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ON POSITIVITY AND BOUNDEDNESS OF SOLUTIONS OF NONLINEAR STOCHASTIC DIFFERENCE EQUATIONS

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Abstract. Consider nonlinear stochastic difference equations

$$X(n+1) = X(n) + hf(X(n)) + \sqrt{h}g(X(n))\xi_{n+1},\ n \in \mathbb{N},\ X(0) = \varsigma \in \mathbb{R},\ (1)$$

where $\{\xi_n\}_{n\in\mathbb{N}}$ are independent $\mathcal{N}(0,1)$-distributed random variables, $h > 0$, viewed as a discretization of Itô stochastic differential equations (SDEs). We discuss the following. If, for all $t \geq 0$, the solution $Y(t)$ of the corresponding SDE is positive, or $Y(t) \in [0, K]$ for some $K > 0$, does the solution $X(n)$ of related discretization (1) possess the same properties with large probability? In general, the answer is no. However in many cases we are able to discretize the SDE related to (1) over a compact interval $[0, T]$ in such a way that an adequate qualitative behavior is observed with an arbitrarily high probability.

1. Introduction.

The problem of keeping reasonable boundaries for analytic solutions under discretization plays an essential role for practically meaningful models, describing the evolution of species in population dynamics, interest rates in mathematical finance and others (see e.g. [6], [9]).

In this paper we consider stochastic difference equations

$$X(n+1) = X(n) + hf(X(n)) + \sqrt{h}g(X(n))\xi_{n+1},\ n = 1, 2, \dots;\ X(0) = \varsigma \in \mathbb{R},\ (2)$$

$h > 0$, as well as Itô stochastic differential equations

$$dY(t) = f(Y(t))dt + g(Y(t))dW(t), \quad t > 0, \quad Y(0) = \varsigma \in \mathbb{R}. \quad (3)$$

Here $(\xi_n)_{n\in\mathbb{N}}$ are i.i.d. $\mathcal{N}(0,1)$-distributed random variables on $(\Omega, \mathcal{F}_n, \mathbb{N})$ with probability distribution function Φ, and $(W(t))_{t\geq 0}$ is an one-dimensional standard Wiener process. The initial value $\varsigma \in \mathbb{R}$ is supposed to be nonrandom. We suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous, nonrandom functions.

The main emphasis of this paper is as follows. Suppose that the initial value $\varsigma \in (0, K)$. Can we guarantee for some natural choice of f and g that the solution X to (2) or / and solution Y to (3) stay in $(0, K)$ with probability close to zero? In this paper we concentrate on some special cases of functions f and g with

$$f(u) = u\mathsf{f}(u), \quad g(u) = u\mathsf{g}(u), \qquad \text{and} \quad (4)$$

$$f(u) = u(K-u)\mathsf{f}(u), \quad g(u) = u(K-u)\mathsf{g}(u), \quad K > 0. \quad (5)$$

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Both cases (4) and (5) are very much related to modeling of populations. In particular, $K > 0$ can be interpreted as the carrying capacity limited by a finite number of natural resources.

We note that (2) can be viewed as an equidistant numerical discretization of Euler-Maruyama type of the stochastic Itô differential equation (3). We recall that Euler-Maruyama (EM) numerical method computes approximations $X(n) \approx Y(nh)$ where Y is solution to (3) and X is a solution to (2) with $\sqrt{h}\xi_{n+1} = W((n+1)h) - W(nh) \in \mathcal{N}(0,h)$, $h > 0$ is the constant step size.

Positivity of solutions to stochastic Itô differential equation of type (3) is discussed in [2] and [10]. In particular, it is shown in [2] (see Proposition 2.4 in [2]) that, if condition (4) holds with locally Lipschitz continuous functions f and g, then the solution $Y(t)$ of equation

$$dY(t) = Y(t)[f(Y(t))dt + g(Y(t))dW(t)], \quad t > 0, \quad Y(0) = \varsigma, \tag{6}$$

with arbitrary initial value $\varsigma > 0$ remains positive until its explosion time (if any). The natural question arises: does the property of positivity remains for the solution of its discretization

$$X(n+1) = X(n)[1 + hf(X(n)) + \sqrt{h}g(X(n))\xi_{n+1}], \quad n = 1,2,\ldots; \quad X(0) = \varsigma, \tag{7}$$

of equation (6)? In general, the answer to this question is "NO". For linear equations, this is shown in [11]. Some results for nonlinear equations are obtained in [4] (the brief description of them is given in Section 3). In Section 3 we discuss when the solution $X(n)$ of the equation (7)(with $h = 1$ for simplicity) leaves the interval $(0, K)$. In particular, $X(n)$ leaves any interval $[0, K]$ with probability 1 if $g(u) > \sigma > 0$. We can also state that, if $f(u) > 0$ and $f(u) \to \infty$ as $u \to \infty$, the probability that the solution X tends to infinity faster than some power-law rate depending on f is positive. A similar result can be proved in the case when $uf(u) < 0$. In both cases we are able to verify that the above mentioned probability becomes closer to 1 when $|X(0)|$ becomes large enough.

In Section 2 we continue the investigation initiated in [10]. We state assumptions which have to be imposed on functions f and g in order to guarantee that solution $Y(t)$ to stochastic differential Itô equations (3) or (6) remains in a given interval $[0, K]$ a.s. or with probability close to 1. We also show that, if $f(x) \geq 0$ and $f(K) \neq 0$ or $g(K) \neq 0$, the solution $Y(t)$ to (6) leaves the interval $[0, K]$ in finite time with non-zero probability. Thus, if we expect that the solution $Y(t)$ does not leave the interval $[0, K]$, it is reasonable to consider the equation

$$dY(t) = Y(t)(K - Y(t))[\mathsf{f}(Y(t))dt + \mathsf{g}(Y(t))dW(t)], \quad t > 0, \quad Y(0) = \varsigma. \tag{8}$$

Equations (8) with $\mathsf{f}(u) \equiv const$ and $\mathsf{g}(u) \equiv const$ are considered in [10]. It is shown there that $Y(t) \in [0, K]$ a.s., if $Y(0) = \varsigma \in [0, K]$. The same proof can be conducted in the case of any continuous functions $\mathsf{f}(u)$ and $\mathsf{g}(u)$ on $[0, K]$.

The Euler-Maruyama-type discretization of equation (8) has the form

$$X(n+1) = X(n)[1 + h(K - X(n))\mathsf{f}(X(n)) + \sqrt{h}(K - X(n))\mathsf{g}(X(n))\xi_{n+1}], \tag{9}$$

$n = 1, 2, \ldots; \quad X(0) = \varsigma$. In Section 4 we show that, for any $K > 0$, solutions $X(n)$ of the equation (9)(with $h = 1$) leave $[0, K]$ a.s. if $\mathsf{g}(u) \geq \sigma > 0$ for all $u \in [0, K]$.

In the case when f and g have a power-law behavior in the neighborhood of zero, i.e.

$$f(u) \sim -a|u|^{1+\alpha}, \quad g^2(u) \sim b|u|^{1+\beta} \quad \text{as} \quad u \to 0, \tag{10}$$

it is shown in [1] that, for some relations between the coefficients α, β, a, b, the solution $X(n)$ remains positive for all $n \in \mathbb{N}$ with probability close to 1 (see also

Section 3.1). Since at the same time this solution $X(n)$ tends to zero, we conclude that $X(n) \in [0, K]$ for some $K > 0$ with probability close to 1.

When we consider the equation (2) as an Euler-Maruyama type discretization of (6) on some finite time-interval $[0, T]$ the situation can be improved. In Section 4.2 we prove that by making the number N of partitions of the interval $[0, T]$ sufficiently large (which means that we make the step size $h = \frac{T}{N}$ sufficiently small) we can guarantee that the solutions $X(n)$ to equation (9) remain in $[0, K]$ with large probability for all $n = 1, 2, \dots, N$. Moreover, results for positivity of solutions of (8) on finite time-intervals $[0, T]$ are proved in [4] and reformulated in Section 3.

On basic notation. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space and $\mathcal{B}$ is the σ-algebra of Borel sets. We suppose that the underlying filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is naturally generated, i.e. $\mathcal{F}_n = \sigma\{\xi_0, \xi_1, \dots \xi_n\}$. Moreover, we use the standard abbreviation "*a.s.*" for the wordings "almost sure" or "almost surely" with respect to the fixed probability measure $\mathbb{P}$ throughout the text. The Borel-Cantelli Lemma and Dynkin's formula are given in the Appendix. For more details on stochastic concepts, the reader may consult [12].

2. Boundedness of solutions of Itô differential equations.

In this section we consider Itô equations of the type

$$dY(t) = Y(t)[f(Y(t))dt + g(Y(t))dW(t)], \quad t > 0, \quad Y(0) = \varsigma \in (0, K), \tag{11}$$

where

$$f : \mathbb{R} \to \mathbb{R} \text{ and } g : \mathbb{R} \to \mathbb{R} \text{ are locally Lipschitz continuous functions.} \tag{12}$$

In subsection 2.1 we adapt results from [2] on positivity of solutions of (11) under (12). In subsection 2.2 we show that, if at least one of the functions $f(K) \neq 0$ or $g(K) \neq 0$, the solutions $Y(t)$ leave the interval $(0, K)$ in finite time with non-zero probability for any $K > 0$.

2.1. On positivity.

By virtue of condition (12), it is guaranteed that there is a unique continuous adapted process Y such that

$$Y(t \wedge \tau_e) = \varsigma + \int_0^{t\wedge\tau_k} f(Y(s))\, ds + \int_0^{t\wedge\tau_k} g(Y(s))\, dW(s), \quad t \geq 0, \quad \text{a.s.}, \tag{13}$$

where $\tau_k = \inf\{t > 0 : |Y(t)| \geq k\}$. The equation (11) has a global solution if the explosion time τ_e defined by

$$\tau_e = \lim_{k\to\infty} \tau_k = \inf\{t > 0 : |Y(t)| \notin [0, \infty)\} \tag{14}$$

obeys the identity $\tau_e = \infty$ (a.s.). We define

$$\vartheta_0 = \inf\{t > 0 : |Y(t)| = 0\}. \tag{15}$$

Proposition 1 ([2])**.** *Suppose that f and g possess the property (12), $\varsigma \neq 0$ and Y is the unique continuous adapted process that obeys (11). For τ_e and ϑ_0 defined by (15), (14), we have $\tau_e \leq \vartheta_0$ a.s.*

2.2. Solution does not remain in $(0, K)$.

Theorem 2.1. *Suppose that f and g obey (12), and*

$$\exists \alpha > 0\ \forall x \in [0, K] \ : \quad f(x) + \alpha g^2(x) > 0. \tag{16}$$

Then the solution $Y(t)$ to (11) leaves $(0, K)$ in finite time with non-zero probability.

Proof. Proposition 1 implies that the solution to (11) is unique, positive for $Y(0) = \varsigma > 0$, and exists until the stopping time τ_e. Since $f, g \in C^0([0,K])$, we know that

$$\exists \beta > 0 \; \forall x \in [0,K] \;: \quad f(x) + \alpha g^2(x) > \beta. \tag{17}$$

Let M be an odd number such that $M > 1 + 2\alpha$. Then, for $x \in [0,K]$, we have

$$f(x) + \frac{M-1}{2} g^2(x) > f(x) + \alpha g^2(x) > \beta.$$

Introduce the differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + u f(u) \frac{\partial}{\partial u} + \frac{1}{2} u^2 g^2(u) \frac{\partial^2}{\partial u^2} \tag{18}$$

and the Liapunov function $V(x) = x^M$. Now, compute

$$\mathcal{L}V(x) = Mx^{M-1} x f(x) + \frac{1}{2} M(M-1) x^{M-2} x^2 g^2(x) = Mx^M \left(f(x) + (M-1) \frac{g^2(x)}{2} \right).$$

Note that, for $x \in [0,K]$ and $b^2 = M\beta$, we find that

$$\mathcal{L}V(x) \geq M\beta x^M = b^2 V(x). \tag{19}$$

Define $W(x,t) = e^{-b^2 t} V(x)$. Then, for $x \in [0,K]$, we have $\mathcal{L}W(x,t) \geq 0$. After applying Dynkin's formula (see Appendix, Lemma 5.2), we can estimate the expected value $\mathbb{E}[W(Y(t \wedge \tau_e), t \wedge \tau_e)] \geq \varsigma^M$ where $\tau_e = \inf\{t > 0 : Y(t) \notin (0,K)\}$ represents the first exit time of Y from $(0,K)$. Thus, we obtain that for all $t > 0$

$$\mathbb{E}\left[e^{-b^2(t\wedge\tau_e)}\right] = \mathbb{E}\left[\frac{e^{-b^2(t\wedge\tau_e)} Y^M(t\wedge\tau_e)}{Y^M(t\wedge\tau_e)}\right] \geq \mathbb{E}\left[\frac{e^{-b^2(t\wedge\tau_e)} Y^M(t\wedge\tau_e)}{\sup_{u\in[0,K]} V(u)}\right] \geq \frac{\varsigma^M}{K^M}. \tag{20}$$

Taking $t \to \infty$ in this inequality leads to $\mathbb{E}\left[e^{-b^2\tau_e}\right] \geq (\varsigma/K)^M > 0$. The latter relation immediately implies that $\mathbb{P}\{e^{-b^2\tau_e} > 0\} = \mathbb{P}\{\tau_e < \infty\} > 0$. □

Remark 1. If

$$\inf_{x\in[0,K]} \frac{f(x)}{g^2(x)} > -\infty \tag{21}$$

then condition (16) holds.

Corollary 1. *Let one of the following conditions be satisfied for some $a, b \neq 0$:*

$$f(x) > 0, \quad x \in [0,K), \quad |g(x)| > |b|, \quad x \in [0,K], \quad \textit{or} \tag{22}$$
$$g(x) > 0, \quad x \in [0,K), \quad f(x) > a^2, \quad x \in [0,K]. \tag{23}$$

Then the solution $Y(t)$ to (11) leaves $(0,K)$ in finite time with non-zero probability.

Corollary 2. *Suppose that $f(u) \geq 0$, f and g have zeros only at 0 or K and $f(K) \neq 0$ or $g(K) \neq 0$. Then $Y(t)$ takes values larger than K in finite time with non-zero probability.*

Example. Conditions of Theorem 2.1 and Corollary 2 are satisfied for $f(u) = a^2 u(K-u)$, $g(u) = ub$ or $g(u) = bu(K-u)$, $f(u) = a^2 u$, $b \neq 0$.

3. **Boundedness of stochastic difference equations (7).** In this section we briefly review the results from [4] applied to the equation (7). Without loss of generality, we refer to f instead of hf and g instead of $\sqrt{h}g$ in (7), since our major conditions are invariant with respect to the choice of step sizes under that replacement, so (7) takes the form

$$X(n+1) = X(n)\Big(1 + f(X(n)) + g(X(n))\xi_{n+1}\Big), \quad n = 1, 2, \dots; \quad X(0) = \varsigma. \quad (24)$$

We discuss how the problem of boundedness of solutions X to (24) from above can be reduced to the "positivity problem" and prove new results on its unboundedness.

3.1. **On non-positivity and positivity of solution.** Let

$$\inf_{u>0}\left\{-\frac{1+f(u)}{|g(u)|}\right\} > -\infty. \quad (25)$$

The following lemma states that under assumption (25) solution $X(n)$ to (24) becomes negative with probability 1.

Lemma 3.1. *Let condition (25) hold. Assume that $X(n)$ is a solution to (24) with positive initial condition $X(0) > 0$. Then there exists an a.s. finite stopping time $\tau_0 : \Omega \to \mathbb{N}$ such that*

$$X(\tau_0(\omega), \omega) \le 0 \qquad a.s.$$

However, when we consider stochastic difference equations (7) on finite time intervals $[0, T]$ while making the step size $h = \frac{T}{N}$ sufficiently small, we can guarantee that its solution remains positive with large probability for all $n = 1, 2, \dots, N$. Namely, consider difference equations

$$X(n+1) = X(n)\left(1 + \frac{T}{N}f(X(n)) + \sqrt{\frac{T}{N}}g(X(n))\xi_{n+1}\right), \; n = 1, 2, \dots; \; X(0) = \varsigma, \quad (26)$$

and denote its solution by $X_N(n, \cdot)$. Define

$$P_N = \{\omega \in \Omega : X_N(n, \omega) > 0, \forall n = 1, 2, \dots, N\}, \quad (27)$$

$$Q(N) = \sup_{u>0}\left\{-\frac{1 + \frac{T}{N}f(u)}{\sqrt{\frac{T}{N}}|g(u)|}\right\}. \quad (28)$$

Theorem 3.2. *Let $X_N(n, \cdot)$ be a solution to (26), P_N and $Q(N)$ be defined by (27) and (28) respectively. Assume that*

(i) $Q(N) < \infty$ for each $N \in \mathbb{N}$; *(ii) $Q(N) \to -\infty$ when $N \to +\infty$;*
(iii) $\limsup_{N\to+\infty} \frac{N}{Q^2(N)} < \infty$.

Then, $\forall \gamma > 0 \, \exists \bar{N}(\gamma) \in \mathbb{N} \, \forall N > \bar{N}(\gamma) \, : \, \mathbb{P}(P_N) > 1 - \gamma$.

Remark 2. When f and g satisfy (10), it is proved in [1] that, for some combination of $\alpha > 0$, $\beta > 0$, a and b and sufficiently small initial conditions $X(0) = \varsigma$, the solution $X(n)$ remains positive with probability close to 1. Since at the same time this solution tends to zero, it stays in some interval $[0, K]$ with probability close to 1. Note that in this case (25) does not hold.

Remark 3. The problem of boundedness of solutions to equation

$$X(n+1) = X(n) + f(X(n)) + g(X(n))\xi_{n+1}, \quad n = 1, 2, \dots; \quad X(0) = \varsigma, \quad (29)$$

by some number $K > 0$ can be reduced to the problem of positivity. Namely, let

$$Y(n) = K - X(n), \quad -f(K-u) = \tilde{f}(u), \quad -g(K-u) = \tilde{g}(u).$$

Then the equation (29) takes the form

$$Z(n+1) = Z(n) + \tilde{f}(Z(n)) + \tilde{g}(Z(n))\xi_{n+1}, \quad n = 1, 2, \dots; \quad Z(0) = K - \varsigma. \tag{30}$$

Therefore, X exceeds K if and only if Z becomes negative (a.s.).

3.2. **Solution leaves** $[0, K]$ **a.s.** For some $\sigma > 0$, let

$$g(u) \geq \sigma > 0, \quad u \in \mathbb{R}. \tag{31}$$

Lemma 3.3. *Assume that the condition (31) holds. Then, for any $K > 0$, the solution $X(n)$ to (24) leaves the interval $[0, K]$ with probability 1.*

3.3. $\mathbf{f}(n) > 0$ **for large** n**.** In the following two subsections we consider

$$X(n+1) = X(n)\left(1 + f(X(n)) + \sigma\xi_{n+1}\right), \quad n = 1, 2, \dots; \quad X(0) = \varsigma, \tag{32}$$

where function f is asymptotically positive and $f(n) \to \infty$ more quickly than $\ln^{1/2}(n)$. We can show that $|X(n)|$ increases to infinity on a set with non-zero probability. For the sake of a more precise formulation, suppose that

$$|f(u)| \uparrow \infty, \quad \text{as} \quad |u| \to \infty, \quad |f(u)| > (\ln u)^{\frac{1}{2}+\varepsilon}. \tag{33}$$

Lemma 3.4. *Let (33) hold. Let $f(u) \uparrow \infty$ as $u \to \infty$, and let $X(n)$ be a solution to (32). Then $\mathbb{P}\{X(n) > n, \quad n = 1, 2, \dots\} > 0$.*

3.4. $u\mathbf{f}(u) < 0$ **for large** u**.**

Lemma 3.5. *Let (33) hold, $f(u) \downarrow -\infty$ as $u \to \infty$ and $X(n)$ be a solution to (32). Then $\mathbb{P}\{X(2k+1) < -2(k+1), X(2k) > 2k+1, \quad k = 0, 1, 2, \dots\} > 0$.*

4. **On the behavior of difference equations (9).** In this section we consider equation (9) where, again, we refer to f instead of $h\mathsf{f}$ and g instead of $\sqrt{h}\mathsf{g}$, e.g.

$$X(n+1) = X(n)[1 + (K - X(n))\mathsf{f}(X(n)) + (K - X(n))\mathsf{g}(X(n))\xi_{n+1}], \tag{34}$$

where $\mathsf{f} : \mathbb{R} \to \mathbb{R}$ and $\mathsf{g} : \mathbb{R} \to \mathbb{R}$ are supposed to be continuous, and, for some $b > 0$, we have

$$\mathsf{g}(u) > b > 0, \quad u \in (0, K). \tag{35}$$

We prove that with probability 1 the solution $X(n)$ leaves the interval $[0, K]$ in finite time. We also show that when we treat equation (9) as an Euler-Maruyama-type discretization of (8) on the finite interval $[0, T]$, the solution $X(n) \in (0, K)$ with probability close to 1, if the partition of $[0, T]$ is sufficiently small.

4.1. **Solution does not remain in** $[0, K]$**.**

Theorem 4.1. *Let condition (35) hold and $X(0) = \varsigma \in (0, K)$. Then the solution $X(n)$ to (34) leaves the interval $[0, K]$ in finite time with probability 1.*

Proof. For each $n \in \mathbb{N}$, define

$$\bar{B}_n = \{\omega \in \Omega : X(i, \omega) \in (0, K), \forall i = 1, 2, \dots, n\}. \tag{36}$$

Since $B_n = \{\omega \in \Omega : \exists i \leq n : X(i, \omega) \notin (0, K)\}$, we can conclude from the Borel-Cantelli Lemma (see Appendix) that solution $X(n)$ to (34) leaves the interval $(0, K)$ with probability 1 if

$$\sum_{n=0}^{\infty} \mathbb{P}(\bar{B}_n) < +\infty. \tag{37}$$

For the sake of abbreviation, set $E_n = \{\omega \in \Omega : X(n,\omega) \in (0,K)\}$. We calculate

$$\begin{aligned}
&\mathbb{P}\left\{\omega \in \Omega : X(n+1,\omega) \in (0,K) \middle| X(n) \in (0,K)\right\}\\
=\ &\mathbb{P}\left\{-\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \xi_{n+1} < \frac{1}{X(n)\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} \middle| E_n\right\}\\
=\ &\mathbb{P}\left\{-\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \xi_{n+1} < \frac{1}{X(n)\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} \middle| \right.\\
&\qquad \left. X(n) \in (0,K), X(n) < \frac{K}{2}\right\} \times \mathbb{P}\left\{X(n) < \frac{K}{2}\right\}\\
&+ \mathbb{P}\left\{-\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \xi_{n+1} < \frac{1}{X(n)\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} \middle| \right.\\
&\qquad \left. X(n) \in (0,K), X(n) \geq \frac{K}{2}\right\} \times \mathbb{P}\left\{X(n) \geq \frac{K}{2}\right\}.
\end{aligned}$$

For $X(n) \in \left(0, \frac{K}{2}\right)$, by (35) we obtain that

$$K - X(n) > \frac{K}{2}, \qquad \frac{1}{(K-X(n))\mathsf{g}(X(n))} < \frac{1}{b\frac{K}{2}} = \frac{2}{Kb}$$

and, for $X(n) \in \left(\frac{K}{2}, K\right)$, by (35) we encounter

$$X(n) > \frac{K}{2}, \qquad \frac{1}{X(n)\mathsf{g}(X(n))} < \frac{1}{b\frac{K}{2}} = \frac{2}{Kb}.$$

For all $X(n) \in (0,K)$ by (35) and by continuity of f and g, we also have

$$\frac{\hat{K}_f}{\hat{K}_g} < \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \frac{\tilde{K}_f}{b}$$

with some $\hat{K}_f > 0$, $\hat{K}_g > 0$ and $\tilde{K}_f > 0$. Therefore, for $X(n) \in \left(0, \frac{K}{2}\right)$, we find that

$$-\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\hat{g}(X(n))} > -\frac{2}{Kb} - \frac{\tilde{K}_f}{b} \quad \text{and}$$
$$\frac{1}{X(n)\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \infty.$$

Also, for $X(n) \in \left(\frac{K}{2}, K\right)$, we have

$$-\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} > -\infty \quad \text{and}$$
$$\frac{1}{X(n)\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \frac{2}{Kb} - \frac{\hat{K}_f}{\hat{K}_g}.$$

Taking into account all observations from above, we get

$$\begin{aligned}
&\mathbb{P}\left\{\frac{-1}{(K-X(n))\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} < \xi_{n+1} < \frac{1}{X(n)\mathsf{g}(X(n))} - \frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} \middle| X(n) \in \left(0, \frac{K}{2}\right)\right\}\\
&\leq \mathbb{P}\left\{-\frac{2}{Kb} - \frac{\tilde{K}_f}{b} < \xi_{n+1} < \infty \middle| X(n) \in (0, \frac{K}{2})\right\} = \mathbb{P}\left\{-\frac{2}{Kb} - \frac{\tilde{K}_f}{b} < \xi_{n+1} < \infty\right\}.
\end{aligned} \tag{38}$$

Let ζ be $\mathcal{N}(0,1)$-distributed. We define

$$q_l := \mathbb{P}\left\{-\frac{2}{Kb} - \frac{\tilde{K}_f}{b} < \zeta < \infty\right\}, \tag{39}$$

and note that $q_l \in (0,1)$. By the same procedure as above, we get

$$\mathbb{P}\Big\{\frac{-1}{(K-X(n))\mathsf{g}(X(n))}-\frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))}<\xi_{n+1}<\frac{1}{X(n)\mathsf{g}(X(n))}-\frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))}\Big|X(n)\in\Big(\frac{K}{2},K\Big)\Big\}$$
$$\le \mathbb{P}\left\{-\infty<\xi_{n+1}<\frac{2}{Kb}-\frac{\hat{K}_f}{\hat{K}_g}\middle|X(n)\in(\frac{K}{2},K)\right\}=\mathbb{P}\left\{-\infty<\xi_{n+1}<\frac{2}{Kb}-\frac{\hat{K}_f}{\hat{K}_g}\right\}. \tag{40}$$

Let ζ be $\mathcal{N}(0,1)$-distributed. Define

$$q_r := \mathbb{P}\left\{-\infty<\zeta<\frac{2}{Kb}-\frac{\hat{K}_f}{\hat{K}_g}\right\}, \tag{41}$$

and note that $q_r \in (0,1)$. Then $q := \max\{q_r, q_l\} \in (0,1)$. Consequently, we arrive at

$$\begin{aligned}
&\mathbb{P}\left\{\omega\in\Omega : X(n+1,\omega)\in(0,K)\middle|X(n)\in(0,K)\right\}\\
&=\mathbb{P}\left\{\omega\in\Omega : X(n+1,\omega)\in(0,K)\middle|X(n)\in\left(0,\frac{K}{2}\right)\right\}\mathbb{P}\left\{X(n)<\frac{K}{2}\right\}\\
&\quad+\mathbb{P}\left\{\omega\in\Omega : X(n+1,\omega)\in(0,K)\middle|X(n)\in\left(\frac{K}{2},K\right)\right\}\mathbb{P}\left\{X(n)\ge\frac{K}{2}\right\}\\
&\le q_l\mathbb{P}\left\{X(n)<\frac{K}{2}\right\}+q_r\mathbb{P}\left\{X(n)\ge\frac{K}{2}\right\}\le q<1.
\end{aligned} \tag{42}$$

Therefore, $\mathbb{P}(\bar{B}_n)=\mathbb{P}\{\omega\in\Omega : X(i,\omega)\in(0,K), \forall i=1,2,\dots,n\}$ satisfies

$$\mathbb{P}(\bar{B}_n) \;=\; \prod_{i=1}^{n}\mathbb{P}\left\{\omega\in\Omega : X(i,\omega)\in(0,K)\middle|X(i-1)\in(0,K)\right\}\le q^n.$$

Thus, $\sum_{n=0}^{\infty}\mathbb{P}(\bar{B}_n)\le 1/(1-q)<\infty$ and, by the Borel-Cantelli Lemma, there is a finite stopping time τ_0 with $\tau_0(\omega)=\inf\{i\in\mathbb{N} : X(i,\omega)\notin[0,K]\}$ and $X(\tau)\notin[0,K]$. Note that the events $\{X(n)=0\}$ and $\{X(n)=K\}$ have probability 0 when $\varsigma\neq 0,K$ since ξ is normally distributed and hence X possess continuous probability distributions. Consequently, the proof of Theorem 4.1 is complete. □

Remark 4. A similar proof can be conducted in the case

$$g(u)=u^{\alpha_1}(K-u)^{\alpha_2}\mathsf{g}(u), \quad \mathsf{g}(u)>0, \quad u\in(0,K),$$

for some $\alpha_1,\alpha_2\in[0,1]$. However, when at least one of α_1 or α_2 is larger than 1, the situation is different. In view of Remark 3 we can apply results from [1] and conclude that under some additional assumptions on α_1 and α_2, $X(n)\to K$ as $n\to+\infty$ and X stays in $(0,K)$ with probability close to 1.

4.2. $X(n)\in(0,K)$ **with probability close to 1.** Let $T>0$ and $N\in\mathbb{N}$ are some nonrandom numbers. Consider stochastic difference equation (9) with $h=\frac{T}{N}$

$$X(n+1)=X(n)\left(1+\frac{T}{N}(K-X(n))\mathsf{f}(X(n))+\sqrt{\frac{T}{N}}(K-X(n))\mathsf{g}(X(n))\xi_{n+1}\right) \tag{43}$$

where $n=1,2,\dots,N$ and $X(0)>0$. Define

$$P_N=\{\omega\in\Omega : X_N(n,\omega)\in(0,K), \forall n=1,2,\dots,N\}. \tag{44}$$

Theorem 4.2. *Let $X_N(n,\cdot)$ be the solution to (43) and let P_N be defined by (44). Then, $\forall\gamma>0\ \exists\bar{N}(\gamma)\in\mathbb{N}\ \forall N>\bar{N}(\gamma)\ :\ \mathbb{P}(P_N)>1-\gamma$.*

Proof. Set $E_n = \{\omega \in \Omega : X_N(n,\omega) \in (0,K)\}$. We calculate

$$\mathbb{P}\left\{\omega \in \Omega : X(n+1,\omega) \in (0,K) \middle| E_n\right\} = \mathbb{P}\left\{A_n < \xi_{n+1} < B_n \middle| E_n\right\} \qquad \text{where}$$

$$A_n = -\sqrt{\frac{N}{T}}\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \sqrt{\frac{T}{N}}\frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))}, \quad B_n = \sqrt{\frac{N}{T}}\frac{1}{X(n)\mathsf{g}(X(n))} - \sqrt{\frac{T}{N}}\frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))}.$$

Due to property (35) there is a constant $a > 0$ such that

$$\inf_{u\in(0,K)} \frac{1}{(K-u)\mathsf{g}(u)} > a, \quad \inf_{u\in(0,K)} \frac{1}{u\hat{g}(u)} > a. \tag{45}$$

Due to continuity of f and property (35) we also have $\sup_{u\in(0,K)} |\mathsf{f}(u)/\mathsf{g}(u)| = c$ with some $c > 0$. Therefore, when $X(n) \in (0,K)$, $n = 1,2,\ldots,N$, we find that

$$\begin{aligned} -\sqrt{\frac{N}{T}}\frac{1}{(K-X(n))\mathsf{g}(X(n))} - \sqrt{\frac{T}{N}}\frac{\mathsf{f}(X(n))}{\hat{g}(X(n))} &\leq -\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c, \\ \sqrt{\frac{N}{T}}\frac{1}{X(n)\mathsf{g}(X(n))} - \sqrt{\frac{T}{N}}\frac{\mathsf{f}(X(n))}{\mathsf{g}(X(n))} &\geq \sqrt{\frac{N}{T}}a - \sqrt{\frac{T}{N}}c. \end{aligned}$$

Thus, we can estimate

$$\begin{aligned} &\mathbb{P}\left\{\omega \in \Omega : X_N(n+1,\omega) \in (0,K) \middle| X_N(n) \in (0,K)\right\} \\ \geq\ &\mathbb{P}\left\{-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c < \xi_{n+1} < \sqrt{\frac{N}{T}}a - \sqrt{\frac{T}{N}}c \middle| X_N(n) \in (0,K)\right\} \\ =\ &\mathbb{P}\left\{-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c < \xi_{n+1} < \sqrt{\frac{N}{T}}a - \sqrt{\frac{T}{N}}c\right\} \\ =\ &\Phi\left(\sqrt{\frac{N}{T}}a - \sqrt{\frac{T}{N}}c\right) - \Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right) = 1 - 2\Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right). \end{aligned}$$

We note that $1 - 2\Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right) \to 1$ as $N \to \infty$. Thus, we arrive at

$$\begin{aligned} &\mathbb{P}(P_N) = \mathbb{P}\{\omega \in \Omega : X_N(i,\omega) \in (0,K), \forall i = 1,2,\ldots,N\} \\ =\ &\prod_{i=1}^{N} \mathbb{P}\left\{X_N(i) \in (0,K) \middle| X_N(i-1) \in (0,K)\right\} \geq \left(1 - 2\Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right)\right)^N. \end{aligned}$$

Now we are able to prove that

$$\lim_{N\to\infty}\left(1 - 2\Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right)\right)^N = 1$$

which is equivalent to proving that $\lim_{N\to\infty} N \ln\left(1 - 2\Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right)\right) = 0$.

Since ξ_n have bounded second moments, we have that $\lim_{y\to\infty} y^2\Phi(-y) = 0$. Applying this fact and also the limit $\lim_{x\to 0+} \ln(1-x)/x = -1$, we arrive at

$$\begin{aligned} &\lim_{N\to\infty} N \ln\left(1 - \Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right)\right) = -\lim_{N\to\infty} N\Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right) \\ =\ &-\lim_{N\to\infty}\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right)^2 \Phi\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right) \lim_{N\to\infty} N\left(-\sqrt{\frac{N}{T}}a + \sqrt{\frac{T}{N}}c\right)^{-2} = 0. \end{aligned}$$

Thus $\mathbb{P}(P_N) \to 1$ as $N \to \infty$ and Theorem 4.2 is proved. □

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5. **Appendix.** Below we formulate two famous results: the Borel-Cantelli Lemma and the Dynkin's formula in the most suitable form (see e.g.[7, 8, 12]) for our analysis.

Lemma 5.1. *(***Borel-Cantelli***) Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of events in some probability space. If $\sum_{n=1}^{\infty}\mathbb{P}(E_n)<\infty$, then the probability that infinitely many of E_n occur is 0.*

Lemma 5.2. *(***Dynkin's formula***) Let $Y(t)$ be a continuous adapted process satisfying equation (3) with nonrandom initial condition $Y(0)=\varsigma$. Let $D\in\mathbb{R}$ be an open, connected domain and let $F: D\to\mathbb{R}$ be a twice differentiable function. Let operator $\mathcal{L}$ be defined by*

$$\mathcal{L}=\frac{\partial}{\partial t}+f(u)\frac{\partial}{\partial u}+\frac{1}{2}g^2(u)\frac{\partial^2}{\partial u^2}.$$

Let $\tau=\inf\{t: Y(t)\notin D\}$ and let $\tau(t)=\tau\wedge t$. Then for all $\varsigma\in D$ and all $t\geq 0$ we have

$$\mathbb{E}F(Y(\tau(t)))=F(\varsigma)+\mathbb{E}\left[\int_0^{\tau(t)}\mathcal{L}F(Y(s))ds\right].$$

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