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DYNAMICALLY CONSISTENT DISCRETE-TIME LOTKA-VOLTERRA COMPETITION MODELS

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Abstract. Sufficient conditions are given such that the discrete time competition models constructed by applying nonstandard finite difference (NSFD) schemes for the Lotka-Volterra competition models are dynamically consistent. The derived discrete models preserve the positivity of solutions, local stability conditions, boundedness, and the monotonicity of the continuous Lotka-Volterra system. In other words, we are able to construct discrete-time competition models that behave just like the continuous-time Lotka-Volterra competition models.

1. **Introduction.** The following system of differential equations is Lotka-Volterra competition model:

$$\begin{aligned} \frac{dx}{dt} &= x(r_1 - a_{11}x - a_{12}y), \\ \frac{dy}{dt} &= y(r_2 - a_{21}x - a_{22}y), \end{aligned} \tag{1}$$

where

$$r_1 > 0, r_2 > 0, a_{11} > 0, a_{22} > 0, a_{12} \geq 0, \text{ and } a_{21} \geq 0. \tag{2}$$

The two variables $x(t)$ and $y(t)$ represent the number of individuals or population density in species x and y at time t; the parameters r_i's are the intrinsic growth rates for the two species x and y; a_{12} and a_{21} are the interspecific acting coefficients. The dynamics of the model is well-known [1] and we will briefly mention the main properties of the system.

(P1) The solutions are positive if the initial conditions are positive. In other words, the positive cone is *positively invariant*.

(P2) The system (1) has at most four equilibria. They are the extinct equilibrium $E_0 = (0, 0)$; the exclusive equilibria $E_1 = (r_1/a_{11}, 0)$ and $E_2 = (0, r_2/a_{22})$; and the possible coexistence equilibrium $E_3 = ((a_{22}r_1 - a_{12}r_2)/(a_{11}a_{22} - a_{12}a_{21}), (a_{11}r_2 - a_{21}r_1)/(a_{11}a_{22} - a_{12}a_{21}))$.

(P3) There are four possible outcomes related to the equilibria. (i) E_0 is a repeller and always unstable. (ii) E_1 is locally asymptotically stable if $\dfrac{a_{11}}{a_{21}} < \dfrac{r_1}{r_2}$. (iii)

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E_2 is locally asymptotically stable if $\dfrac{r_1}{r_2} < \dfrac{a_{12}}{a_{22}}$. (iv) E_3 is locally asymptotically stable if

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

(P4) The system is monotonic.
(P5) The solutions are eventually bounded.
(P6) The nonnegative x-axis and the nonnegative y-axis are positively invariant.

Furthermore, if E_3 is stable, it is globally asymptotically stable.

Numerical schemes can be used to convert differential equations into difference equations. If the corresponding difference equations possess the same dynamical behavior as the continuous equations, such as local stability, bifurcations, and/or chaos, then they are said to be *dynamically consistent* [4]. More specifically, Mickens [5] defines dynamic consistency (DC) as the following:

Definition 1.1. [5] Consider the differential equation $x' = f(x)$. Let a finite difference scheme for the equation be $x_{k+1} = F(x_k, h)$. Let the differential equation and/or its solutions have property P. The discrete model equation is dynamically consistent with the differential equation if it and/or its solutions also have property P.

The following notation will be used:

$$X = x(t+h),\ x = x(t),\ Y = y(t+h),\ y = y(t).$$

Liu and Elaydi [4] used the following numerical scheme:

$$\begin{aligned} \frac{X-x}{\phi} &= r_1x - a_{11}xX - a_{12}yX, \\ \frac{Y-y}{\phi} &= r_2y - a_{21}xY - a_{22}yY, \end{aligned} \tag{3}$$

where $\phi = h + O(h^2)$. Cushing et al. [2] showed that the dynamics of the discrete system (3) are similar to the continuous model (1). This discrete-time model (3) is dynamically consistent with the continuous model with properties (P1)–(P6); E_3 is globally stable if it exists.

The author [6] showed similar methods that produce discrete-time competition models that preserve local stability. Two of the methods shown in [6] preserve all six properties (P1)–(P6). A different NSFD scheme for model (1) was investigated by the author and Gelca [8]; this method was deduced in [7]. We showed that there are many NSFD schemes for model (1) that are dynamically consistent with all six properties (P1)–(P6).

Our object is to extend and generalize our previous findings in [6, 8] and to find general NSFD schemes for the continuous model (1) such that the resulting discrete-time models are dynamically consistent with the continuous model with respect to all six properties (P1)–(P6). Specifically, we are looking for NSFD schemes in the following form:

$$\begin{aligned} \frac{X-x}{\phi} &= r_1x - a_{11}xX - a_{12}(b_1xy + b_2Xy + b_3xY + b_4XY), \\ \frac{Y-y}{\phi} &= r_2y - a_{21}(d_1xy + d_2Xy + d_3xY + d_4XY) - a_{22}yY, \end{aligned} \tag{4}$$

where $\phi = h + O(h^2)$ and

$$\begin{aligned} b_1 + b_2 + b_3 + b_4 &= 1, \\ d_1 + d_2 + d_3 + d_4 &= 1. \end{aligned} \tag{5}$$

This system can also be presented as the implicit map $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}$:

$$\begin{aligned} X &= x \cdot \frac{1 + r_1\phi - a_{12}\phi\,(b_1 y + b_3 Y)}{1 + a_{11}\phi\, x + a_{12}\phi\,(b_2 y + b_4 Y)}, \\ Y &= y \cdot \frac{1 + r_2\phi - a_{21}\phi\,(d_1 x + d_2 X)}{1 + a_{21}\phi\,(d_3 x + d_4 X) + a_{22}\phi\, y}. \end{aligned} \tag{6}$$

Let us describe briefly the contents of the paper. In Section 2, assumptions on b_i's and d_i's such that the discrete-time model (4) has the positivity property and the same local dynamics with the continuous model (1) are given and properties (P1)–(P3) are preserved. In Section 3, criteria for monotonicity, property (P4), global stability, and the boundedness of solutions (P5) are given. For system (6), property (P6) is obvious. Finally, in Section 4, the discrete-time models satisfying all of the criteria for (P1)–(P6) are given as the following

$$\begin{aligned} \frac{X - x}{\phi} &= r_1 x - a_{11} X x - a_{12}[\alpha X y + (1-\alpha) X Y], \\ \frac{Y - y}{\phi} &= r_2 y - a_{21}[\beta x Y + (1-\beta) X Y] - a_{22} Y y, \end{aligned} \tag{7}$$

where $0 \le \alpha \le 1$, $0 \le \beta \le 1$, and $\alpha + \beta \ge 1$, Discussion and possible future work are presented in Section 5.

2. **Positivity and Local Stability.**

For the discrete-time L-V competition system (4) to preserve positivity we need certain conditions.

Lemma 2.1. *Consider the system of difference equations (4) with assumption (5). In addition, we assume*

$$\begin{aligned} b_1 \le 0,\ b_2 \ge 0,\ b_3 \le 0, \\ d_1 \le 0,\ d_2 \le 0,\ d_3 \ge 0,\ d_4 \ge 0, \end{aligned} \tag{8}$$

and the parameter b_4 satisfies either cases:

$$\begin{cases} b_4 > 0, \\ b_4 = 0, \quad b_3 d_4 = 0,\ b_2 d_3 - b_1 d_4 - b_3 d_2 \ge 0, \end{cases} \tag{9}$$

then if there is a point (x_0, y_0) such that $X(x_0, y_0) > 0$ and $Y(x_0, y_0) > 0$, then for all $x, y > 0$ we have $X(x, y) > 0$ and $Y(x, y) > 0$.

Proof. The discrete system (4) is equivalent to (6). Let

$$\tilde{b}_1 = -b_1 \ge 0,\ \tilde{b}_3 = -b_3 \ge 0,\ \tilde{d}_1 = -d_1 \ge 0,\ \tilde{d}_2 = -d_2 \ge 0.$$

Then system (6) becomes

$$X = x \cdot \frac{1 + r_1\phi + a_{12}\phi\,(\tilde{b}_1 y + \tilde{b}_3 Y)}{1 + a_{11}\phi\, x + a_{12}\phi\,(b_2 y + b_4 Y)}, Y = y \cdot \frac{1 + r_2\phi + a_{21}\phi\,(\tilde{d}_1 x + \tilde{d}_2 X)}{1 + a_{21}\phi\,(d_3 x + d_4 X) + a_{22}\phi\, y}.$$

Substituting X into the equation for Y and rearranging we obtain the quadratic equation in Y: $AY^2 + BY + C = 0$, where

$$\begin{aligned} A &= a_{12}\phi(b_4 + b_4 a_{22}\phi y + a_{21}\phi x b_4 d_3 + a_{21}\phi x \tilde{b}_3 d_4), \\ B &= 1 + \phi(a_{11}x + a_{22}y + a_{21}d_3 x + a_{21}d_4 x + a_{12}b_2 y - a_{12}b_4 y) \\ &\qquad + \phi^2[a_{12}a_{21}xy(\tilde{b}_1 d_4 + b_2 d_3 - \tilde{b}_3\tilde{d}_2 - b_4\tilde{d}_1) - a_{12}r_2 b_4 y \\ &\qquad + a_{21}r_1 d_4 x + a_{11}a_{21}d_3 x^2 + a_{11}a_{22}xy + a_{12}a_{22}b_2 y^2)], \\ C &= -y[1 + \phi(r_2 + a_{11}x + a_{12}b_2 y + a_{21}\tilde{d}_1 x + a_{21}\tilde{d}_2 x) + \phi^2(r_2 a_{11}x + \\ &\qquad r_1 a_{21}\tilde{d}_2 x + r_2 a_{12}b_2 y + a_{11}a_{21}\tilde{d}_1 x^2 + a_{12}a_{21}\tilde{b}_1\tilde{d}_2 xy + a_{12}a_{21}b_2\tilde{d}_1 xy)]. \end{aligned} \tag{10}$$

The coefficients satisfy $A \geq 0$ and $C < 0$. When $A > 0$ and $C < 0$, one of the solutions is strictly positive, and the other is strictly negative. When $A = 0$, which is under the assumption (9), then we have $B > 0$, and therefore the solution is strictly positive. Since Y is positive at one point, by continuity it must be positive everywhere. A similar argument shows that X is positive everywhere, and the lemma is proved. □

Treat the right-hand side functions in system (6) as $f(x, y, Y)$ and $g(x, y, X)$. Then the Jacobian matrix of X and Y with respect to x and y is

$$J = \begin{pmatrix} \dfrac{f_x + f_Y g_x}{1 - f_Y g_X} & \dfrac{f_y + f_Y g_y}{1 - f_Y g_X} \\ \dfrac{g_x + g_X f_x}{1 - f_Y g_X} & \dfrac{g_y + g_X f_y}{1 - f_Y g_X} \end{pmatrix}$$

where $f_x = \frac{\partial f}{\partial x}$ and $f_X = \frac{\partial f}{\partial X}$ and so on. The local stability results at E_0, E_1, and E_2 are easily obtained.

Lemma 2.2. *Consider the discrete system (4) under the assumptions (5), (8), and (9). Then (i) E_0 is unstable and a repeller, (ii) E_1 is locally asymptotically stable if $a_{11}/a_{21} < r_1/r_2$, and (iii) E_2 is locally asymptotically stable if $r_1/r_2 < a_{12}/a_{22}$.*

Proof. The Jacobian matrix at $E_0 = (0, 0)$ is

$$J(E_0) = \begin{pmatrix} 1 + r_1\phi & 0 \\ 0 & 1 + r_2\phi \end{pmatrix}.$$

Both eigenvalues are greater than one. E_0 is a repeller.

The Jacobian matrix at E_1 is

$$J(E_1) = \begin{pmatrix} \dfrac{1}{1 + r_1\phi} & * \\ 0 & 1 - \dfrac{\phi(a_{21}r_1 - a_{11}r_2)}{a_{11} + a_{21}r_1\phi(1 - d_1 - d_2)} \end{pmatrix},$$

where "$*$" represents a nonzero element. Eigenvalue $1/(1 + r_1\phi)$ is less than 1. The second eigenvalue λ_2 satisfies

$$|\lambda_2| < 1 \Leftrightarrow (a_{21}r_1 - a_{11}r_2)[2a_{11} + a_{11}r_2\phi + a_{21}r_1\phi(1 - 2d_1 - 2d_2)] > 0$$

By assumption (8), we easily obtain that $1 - 2d_1 - 2d_2 \geq 0$. Therefore, we have E_1 is stable if

$$|\lambda_2| < 1 \Leftrightarrow a_{21}r_1 - a_{11}r_2 > 0.$$

Similarly, we can prove that E_2 is stable. □

Now, we consider the stability conditions at the positive equilibrium $E_3 = (x^*, y^*)$. The positive equilibrium E_3 of (6) satisfies At the positive steady state E_3, the partial derivatives can be evaluated:

$$\begin{aligned} f_x &= 1 - \frac{a_{11}\phi\, x^*}{D_1},\ f_y = -\frac{a_{12}\phi(b_1+b_2)x^*}{D_1},\ f_Y = -\frac{a_{12}\phi(b_3+b_4)x^*}{D_1}, \\ g_x &= -\frac{a_{21}\phi(d_1+d_3)y^*}{D_2},\ g_y = 1 - \frac{a_{22}\phi\, y^*}{D_2},\ g_X = -\frac{a_{21}\phi(d_2+d_4)y^*}{D_2}, \end{aligned} \tag{11}$$

where

$$D_1 = 1 + a_{11}\phi x^* + a_{12}\phi(b_2+b_4)y^*$$

and

$$D_2 = 1 + a_{21}\phi(d_3+d_4)x^* + a_{22}\phi y^*.$$

Theorem 2.3. *Consider the discrete system (6) under the assumptions (5), (8), and (9) and the additional assumptions*

$$(b_2+b_4)(d_3+d_4) - (b_3+b_4)(d_2+d_4) \geq 0, \tag{12}$$

$$(b_1+b_2)(d_2+d_4) + (b_3+b_4)(d_1+d_3) \geq 0. \tag{13}$$

$$(b_1+b_2)(d_1+d_3) - (b_3+b_4)(d_2+d_4) \geq 0. \tag{14}$$

Then the coexistence equilibrium E_3 is locally asymptotically stable if

$$\frac{a_{11}}{a_{21}} > \frac{r_1}{r_2} > \frac{a_{12}}{a_{22}}.$$

Proof. By assumption (8) and (9), we have $b_2 + b_4 \geq 0$ and $d_3 + d_4 \geq 0$. Since $x^* > 0$, $y^* > 0$, and $a_{ij} > 0$ for $i, j = 1, 2$, we have $D_1 > 1$ and $D_2 > 1$. We can show that $1 - f_Y g_X$ is positive since

$$1 - f_Y g_X = \frac{D_1 D_2 - a_{12}a_{21}\phi^2(b_3+b_4)(d_2+d_4)x^*y^*}{D_1 D_2}.$$

Expand the multiplication in the numerator we obtain positive terms plus the term

$$[(b_2+b_4)(d_3+d_4) - (b_3+b_4)(d_2+d_4)]a_{12}a_{21}\phi^2 x^* y^*$$

which is ≥ 0 because of assumption (12). Therefore, $1 - f_Y g_X > 0$.

The positive equilibrium (x^*, y^*) is stable if Jury criteria are satisfied: $|\text{trace}(J)| < 1 + \det(J) < 2$. First, we show that $\text{trace}(J) > 0$. The trace of the Jacobian matrix is

$$\frac{f_x + g_y + g_X f_y + f_Y g_x}{1 - f_Y g_X}.$$

It is easy to verify that $0 < f_x < 1$, $0 < g_y < 1$. And since

$$g_X f_y + f_Y g_x = [(b_1+b_2)(d_2+d_4) + (b_3+b_4)(d_1+d_3)] \cdot \frac{a_{12}a_{21}\phi^2 x^* y^*}{D_1 D_2} \geq 0$$

by assumption (13), the numerator of the trace is positive. Since the denominator is positive, it follows that $\text{trace}(J) > 0$.

Jury criteria are reduced to $1 - \det(J) > 0$ and $1 + \det(J) - \text{trace}(J) > 0$. Since

$$f_y g_x - f_Y g_X = [(b_1+b_2)(d_1+d_3) - (b_3+b_4)(d_2+d_4)] \cdot \frac{a_{12}a_{21}\phi^2 x^* y^*}{D_1 D_2} \geq 0$$

by assumption (14), and we know that $0 < f_x < 1$ and $0 < g_y < 1$, so we obtain

$$1 - \det(J) = \frac{(1 - f_x g_y) + (f_y g_x - f_Y g_X)}{1 - f_Y g_X} > 0.$$

Since

$$1+\det(J)-\text{trace}(J)=\frac{(a_{11}a_{22}-a_{12}a_{21})\phi\ x^*y^*}{D_1D_2(1-f_Yg_X)}$$

and the denominator $D_1D_2(1-f_Yg_X)$ is positive, we obtain

$$1+\det(J)-\text{trace}(J)>0\Leftrightarrow\frac{a_{11}}{a_{21}}>\frac{a_{12}}{a_{22}}.$$

The proof is complete. □

3. **Monotonicity, Global Stability, and Boundedness.** Next, we want to show that the discrete system (6) also preserves monotonicity. The system (6) can also be written as:

$$X=f_1(x,y),\ Y=g_1(x,y),$$

where $g_1(x,y)=\frac{-B+\sqrt{B^2-4AC}}{2A}$ and $A,\ B,\ C$ are the coefficients of a quadratic equation for Y given in (10). $f_1(x,y)$ is obtained by substituting Y into the equation for X in (6). Then this system defines a two-dimensional map: $T:\mathbb{R}^2_+\to\mathbb{R}^2_+$ by

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}f_1(x,y)\\g_1(x,y)\end{pmatrix}.$$

Define an order relation "$\ll$" as follows.

$$\begin{pmatrix}x_1\\y_1\end{pmatrix}\ll\begin{pmatrix}x_2\\y_2\end{pmatrix}\text{ if } x_1\le x_2 \text{ and } y_1\ge y_2.$$

By definition of the map, if we can show that

$$\begin{pmatrix}x_1\\y_1\end{pmatrix}\ll\begin{pmatrix}x_2\\y_2\end{pmatrix}\Rightarrow T\begin{pmatrix}x_1\\y_1\end{pmatrix}\ll T\begin{pmatrix}x_2\\y_2\end{pmatrix},$$

then the map T generates a discrete monotone flow on $\mathbb{R}^2_+$.

3.1. **Monotonicity.** To prove that T defined in equations (6) generates a discrete monotone flow, we need to find the criteria for equations (6) such that, for fixed y, X is an increasing function of x and Y a decreasing function of x.

Consider (6). Fix y and set $x=t$, $A_1=1+r_1\phi+a_{12}\tilde{b}_1\phi y$, $B_1=a_{12}\tilde{b}_3\phi$, $C_1=1+a_{12}\phi b_2y$, $D_1=a_{11}\phi$, $E_1=a_{12}\phi b_4$, $A_2=(1+r_2\phi)y$, $B_2=a_{21}\phi\tilde{d}_1y$, $C_2=a_{21}\phi\tilde{d}_2y$, and $D_2=1+a_{22}\phi y$, $E_2=a_{21}\phi d_3$, and $F_2=a_{21}\phi d_4$. Then equations (6) become

$$X=X(t)=\frac{A_1t+B_1Yt}{C_1+D_1t+E_1Y},$$
$$Y=Y(t)=\frac{A_2+B_2t+C_2X}{D_2+E_2t+F_2X}.$$

We have

$$(C_1+D_1t+E_1Y)X=A_1t+B_1Yt,\quad (D_2+E_2t+F_2X)Y=A_2+B_2t+C_2X.$$

Differentiating with respect to t we obtain the following system in X' and Y'

$$(C_1+D_1t+E_1Y)X'+(E_1X-B_1t)Y'=A_1+B_1Y-D_1X,$$
$$(F_2Y-C_2)X'+(D_2+E_2t+F_2X)Y'=B_2-E_2Y.$$

So

$$X'=\frac{(A_1+B_1Y-D_1X)(D_2+E_2t+F_2X)-(B_2-E_2Y)(E_1X-B_1t)}{(C_1+D_1t+E_1Y)(D_2+E_2t+F_2X)-(F_2Y-C_2)(E_1X-B_1t)}.$$

The denominator of X' is positive if

$$B_1C_2 = 0 \tag{15}$$

since the term E_1F_2XY is canceled if we expand the multiplication. Using Lemma 1 and the fact that $A_1 + B_1Y - D_1X = (E_1XY + C_1X)/t > 0$ we obtain that the numerator is positive if

$$B_2E_1 = 0. \tag{16}$$

If (15) and (16) are satisfied, then we have $X' > 0$ and X is increasing with t.

On the other hand, if we can show that $Y' < 0$, then Y is decreasing with t. Assume $X' > 0$, then

$$Y' = \frac{C_2(D_2 + E_2t) - F_2(A_2 + B_2t)}{(D_2 + E_2t + F_2X)^2}X' + \frac{B_2(D_2 + F_2X) - E_2(A_2 + C_2X)}{(D_2 + E_2t + F_2X)^2},$$

which is less than zero if

$$B_2 = C_2 = 0. \tag{17}$$

In order for all three conditions (15), (16), and (17) to be true, we only require $B_2 = C_2 = 0$, that is,

$$d_1 = d_2 = 0. \tag{18}$$

The above results are summarized in the following lemma.

Lemma 3.1. *Consider the system (6) under the assumptions (5), (8), (9), (12), (13), and (14). Additionally, we assume $d_1 = d_2 = 0$. Then if $0 < x_1 \le x_2$ and $y > 0$ then $X(x_1, y) \le X(x_2, y)$ and $Y(x_1, y) \ge Y(x_2, y)$.*

Similarly, by switching X and Y, respectively x and y in Lemma 3, we can show that for fixed x, X is a decreasing function of y and Y an increasing function of y. The result is stated in the following lemma.

Lemma 3.2. *Consider the system (6) under the assumptions (5), (8), (9), (12), (13), and (14). Additionally, we assume*

$$b_1 = b_3 = 0. \tag{19}$$

If $0 < y_2 \le y_1$ and $x > 0$ then $X(x, y_1) \le X(x, y_2)$ and $Y(x, y_1) \ge Y(x, y_2)$.

The following theorem states that the map T defined in equations (6) generates a discrete monotone flow in the 2-dimensional space.

Theorem 3.3. *Consider the system (6) under the assumptions (5), (8), (9), (12), (13), and (14). Additionally, we assume $b_1 = b_3 = d_1 = d_2 = 0$. If $0 < x_1 \le x_2$ and $0 < y_2 \le y_1$, then $X(x_1, y_1) \le X(x_2, y_2)$ and $Y(x_1, y_1) \ge Y(x_2, y_2)$. That is, the system (6) generates a monotone map.*

Proof. By Lemmas 3 and 4, we obtain

$$X(x_1, y_1) \le X(x_2, y_1) \le X(x_2, y_2) \text{ and } Y(x_1, y_1) \ge Y(x_2, y_1) \ge Y(x_2, y_2).$$

□

We can show that the dynamical system (6) is *dissipative* since all positive trajectories eventually lie in a bounded set.

Lemma 3.4. *Consider the system (6) under the assumptions (5), (8), (9), and (18), and (19). Then given $x > 0$ and $y > 0$, we have $0 < X \le \frac{1+r_1\phi}{a_{11}\phi}$ and $0 < Y \le \frac{1+r_2\phi}{a_{22}\phi}$.*

Proof. In Lemma 1, we have shown that X and Y are positive. Then from equations (6), we have

$$X \leq \frac{(1+r_1\phi)x}{a_{11}\phi\ x} = \frac{1+r_1\phi}{a_{11}\phi} \text{ and } Y \leq \frac{(1+r_2\phi)y}{a_{22}\phi\ y} = \frac{1+r_2\phi}{a_{22}\phi}.$$

□

Because the discrete system (6) is monotone and all solutions are bounded and every attractor contains a stable equilibrium, there are no attracting periodic orbits other than equilibria [3]. We have the following global results.

Theorem 3.5. *Consider the system (6) under the assumptions (5), (8), (9), (12), (13), and (14). Additionally, we assume* $b_1 = b_3 = d_1 = d_2 = 0$*. All solutions of the discrete system (6) converge to an equilibrium. Hence, if the interior equilibrium* E^* *is locally asymptotically stable, it is also globally asymptotically stable.*

4. Examples of Dynamically Consistent Models.

In order for the discrete system (6) to generate monotone flows, preserving positivity and boundedness of solutions, maintaining elementary stability, all of the assumptions (5), (8), (9), (12), (13), (14), (18), and (19) need to hold. The assumptions are reduced to

$$(b_1, b_2, b_3, b_4) = (0, \alpha, 0, 1-\alpha),\ (d_1, d_2, d_3, d_4) = (0, 0, \beta, 1-\beta), \tag{20}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $\alpha + \beta \geq 1$. The system (4) now becomes (7).

The author and Gelca [8] applied the following method to the Lotka-Volterra model (1) as the following

$$\begin{aligned} \frac{X-x}{\phi} &= r_1 x - a_{11}Xx - a_{12}[\alpha Xy + (1-\alpha)XY], \\ \frac{Y-y}{\phi} &= r_2 y - a_{21}[(1-\alpha)xY + \alpha XY] - a_{22}Yy. \end{aligned} \tag{21}$$

This system is a special case of (7) when $\beta = 1 - \alpha$.

In [6], the author gave two NSFD methods that are dynamically consistent to the continuous model with respect to properties (P1)–(P6). The two dynamically consistent methods are when $\alpha = 1$ and $\beta = 0$:

$$\begin{aligned} \frac{X-x}{h} &= r_1 x - a_{11}xX - a_{12}Xy, \\ \frac{Y-y}{h} &= r_2 y(t) - a_{21}XY - a_{22}yY, \end{aligned}$$

and when $\alpha = 0$ and $\beta = 1$:

$$\begin{aligned} \frac{X-x}{h} &= r_1 x - a_{11}xX - a_{12}XY, \\ \frac{Y-y}{h} &= r_2 y - a_{21}xY - a_{22}yY. \end{aligned}$$

Note that when $\alpha = \beta = 1$, we obtain Liu and Elaydi's model (3).

5. **Conclusion.** We give sufficient conditions for the NSFD methods (4) for the Lotka-Volterra competition model to produce dynamically consistent discrete-time competition models. The NSFD methods are given in the form

$$\frac{X-x}{\phi} = r_1 x - a_{11}Xx - a_{12}[\alpha Xy + (1-\alpha)XY],$$
$$\frac{Y-y}{\phi} = r_2 y - a_{21}[\beta xY + (1-\beta)XY] - a_{22}Yy,$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, and $\alpha + \beta \geq 1$. The discrete models preserve positivity, boundedness, and monotonicity of solutions. In addition, these methods are elementary stable; the local stability conditions are the same between the continuous models and the discrete models regardless of the step size. Our results generalize Liu and Elaydi's model in [4] and our own in [6, 8].

We only analyze the competition models, the construction of NSFD schemes for cooperative models or predator-prey models where x^2 and y^2 appear in the equations could be done similarly. Also, we would like to know how to develop these schemes to higher-dimensional systems. One of our future works is to find explicit schemes that are dynamically consistent.

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REFERENCES

[1] L.J.S. Allen, "An Introduction to Mathematical Biology," Prentice Hall, New Jersey, 2007.

[2] J. M. Cushing, Sheree LeVarge, Nakul Chitnis, and Shandelle M. Henson, *Some Discrete Competition Models and the Competitive Exclusion Principle*, Journal of Difference Equations and Applications, **10** (2004), 1139–1151.

[3] M. W. Hirsch and H. Smith, *Monotone dynamical systems*, in "Handbook of Differential Equations, Ordinary Differential Equations, volume 2," (eds. A. Canada, P. Drabek, and A. Fonda), Elservier B.V., Amsterdam 2005.

[4] P. Liu and S. N. Elaydi, *Discrete competitive and cooperative models of Lotka-Volterra type*, Journal of Computational Analysis and Applications, **3** (2001), 53–73.

[5] R. E. Mickens, "Advances in the Applications of Nonstandard Finite Difference Schemes," World Scientific, Singapore, 2005.

[6] L.-I. W. Roeger, *Dynamically consistent discrete Lotka-Volterra competition models derived from nonstandard finite-difference schemes*, Discrete and Continuous Dynamical Systems Series B, **9** (2008), 415–429.

[7] L.-I. W. Roeger, *Periodic solutions preserved by nonstandard finite-difference schemes for Lotka-Volterra system: a different approach*, Journal of Difference Equations and Applications, **14** (2008), 481–493.

[8] L.-I. W. Roeger and R. Gelca, *Dynamically consistent discrete monotone competition models by a nonstandard finite difference scheme*, unpublished work.

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