise from particular choices of the polynomials f and g. Here we treat the ncNLS and ncSG equations as a model.

We start with some preparational calculations. By (6), S is off-diagonal, implying $\{JS, S\} = \{J, S\}S = 2JS^{(d)}\, S = 0$. Thus

$$\begin{aligned}
\mathcal{R}(JS) &= S_x - J\Big\{S, \int_{-\infty}^{x} \{\, JS, S \,\}\, d\xi\Big\} \;=\; S_x, \\
\mathcal{R}^2(JS) &= \mathcal{R}(S_x) \\
&= J\Big(S_{xx} - \{S, \int_{-\infty}^{x} \{\, S_x, S \,\}\, d\xi \,\}\Big) \\
&= J\Big(S_{xx} - \{S, \int_{-\infty}^{x} (S^2)_x d\xi \,\}\Big) \\
&= J\Big(S_{xx} - 2S^3\Big). \qquad (22)
\end{aligned}$$

On the other hand, we get

$$\mathcal{R}(S_t) \;=\; J\Big(S_{xt} - \{\, S, \int_{-\infty}^{x} (S^2)_t \, d\xi \,\}\Big). \qquad (23)$$

The ncNLS equation. Here one chooses $f(z) = -\mathrm{i}z^2$ and $g(z) = 1$. Using (22) we immediately see that (3) becomes

$$S_t = -\mathrm{i}J\,\big(\, S_{xx} - 2S^3 \,\big)\,. \qquad (24)$$

Let now E, F be the eigenspaces for the eigenvalues $+1$, -1 of the involution J. Then $G = E \oplus F$, and the conditions $[J, C] = 0$, $\{J, N\} = 0$ mean that C is diagonal and N is off-diagonal with respect to this decomposition, say

$$C = \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}, \qquad N(x,t) = \begin{pmatrix} 0 & L(x,t) \\ -M(x,t) & 0 \end{pmatrix}$$

with constant operators $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$ and operator-functions $L(x,t) \in \mathcal{L}(F, E)$, $M(x,t) \in \mathcal{L}(E, F)$.

We now transcribe (6). To calculate the inverse operator $(I + N)^{-1}$, we use

$$\begin{aligned}
\begin{pmatrix} I_E + LM & 0 \\ 0 & I_F + ML \end{pmatrix} &= (I - N^2) \;=\; (I - N)(I + N) \\
&= \begin{pmatrix} I_E & -L \\ M & I_F \end{pmatrix}\begin{pmatrix} I_E & L \\ -M & I_F \end{pmatrix},
\end{aligned}$$

which yields

$$\begin{aligned}
(I + N)^{-1} &= (I - N^2)^{-1}(I - N) \\
&= \begin{pmatrix} (I_E + LM)^{-1} & 0 \\ 0 & (I_F + ML)^{-1} \end{pmatrix}\begin{pmatrix} I_E & -L \\ M & I_F \end{pmatrix}.
\end{aligned}$$

Therefore, $(I + N)^{-1}C(I + N) = (I - N^2)^{-1}\Big((I - N)C(I + N)\Big)$, and the term in the brackets is evaluated as

$$\begin{aligned}
(I - N)C(I + N) &= \begin{pmatrix} I_E & -L \\ M & I_F \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}\begin{pmatrix} I_E & L \\ -M & I_F \end{pmatrix} \\
&= \begin{pmatrix} A - LBM & AL + LB \\ MA + BM & MAL - B \end{pmatrix}.
\end{aligned}$$

We get

$$\begin{aligned} J(I+N)^{-1}C(I+N) \;&=\; J(I-N^2)^{-1}\Big((I-N)C(I+N)\Big) \\ &= \begin{pmatrix} (I_E+LM)^{-1}(A-LBM) & (I_E+LM)^{-1}(AL+LB) \\ -(I_F+ML)^{-1}(MA+BM) & -(I_F+ML)^{-1}(MAL-B) \end{pmatrix}. \end{aligned}$$

This shows that

$$\begin{aligned} S &= \frac{1}{2}\Big[J,(I+N)^{-1}C(I+N)\Big] \\ &= \Big(J(I+N)^{-1}C(I+N)\Big)^{(od)} \\ &= \begin{pmatrix} 0 & (I_E+LM)^{-1}(AL+LB) \\ -(I_F+ML)^{-1}(MA+BM) & 0 \end{pmatrix}, \end{aligned}$$

and it is straightforward to translate (24) to the corresponding nc system for $Q(x,t)$, $R(x,t)$. We thus deduce from Theorem 3.1 the following

Proposition 6.1. *Let E, F be a Banach spaces and $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$. Assume that $L = L(x,t) \in \mathcal{L}(F,E)$ and $M = M(x,t) \in \mathcal{L}(E,F)$ are operator-valued functions which, on a strip $\mathbb{R}\times(t_1,t_2)$, are sufficiently smooth and behave sufficiently well for $x \to -\infty$, and solve the base equations $L_x = AL$, $L_t = -\mathrm{i}A^2L$, and $M_x = BM$, $M_t = \mathrm{i}B^2M$. Moreover, assume that $(I_E + LM)$ (and thus also $(I_F + ML)$) is invertible on $\mathbb{R}\times(t_1,t_2)$.*

Then, on $\mathbb{R}\times(t_1,t_2)$ a solution of the ncNLS system

$$\begin{aligned} R_t &= -\mathrm{i}R_{xx} + 2\mathrm{i}RQR, \\ Q_t &= \quad \mathrm{i}Q_{xx} - 2\mathrm{i}QRQ. \end{aligned}$$

is given by $R = (I_E + LM)^{-1}(AL + LB)$ and $Q = -(I_F + ML)^{-1}(BM + MA)$.

Remark 2. a) The scalar one-soliton solution is obtained by taking everything scalar-valued and setting $b = \overline{a}$. Already here it becomes apparent that the general assumption $A = B$ in [6] is not well-suited for applications.

b) The scalar NLS equation is obtained imposing the relation $r = -\overline{q}$ to the NLS system. In the nc setting there are several options to translate this reduction ([2], [5], [17], [18], [20], [21]).

The ncSG equation. Here $f(z) = 1$, $g(z) = z$. In this case (3) becomes

$$S_{tx} - S\int_{-\infty}^{x}(S^2)_t\, d\xi - \int_{-\infty}^{x}(S^2)_t\, d\xi\; S = S,$$

where we have used (23). The right reduction from the system to a single equation in this case amounts to imposing the relation $R = -Q$. This yields

Proposition 6.2. *Let E be a Banach space and $A \in \mathcal{L}(E)$ invertible. Assume that $L = L(x,t) \in \mathcal{L}(E)$ is an operator-valued function which, on a strip $\mathbb{R}\times(t_1,t_2)$, is sufficiently smooth and behaves sufficiently well for $x \to -\infty$, and solves the base equations $L_x = AL$ and $L_t = A^{-1}L$. Moreover, assume that $(I + L^2)$ is invertible on $\mathbb{R}\times(t_1,t_2)$.*

Then, on $\mathbb{R}\times(t_1,t_2)$ a solution of the ncSG equation

$$R_{tx} + R\int_{-\infty}^{x}(R^2)_t\, d\xi + \int_{-\infty}^{x}(R^2)_t\, d\xi\; R = R \tag{25}$$

is given by $R = (I + L^2)^{-1}(AL + LA)$.

Remark 3. For the SG equation in standard form $u_{xt} = \sin(u)$, the corresponding formulas are exploited in [19] to give a rigorous asymptotic description of general multipole solutions.

7. **Matrix soliton solutions.** In this section we construct N-soliton solutions for the matrix SG equation. They will turn out to be precisely analogous to the multisoliton solutions derived in [12] for the matrix KdV equation.

Step 1. We claim: If R is an $\mathcal{L}\big((\mathbb{C}^m)^N\big)$-solution of the ncSG equation (25) which is of the particular form

$$R(x,t) = R_0(x,t)\ CD^T \tag{26}$$

for some constant operators $C,\ D \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^{mN})$ with $D^TD = I_m$, then an $\mathcal{L}(\mathbb{C}^m)$-solution of (25) is given by $\widetilde{R}(x,t) = D^TR(x,t)D$.

Indeed, we only have to check the effect of multiplying a solution from the left with D^T and from the right with D for nonlinear terms. From $\widetilde{R} = D^TR_0C$ we get $\widetilde{R}^2 = D^TR_0CD^TR_0C = D^TR_0CD^TR_0CD^TD = D^TR^2D$. Similarly one deals with $\{\widetilde{R}, \int_{-\infty}^x (\widetilde{R}^2)_t d\xi\}$.

Step 2. Consider the following choices:

$$A = \begin{pmatrix} \lambda_1 I_m & & 0 \\ & \ddots & \\ 0 & & \lambda_N I_m \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ \vdots \\ C_N \end{pmatrix}, \quad D = \tfrac{1}{\sqrt{m}} \begin{pmatrix} I_m \\ \vdots \\ I_m \end{pmatrix}, \tag{27}$$

with $0 < \lambda_1 < \ldots < \lambda_N$ and $m\times m$-matrices $C_1, \ldots, C_N$. Note that the assumptions of Proposition 6.2 can be satisfied by choosing $L(x,t) = \exp(Ax + A^{-1}t)B$. Then the solution reads

$$R(x,t) = \big(I + L(x,t)^2\big)^{-1} \exp(Ax + A^{-1}t)\big(AB + BA\big).$$

If we now set $B = \frac{1}{\sqrt{m}}\Big(\frac{1}{\lambda_i+\lambda_j}C_i\Big)_{i,j=1}^N$, then one easily verifies $AB + BA = CD^T$. Thus we have constructed the following solution for the matrix SG equation (with values in the $m \times m$-matrices) according to Step 1

$$\widetilde{R}(x,t) \;=\; \tfrac{1}{\sqrt{m}} \begin{pmatrix} I_m & \cdots & I_m \end{pmatrix} \big(I + L(x,t)^2\big)^{-1} \begin{pmatrix} \exp(\lambda_1 x + \frac{1}{\lambda_1}t)C_1 \\ \vdots \\ \exp(\lambda_N x + \frac{1}{\lambda_N}t)C_N \end{pmatrix}, \tag{28}$$

$$\text{where } L(x,t) = \tfrac{1}{\sqrt{m}} \left(\frac{\exp(\lambda_i x + \frac{1}{\lambda_i}t)}{\lambda_i + \lambda_j} C_i \right)_{i,j=1}^N .$$

Remark 4. a) We may convince ourselves that (28) is the exact SG analogue to Goncharenko's KdV multisoliton solutions. To this end, we apply the solution formula for the KdV equation in [8], Proposition 2.2, to the same data (27). Then a straightforward calculation leads to Goncharenko's solution written as in [12], formula (10).

b) In [14] an alternative derivation of multisoliton solutions for the matrix SG equation via quasideterminants is given. We also refer to this article for an interesting asymptotic discussion of the first relevant cases.

c) Solving the base equation in Proposition 6.2 with the ansatz $L(x,t) = \exp(Ax+ A^{-1}t)B$, the task to satisfy (26) amounts to finding, for C, D given, $B \in \mathcal{L}(\mathbb{C}^{mN})$

such that $AB + BA = CD^T$. As explained for example in [8], [19], it follows from the theory of elementary operators that this is always possible. This means that even in the finite-matrix setting our method leads to much richer solution families. It would be interesting to explore structural properties. For the scalar case $m = 1$ the solutions coming from finite matrices are exactly the multipole solutions, for which a complete asymptotic analysis is given in [19].

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