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ON NORMAL STABILITY FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. We show convergence of solutions to equilibria for quasilinear and fully nonlinear parabolic evolution equations in situations where the set of equilibria is non-discrete, but forms a finite-dimensional C^1-manifold which is normally stable.

1. **Introduction.** In this short note we consider quasilinear as well as fully nonlinear parabolic equations and we study convergence of solutions towards equilibria in situations where the set of equilibria forms a C^1-manifold.

Our main result can be summarized as follows: suppose that for a nonlinear evolution equation we have a C^1-*manifold of equilibria* $\mathcal{E}$ such that at a point $u_* \in \mathcal{E}$, the kernel $N(A)$ of the linearization A is isomorphic to the tangent space of $\mathcal{E}$ at u_*, the eigenvalue 0 of A is semi-simple, and the remaining spectral part of the linearization A is stable. Then solutions starting nearby u_* exist globally and converge to some point on $\mathcal{E}$. This situation occurs frequently in applications. We call it the *generalized principle of linearized stability*, and the equilibrium u_* is then termed *normally stable*.

A typical example for this situation to occur is the case where the equations under consideration involve symmetries, i.e. are invariant under the action of a Lie-group.

The situation where the set of equlibria forms a C^1-manifold occurs for instance in phase transitions [13, 25], geometric evolution equations [12, 14], free boundary problems in fluid dynamics [15, 16], stability of traveling waves [26], and models of tumor growth, to mention just a few.

A standard method to handle situations as described above is to refer to *center manifold theory*. In fact, in that situation the center manifold of the problem in question will be unique, and it coincides with $\mathcal{E}$ near u_*. Thus the so-called

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shadowing lemma in center manifold theory implies the result. Center manifolds are well-studied objects in the theory of nonlinear evolution equations. For the parabolic case we refer to the monographs [17, 20], and to the publications [5, 6, 10, 19, 21, 27, 28].

However, the theory of center manifolds is a technically difficult matter. Therefore it seems desirable to have a simpler, direct approach to the generalized principle of linearized stability which avoids the technicalities of center manifold theory.

Such an approach has been introduced in [26] in the framework of L_p-maximal regularity. It turns out that within this approach the effort to prove convergence towards equilibria in the normally stable case is only slightly larger than that for the proof of the standard linearized stability result - which is simple.

The purpose of this paper is to extend the approach given in [26] to cover a broader setting and a broader class of nonlinear parabolic equations, including fully nonlinear equations. This approach is flexible and general enough to reproduce the results contained in [7, 12, 13, 14, 15, 16, 25, 26], and it will have applications to many other problems.

Our approach makes use of the concept of maximal regularity in an essential way. As general references for this theory we refer to the monographs [1, 11, 20].

2. **Abstract nonlinear problems in a general setting.** Let X_0 and X_1 be Banach spaces, and suppose that X_1 is densely embedded in X_0. Suppose that $F : U_1 \subset X_1 \to X_0$ satisfies

$$F \in C^k(U_1, X_0) \quad \text{for some } k \in \mathbb{N},\ k \geq 1, \tag{1}$$

where U_1 is an open subset of X_1. Then we consider the autonomous (fully) nonlinear problem

$$\dot{u}(t) + F(u(t)) = 0, \quad t > 0, \quad u(0) = u_0, \tag{2}$$

for $u_0 \in U_1$. In the sequel we use the notation $|\cdot|_j$ to denote the norm in the respective spaces X_j for $j = 0, 1$. Moreover, for any normed space X, $B_X(u, r)$ denotes the open ball in X with radius $r > 0$ around $u \in X$.

Let $\mathcal{E} \subset U_1$ denote the set of equilibrium solutions of (2), which means that

$$u_* \in \mathcal{E} \quad \text{if and only if} \quad F(u_*) = 0.$$

Given an element $u_* \in \mathcal{E}$, we assume that u_* is contained in an m-dimensional manifold of equilibria. This means that there is an open subset $U \subset \mathbb{R}^m$, $0 \in U$, and a C^1-function $\Psi : U \to X_1$ such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u_*$,
- the rank of $\Psi'(0)$ equals m, and (3)
- $F(\Psi(\zeta)) = 0, \quad \zeta \in U.$

We assume further that near u_* there are no other equilibria than those given by $\Psi(U)$, i.e. $\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)$, for some $r_1 > 0$.

Let $u_* \in \mathcal{E}$ be given and set $A := F'(u_*)$. Then we assume that $A \in \mathcal{H}(X_1, X_0)$, by which we mean that $-A$, considered as a linear operator in X_0 with domain X_1, generates a strongly continuous analytic semigroup $\{e^{-At}; t \geq 0\}$ on X_0. In particular we may take the graph norm of A as the norm in X_1.

For the deviation $v := u - u_*$ from u_*, equation (2) can be restated as

$$\dot{v}(t) + Av(t) = G(v(t)), \quad t > 0, \quad v(0) = v_0, \tag{4}$$

where $v_0 = u_0 - u_*$, and $G(z) := Az - F(z + u_*)$, $z \in V_1 := U_1 - u_*$. It follows from (1) that $G \in C^k(V_1, X_0)$. Moreover, we have $G(0) = 0$ and $G'(0) = 0$. Setting $\psi(\zeta) = \Psi(\zeta) - u_*$ results in the following equilibrium equation for problem (4)

$$A\psi(\zeta) = G(\psi(\zeta)), \qquad \text{for all } \ \zeta \in U. \tag{5}$$

Taking the derivative with respect to ζ and using the fact that $G'(0) = 0$ we conclude that $A\psi'(0) = 0$ and this implies that the tangent space of $\mathcal{E}$ at u_* is contained in $N(A)$, the kernel of A.

For $J = [0, a)$, $a \in (0, \infty]$, we consider a pair of Banach spaces $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ such that $\mathbb{E}_0(J) \hookrightarrow L_{1,\mathrm{loc}}(J; X_0)$ and

$$\mathbb{E}_1(J) \hookrightarrow H^1_{1,\mathrm{loc}}(J; X_0) \cap L_{1,\mathrm{loc}}(J; X_1),$$

respectively. Denoting by $X_\gamma = \gamma\mathbb{E}_1$ the temporal trace space of $\mathbb{E}_1(J)$ we assume that

(A1) $\gamma\mathbb{E}_1$ is independent of J, and the embedding $\mathbb{E}_1(J) \hookrightarrow BUC(J; X_\gamma)$ holds. In addition, we assume that there is a constant $c_0 > 0$ independent of $J = [0, a)$, $a \in (0, \infty]$, such that

$$\sup_{t \in J} \|w(t)\|_\gamma \le c_0 \|w\|_{\mathbb{E}_1(J)}, \quad \text{for all } w \in \mathbb{E}_1(J), \ w(0) = 0. \tag{6}$$

We refer to [1, Section III.1.4] for further information on trace spaces. Moreover, we assume that

(A2) $\tilde{w} \in \mathbb{E}_1(J)$ and $|w(t)|_0 \le |\tilde{w}(t)|_1$, $t \in J$, imply $\|w\|_{\mathbb{E}_0(J)} \le \|\tilde{w}\|_{\mathbb{E}_1(J)}$; for $\omega > 0$ fixed, there exists a constant $c_1 > 0$ not depending on J and such that

$$\begin{aligned} &\int_J e^{-\omega s} |w(s)|_1 \, ds \le c_1 \|w\|_{\mathbb{E}_1(J)}, \quad \text{for all } w \in \mathbb{E}_1(J), \\ &\int_t^\infty e^{-\omega s} |w(s)|_1 \, ds \le c_1 e^{-\omega t} \|w\|_{\mathbb{E}_1(\mathbb{R}_+)}, \quad \text{for all } w \in \mathbb{E}_1(\mathbb{R}_+) \text{ and } t \ge 0. \end{aligned} \tag{7}$$

Our *key assumption* is that $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ is a pair of maximal regularity for A. To be more precise we assume that

(A3) the linear Cauchy problem $\dot{w} + Aw = g$, $w(0) = w_0$ has for each $(g, w_0) \in \mathbb{E}_0(I) \times \gamma\mathbb{E}_1(I)$ a unique solution $w \in \mathbb{E}_1(I)$, where $I = [0, T)$ is a finite interval.

We impose the following assumption for the sake of convenience. For all examples that we have in mind the condition can be derived from (A3).

Suppose that $\sigma(A)$, the spectrum of A, admits a decomposition $\sigma(A) = \sigma_s \cup \sigma'$, where $\sigma_s \subset \{z \in \mathbb{C} : \operatorname{Re} z > \omega\}$ for some $\omega > 0$ and $\sigma' \subset \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$. Let P_s denote the spectral projection corresponding to the spectral set σ_s and let $A_s := P_s A P_s$. Then we assume that

(A4) there exists a constant $M_0 > 0$ such that for any $J = [0, a)$, $a \in (0, \infty]$, any $\sigma \in [0, \omega]$, and any function g with $e^{\sigma t} P_s g \in \mathbb{E}_0(J)$ there is a unique solution w of $\dot{w} + A_s w = P_s g$, $t \in J$, $w(0) = 0$, satisfying

$$\|e^{\sigma t} w\|_{\mathbb{E}_1(J)} \le M_0 \|e^{\sigma t} P_s g\|_{\mathbb{E}_0(J)};$$

there exists a constant $M_1 > 0$ such that for any $J = [0, a)$, $a \in (0, \infty]$, and for any $z \in X_\gamma$ there holds

$$\|e^{\sigma t} e^{-A_s t} P_s z\|_{\mathbb{E}_1(J)} + \sup_{t \in J} |e^{\sigma t} e^{-A_s t} P_s z|_\gamma \le M_1 |P_s z|_\gamma, \quad \sigma \in [0, \omega].$$

We again refer to [1, Chapter III] for more background information on the notion of maximal regularity. In order to cover the case $X_\gamma \neq X_1$ we assume the following *structure condition* on the nonlinearity G:

(A5) there exists a uniform constant C_1 such that for any $\eta > 0$ there is $r > 0$ such that

$$|G(z_1) - G(z_2)|_0 \le C_1(\eta + |z_2|_1)|z_1 - z_2|_1, \quad z_1,\, z_2 \in X_1 \cap B_{X_\gamma}(0, r).$$

Observe that condition (A5) trivially holds in the case $X_\gamma = X_1$, since $G'(0) = 0$. A short computation shows that condition (A5) is also satisfied if F has a quasilinear structure, i.e. if

$$F(u) = B(u)u + f(u) \quad \text{for } u \in U_\gamma, \quad (B, f) \in C^1(U_\gamma, \mathcal{B}(X_1, X_0) \times X_0), \tag{8}$$

where $U_\gamma \subset X_\gamma$ is an open set.

Lastly, concerning *solvability* of the nonlinear problem (4) we will assume that

(A6) given $b > 0$ there exists $r_2 > 0$ such that for any $v_0 \in B_{X_\gamma}(0, r_2)$ problem (4) admits a unique solution $v \in \mathbb{E}_1([0, b])$.

Note that since $v = 0$ is an equilibrium of (4), condition (A6) is satisfied whenever one has existence and uniqueness of local solutions in the described class as well as continuous dependence of the maximal time of existence on the initial data.

We conclude this section by describing three important examples of admissible pairs $(\mathbb{E}_0(J), \mathbb{E}_1(J))$.

Example 1: (L_p-maximal regularity.)
In our first example, the spaces $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ are given by

$$\mathbb{E}_0(J) := L_p(J; X_0), \quad \mathbb{E}_1(J) := H^1_p(J; X_0) \cap L_p(J; X_1). \tag{9}$$

The trace space is a real interpolation space given by $\gamma\mathbb{E}_1 = X_\gamma = (X_0, X_1)_{1-1/p,p}$ and we have $\mathbb{E}_1(J) \hookrightarrow BUC(J; X_\gamma)$, see for instance [1, Theorem III.4.10.2]. For a proof of (6) we refer to [23, Proposition 6.2]. This yields Assumption (A1). For Assumption (A2) we note that

$$\int_J e^{-\omega s}|w(s)|_1\, ds \le c_1\Big(\int_J |w(s)|_1^p\, ds\Big)^{1/p} \le c_1\|w\|_{\mathbb{E}_1(J)}$$

for all $w \in \mathbb{E}_1(J)$ by Hölder's inequality. Moreover,

$$\int_t^\infty e^{-\omega s}|w(s)|_1\, ds \le \Big(\int_t^\infty e^{-\omega s p'}\, ds\Big)^{1/p'}\Big(\int_t^\infty |w(s)|^p\, ds\Big)^{1/p'} \le c_1 e^{-\omega t}\|w\|_{\mathbb{E}_1(\mathbb{R}_+)}$$

for $t \ge 0$ and $w \in \mathbb{E}_1(\mathbb{R}_+)$. We refer to [11, 18, 24], [1, Section III.4.10] and the references therein for conditions guaranteeing that the crucial Assumption (A3) on maximal regularity is satisfied. It is clear that the property of maximal regularity is passed on from A to A_s in the spaces $\mathbb{E}^s_0(J) := L_p(J; X^s_0)$, $\mathbb{E}^s_1(J) := H^1_p(J; X^s_0) \cap L_p(J; X^s_1)$, and this implies Assumption (A4), see for instance [1, Remark III.4.10.9(a)]. Assumption (A5) is satisfied in case that the nonlinear mapping F has a *quasilinear structure*, see [26]. Assumption (A6) follows in case that F has a quasilinear structure from (A3) and [22, Theorem 3.1], see also [2, Theorem 2.1, Corollary 3.3]. We remark that the case of L_p-maximal regularity has been considered in detail in [26].

Example 2: (Continuous maximal regularity).
Let $J=[0,a)$ with $0<a\le\infty$ and set $\dot{J}:=(0,a)$. For $\mu\in(0,1)$ and X a Banach space we set

$$\begin{aligned} BUC_{1-\mu}(J;X) &:= \big\{u\in C(\dot{J};X) : [t\mapsto t^{1-\mu}u]\in BUC(\dot{J};X),\\ &\qquad\qquad \lim_{t\to 0^+} t^{1-\mu}|u(t)|_X=0\big\},\\ BUC_0(J;X) &:= BUC(J;X). \end{aligned}$$

$BUC_{1-\mu}(J;X)$ is turned into a Banach space by the norm

$$\|u\|_{C_{1-\mu}(J;X)} := \sup_{t\in\dot{J}} t^{1-\mu}|u(t)|_X, \quad \mu\in(0,1].$$

Finally, we set $BUC^1_{1-\mu}(J;X) := \{u\in C^1(\dot{J};X) : u,\, \dot{u}\in BUC_{1-\mu}(J;X)\}$. With these preparations we define

$$\begin{aligned} \mathbb{E}_0(J) :&= BUC_{1-\mu}(J;X_0),\\ \mathbb{E}_1(J) :&= BUC^1_{1-\mu}(J;X_0)\cap BUC_{1-\mu}(J;X_1) \end{aligned} \tag{10}$$

endowed with the canonical norms.

Supposing that $\mathcal{H}(X_1,X_0)\neq\emptyset$ the trace space $\gamma\mathbb{E}_1$ is the continuous interpolation space $\gamma\mathbb{E}_1=(X_0,X_1)^0_{\mu,\infty}=: D_A(\mu)$, and we have the embedding $\mathbb{E}_1(J)\hookrightarrow BUC(J;\gamma\mathbb{E}_1)$, see [1, Theorem III.2.3.3]. A proof for estimate (6) can be found in [8, Lemma 2.2(c)], and this shows that Assumption (A1) is satisfied. Assumption (A2) holds as

$$\int_J e^{-\omega s}|w(s)|_1\,ds = \int_J \frac{e^{-\omega s}}{s^{1-\mu}} s^{1-\mu}|w(s)|_1\,ds \le c_1\|w\|_{C_{1-\mu}(J;X_1)} \le c_1\|w\|_{\mathbb{E}_1(J)}$$

for all $w\in\mathbb{E}_1(J)$, and

$$\int_t^\infty e^{-\omega s}|w(s)|_1\,ds = \int_t^\infty \frac{e^{-\omega s}}{s^{1-\mu}} s^{1-\mu}|w(s)|_1\,ds \le c_1 e^{-\omega t}\|w\|_{\mathbb{E}_1(\mathbb{R}_+)}$$

for $t\ge 0$ and $w\in\mathbb{E}_1(\mathbb{R}_+)$.

It turns out that maximal regularity cannot hold in the class (10) if $X_1\neq X_0$ and X_0 is reflexive. On the other side, there is an interesting class of spaces (X_0,X_1) where Assumption (A3) is indeed satisfied for the pair $(\mathbb{E}_0(J),\mathbb{E}_1(J))$ given in (10), see [3, 8, 9, 20] and [1, Theorem III.3.4.1]. A_s inherits the property of maximal regularity from A, and this implies Assumption (A4), see [1, Remark III.3.4.2(b)]. Assumption (A5) holds in the case $\mu=1$ for any function $G\in C^1(U_1,X_0)$ with $G(0)=G'(0)=0$. It also holds for $\mu\in(0,1)$ if the nonlinear function F given in (2) satisfies (8).

If $\mu=1$ and $k\ge 1$ then it follows from (A3) and [3, Theorem 2.7, Corollary 2.9], see also [20, Section 8.4], that Assumption (A6) is satisfied.

If $\mu\in(0,1)$, $k\ge 1$ and F has a *quasilinear* structure, see (8), then Assumption (A6) follows from (A3) and [8, Theorem 5.1], see also [8, Theorem 6.1].

Example 3: (Hölder maximal regularity.)
Suppose $\rho\in(0,1)$, $I\subset\mathbb{R}_+$, $J\subset\mathbb{R}_+$ are intervals with $0\in J$. Then we set

$$\begin{aligned} [u]_{C^\rho(I;X)} &:= \sup\Big\{\frac{|u(t)-u(s)|}{|t-s|^\rho} : s,t\in I,\ s\neq t\Big\},\\ [\![u]\!]_{C^\rho_\rho(J;X)} &:= \sup_{2\varepsilon\in\dot{J}} \varepsilon^\rho [u]_{C^\rho([\varepsilon,2\varepsilon];X)}, \end{aligned}$$

and

$$\|u\|_{C^\rho_\rho(J;X)} := \|u\|_{BC(I;X)} + [\![u]\!]_{C^\rho_\rho(J;X)},$$
$$BC^\rho_\rho(J;X) := \{u \in C^\rho(J;X) : \|u\|_{C^\rho_\rho(J;X)} < \infty\}.$$

Moreover, we set

$$BUC^\rho_\rho(J;X) := \{u \in BUC(J;X) \cap BC^\rho_\rho(J;X) : \lim_{\varepsilon\to 0^+} \varepsilon^\rho [u]_{C^\rho([\varepsilon,2\varepsilon];X)} = 0\}$$

and equip it with the norm $\|\cdot\|_{C^\rho_\rho(J;X)}$. For the pair $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ we take

$$\begin{aligned}\mathbb{E}_0(J) &:= BUC^\rho_\rho(J;X_0),\\ \mathbb{E}_1(J) &:= BUC^{1+\rho}_\rho(J;X_0) \cap BUC^\rho_\rho(J;X_1),\end{aligned} \tag{11}$$

where $BUC^{1+\rho}_\rho(J;X) := \{u \in BUC^\rho_\rho(J;X_0) : \dot u \in BUC^\rho_\rho(J;X_0)\}$. The spaces in (11) are given their canonical norms, turning them into Banach spaces.

We have $\gamma\mathbb{E}_1(J) = X_1$ and it is clear from the definition of (the norm of) $\mathbb{E}_1(J)$ that $\mathbb{E}_1(J) \hookrightarrow BUC(J;X_1)$, and that (6) is satisfied for any $w \in \mathbb{E}_1(J)$. This shows that Assumption (A1) holds. By similar arguments as above we see that Assumption (A2) is satisfied as well.

For the crucial Assumption (A3) we refer to [1, Theorem III.2.5.6] with $\mu = 1$; see also [20, Corollary 4.3.6(ii)]. It is worthwhile to mention that this maximal regularity result is true for *any* $A \in \mathcal{H}(X_1, X_0)$ and any pair (X_0, X_1). Assumption (A4) follows then as above, see [1, Theorem III.2.5.5]. Assumption (A5) holds for *any* function $G \in (U_1, X_0)$ with $G(0) = G'(0) = 0$.

Finally, it follows from Theorem 8.1.1 and Theorem 8.2.3 in [20] that Assumption (A6) holds for the fully nonlinear problem (2) in case that $k \geq 2$. (In fact, it suffices to require that the derivative F' of F be locally Lipschitz continuous.)

3. **The main result.**

In this section we state and prove our main theorem about convergence of solutions for the nonlinear equation (2) towards equilibria.

Theorem 3.1. *Let $u_* \in X_1$ be an equilibrium of (2), and assume that the above conditions (A1)-(A6) are satisfied. Suppose that u_* is normally stable, i.e. assume that*

(i) near u_* the set of equilibria $\mathcal{E}$ is a C^1-manifold in X_1 of dimension $m \in \mathbb{N}$,
(ii) the tangent space for $\mathcal{E}$ at u_* is given by $N(A)$,
(iii) 0 is a semi-simple eigenvalue of A, i.e. $N(A) \oplus R(A) = X_0$,
(iv) $\sigma(A) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > \omega\}$ for some $\omega > 0$.

Then u_* is stable in X_γ, and there exists $\delta > 0$ such that the unique solution $u(t)$ of (2) with initial value $u_0 \in X_\gamma$ satisfying $|u_0 - u_*|_\gamma < \delta$ exists on $\mathbb{R}_+$ and converges at an exponential rate to some $u_\infty \in \mathcal{E}$ in X_γ as $t \to \infty$.

Proof. The proof to Theorem 3.1 will be carried out in several steps, as follows.

(a) We denote by P_l, $l \in \{c, s\}$, the spectral projections corresponding to the spectral sets σ_s and $\sigma_c := \{0\}$, respectively, and let $A_l = P_l A P_l$ be the part of A in $X_0^l = P_l(X_0)$ for $l \in \{c, s\}$. Note that $A_c = 0$. We set $X_j^l := P_l(X_j)$ for $l \in \{c, s\}$ and $j \in \{0, \gamma, 1\}$. It follows from our assumptions that $X_0^c = X_1^c$. In the following we set $X^c := X_0^c$ and equip X^c with the norm of X_0. Moreover, we take as a norm on X_j

$$|v|_j := |P_c v|_0 + |P_s v|_j \quad \text{for} \quad j = 0, \gamma, 1. \tag{12}$$

(b) Next we show that the manifold $\mathcal{E}$ can be represented as the (translated) graph of a function $\phi : B_{X^c}(0,\rho_0) \to X_1^s$ in a neighborhood of u_*. In order to see this we consider the mapping

$$g : U \subset \mathbb{R}^m \to X^c, \quad g(\zeta) := P_c\psi(\zeta), \quad \zeta \in U,$$

where ψ is introduced in the line before formula (5). It follows from our assumptions that $g'(0) = P_c\psi'(0) : \mathbb{R}^m \to X^c$ is an isomorphism. By the inverse function theorem, g is a C^1-diffeomorphism of a neighborhood of 0 in $\mathbb{R}^m$ onto a neighborhood, say $B_{X^c}(0,\rho_0)$, of 0 in X^c. Let $g^{-1} : B_{X^c}(0,\rho_0) \to U$ be the inverse mapping. Then $g^{-1} : B_{X^c}(0,\rho_0) \to U$ is C^1 and $g^{-1}(0) = 0$. Next we set $\Phi(x) := \psi(g^{-1}(x))$ for $x \in B_{X^c}(0,\rho_0)$ and we note that

$$\Phi \in C^1(B_{X^c}(0,\rho_0), X_1^s), \quad \Phi(0) = 0, \quad \{u_* + \Phi(x) : x \in B_{X^c}(0,\rho_0)\} = \mathcal{E} \cap W,$$

where W is an appropriate neighborhood of u_* in X_1. Clearly,

$$P_c\Phi(x) = ((P_c \circ \psi) \circ g^{-1})(x) = (g \circ g^{-1})(x) = x, \quad x \in B_{X^c}(0,\rho_0),$$

and this yields $\Phi(x) = P_c\Phi(x) + P_s\Phi(x) = x + P_s\Phi(x)$ for $x \in B_{X^c}(0,\rho_0)$. Setting $\phi(x) := P_s\Phi(x)$ we conclude that

$$\phi \in C^1(B_{X^c}(0,\rho_0), X_1^s), \quad \phi(0) = \phi'(0) = 0, \tag{13}$$

and that $\{u_* + x + \phi(x) : x \in B_{X^c}(0,\rho_0)\} = \mathcal{E} \cap W$, where W is a neighborhood of u_* in X_1. This shows that the manifold $\mathcal{E}$ can be represented as the (translated) graph of the function ϕ in a neighborhood of u_*. Moreover, the tangent space of $\mathcal{E}$ at u_* coincides with $N(A) = X^c$. By applying the projections P_l, $l \in \{c,s\}$, to equation (5) and using $x + \phi(x) = \psi(g^{-1}(x))$ for $x \in B_{X^c}(0,\rho_0)$, and $A_c \equiv 0$, we obtain the following equivalent system of equations for the equilibria of (4)

$$P_cG(x + \phi(x)) = 0, \quad P_sG(x + \phi(x)) = A_s\phi(x), \quad x \in B_{X^c}(0,\rho_0). \tag{14}$$

Finally, let us also agree that ρ_0 has already been chosen small enough so that

$$|\phi'(x)|_{\mathcal{B}(X^c, X_1^s)} \le 1, \quad |\phi(x)|_1 \le |x|, \quad x \in B_{X^c}(0,\rho_0) \tag{15}$$

where $|\cdot|$ denotes the norm in X^c. This can always be achieved, thanks to (13).

(c) Introducing the new variables

$$\begin{aligned} x &= P_cv = P_c(u - u_*), \\ y &= P_sv - \phi(P_cv) = P_s(u - u_*) - \phi(P_c(u - u_*)) \end{aligned}$$

we then obtain the following system of evolution equations in $X^c \times X_0^s$

$$\begin{cases} \dot{x} = T(x,y), \quad x(0) = x_0, \\ \dot{y} + A_sy = R(x,y), \quad y(0) = y_0, \end{cases} \tag{16}$$

with $x_0 = P_cv_0$ and $y_0 = P_sv_0 - \phi(P_cv_0)$, where the functions T and R are given by

$$\begin{aligned} T(x,y) &= P_cG(x + \phi(x) + y), \\ R(x,y) &= P_sG(x + \phi(x) + y) - A_s\phi(x) - \phi'(x)T(x,y). \end{aligned}$$

Using the equilibrium equations (14), the expressions for R and T can be rewritten as

$$\begin{aligned} T(x,y) &= P_c\big(G(x + \phi(x) + y) - G(x + \phi(x))\big), \\ R(x,y) &= P_s\big(G(x + \phi(x) + y) - G(x + \phi(x))\big) - \phi'(x)T(x,y). \end{aligned} \tag{17}$$

Equation (17) immediately yields

$$T(x,0) = R(x,0) = 0 \quad \text{for all} \;\; x \in B_{X^c}(0,\rho_0),$$

showing that the equilibrium set $\mathcal{E}$ of (2) near u_* has been reduced to the set $B_{X^c}(0,\rho_0)\times\{0\}\subset X^c\times X_1^s$.

Observe also that there is a unique correspondence between the solutions of (2) close to u_* in X_γ and those of (16) close to 0. We call system (16) the *normal form* of (2) near its normally stable equilibrium u_*.

(d) Taking $z_1 = x+\phi(x)+y$ and $z_2 = x+\phi(x)$ it follows from (A5), (15) and (17) that

$$|T(x,y)|,\ |R(x,y)|_0 \le C_1\big(\eta + |x+\phi(x)|_1\big)|y|_1 \le \beta|y|_1, \tag{18}$$

with $\beta := C_2(\eta + r)$, where the constants C_1 and C_2 are independent of η, r and x, y, provided that $x\in \bar B_{X^c}(0,\rho)$, $y\in \bar B_{X^s_\gamma}(0,\rho)\cap X_1$ and $\rho\in(0,r/3]$ with $r<3\rho_0$. Suppose that η and, accordingly, r were already chosen small enough so that

$$M_0\beta = M_0C_2(\eta+r)\le 1/2. \tag{19}$$

(e) Suppose now that $v_0\in B_{X_\gamma}(0,\delta)$, where $\delta<r_2$ will be chosen later. By (A6), problem (4) has a unique solution on some maximal interval of existence $[0,t_*)$. Let η and r be fixed so that (19) holds and set $\rho = r/3$. Let then t_1 be the exit time for the ball $\bar B_{X_\gamma}(0,\rho)$, that is

$$t_1 := \sup\{t\in(0,t_*) : |v(\tau)|_\gamma\le\rho,\ \tau\in[0,t]\}.$$

Suppose $t_1<t_*$ and set $J_1=[0,t_1)$. The definition of t_1 implies that $|x(t)|\le\rho$ for all $t\in J_1$. Assuming without loss of generality that the embedding constant of $X_1\hookrightarrow X_\gamma$ is less or equal to one, we obtain from (12)

$$\begin{aligned}\rho\ge |v(t)|_\gamma &= |x(t)+\phi(x(t))+y(t)|_\gamma = |x(t)| + |\phi(x(t))+y(t)|_\gamma\\ &\ge |x(t)| + |y(t)|_\gamma - |\phi(x(t))|_\gamma \ge |y(t)|_\gamma\end{aligned}$$

for $t\in J_1$, since $\phi(x)$ is non-expansive for $|x|\le\rho_0$. In conclusion we have shown that $|x(t)|, |y(t)|_\gamma\le\rho$ for all $t\in J_1$, so that the estimate (18) holds for $(x(t),y(t))$, $t\in J_1$. Then, by (A4) and (18), we have for $\sigma\in[0,\omega]$

$$\begin{aligned}\|e^{\sigma t}y\|_{\mathbb{E}_1(J_1)} &\le \|e^{\sigma t}e^{-A_s t}y_0\|_{\mathbb{E}_1(J_1)} + M_0\|e^{\sigma t}R(x,y)\|_{\mathbb{E}_0(J_1)}\\ &\le M_1|y_0|_\gamma + M_0\beta\|e^{\sigma t}y\|_{\mathbb{E}_1(J_1)},\end{aligned}$$

which implies

$$\|e^{\sigma t}y\|_{\mathbb{E}_1(J_1)}\le 2M_1|y_0|_\gamma,\quad \sigma\in[0,\omega], \tag{20}$$

thanks to (19). Using (A1), (A4) and (20) we then have for $t\in J_1$,

$$\begin{aligned}|e^{\omega t}y(t)|_\gamma &\le |e^{\omega t}y(t) - e^{\omega t}e^{-A_s t}y_0|_\gamma + |e^{\omega t}e^{-A_s t}y_0|_\gamma\\ &\le c_0\|e^{\omega t}y - e^{\omega t}e^{-A_s t}y_0\|_{\mathbb{E}_1(J_1)} + M_1|y_0|_\gamma\\ &\le (3c_0M_1+M_1)|y_0|_\gamma,\end{aligned}$$

which yields with $M_2 = 3c_0M_1+M_1$,

$$|y(t)|_\gamma\le M_2e^{-\omega t}|y_0|_\gamma,\quad t\in J_1.$$

Using (7) we deduce further from the equation for x and the estimate for T in (18), and from (19)–(20) that

$$\begin{aligned}|x(t)| &\le |x_0| + \int_0^t |T(x(s),y(s))|\,ds \le |x_0| + \beta\int_0^t |y(s)|_1\,ds\\ &\le |x_0| + \beta c_1\|e^{\omega t}y\|_{\mathbb{E}_1(J_1)} \le |x_0| + M_3|y_0|_\gamma,\quad t\in J_1,\end{aligned}$$

where $M_3 = M_1 c_1/M_0$. Since $v(t) = x(t) + \phi(x(t)) + y(t)$, the previous estimates and (15) imply that for some constant $M_4 \geq 1$,

$$|v(t)|_\gamma \leq M_4 |v_0|_\gamma, \quad t \in J_1.$$

Choosing $\delta = \min\{\rho, r_2\}/(2M_4)$, we have $|v(t_1)|_\gamma \leq \min\{\rho, r_2\}/2$, a contradiction to the definition of t_1, and hence $t_1 = t_*$. The above argument then yields uniform bounds $\|v\|_{\mathbb{E}_1(J)} \leq C$ and $\sup_{t\in J} |v(t)|_\gamma \leq r_2/2$ for all $J = [0, a)$ with $a < t_*$. In view of (A6), it follows that $t_* = \infty$.

(f) Repeating the above estimates on the interval $[0, \infty)$ we obtain

$$|x(t)| \leq |x_0| + M_3 |y_0|_\gamma, \quad |y(t)|_\gamma \leq M_2 e^{-\omega t} |y_0|_\gamma, \quad t \in [0, \infty), \tag{21}$$

for $v_0 \in B_{X_\gamma}(0, \delta)$. Moreover, $\lim_{t\to\infty} x(t) = x_0 + \int_0^\infty T(x(s), y(s))ds =: x_\infty$ exists since the integral is absolutely convergent. This yields existence of

$$v_\infty := \lim_{t\to\infty} v(t) = \lim_{t\to\infty} x(t) + \phi(x(t)) + y(t) = x_\infty + \phi(x_\infty).$$

Clearly, v_∞ is an equilibrium for equation (4), and $u_\infty := u_* + v_\infty \in \mathcal{E}$ is an equilibrium for (2). It follows from (A2), the estimate for T in (18), and from (20) that

$$\begin{aligned}|x(t) - x_\infty| = \Big| \int_t^\infty T(x(s), y(s))\, ds \Big| &\leq \beta \int_t^\infty |y(s)|_1\, ds \\ &\leq \beta c_1 e^{-\omega t} \|e^{\omega t} y\|_{\mathbb{E}_1(\mathbb{R}_+)} \leq M_4 e^{-\omega t} |y_0|_\gamma.\end{aligned}$$

This shows that $x(t)$ converges to x_∞ at an exponential rate. Due to (15), (21) and the exponential estimate for $|x(t) - x_\infty|$ we now get for the solution $u(t) = u_* + v(t)$ of (2)

$$\begin{aligned}|u(t) - u_\infty|_\gamma &= |x(t) + \phi(x(t)) + y(t) - v_\infty|_\gamma \\ &\leq |x(t) - x_\infty|_\gamma + |\phi(x(t)) - \phi(x_\infty)|_\gamma + |y(t)|_\gamma \\ &\leq (2M_4 + M_2) e^{-\omega t} |y_0|_\gamma \\ &\leq M e^{-\omega t} |P_s v_0 - \phi(P_c v_0)|_\gamma\,,\end{aligned} \tag{22}$$

thereby completing the proof of the second part of Theorem 3.1. Concerning stability, note that given $r > 0$ small enough we may choose $0 < \delta \leq r$ such that the solution starting in $B_{X_\gamma}(u_*, \delta)$ exists on $\mathbb{R}_+$ and stays within $B_{X_\gamma}(u_*, r)$. □

Remarks: (a) Theorem 3.1 has been proved in [26] in the setting of L_p-maximal regularity, and applications to quasilinear parabolic problems with nonlinear boundary conditions, to the Mullins-Sekerka problem, and to the stability of travelling waves for a quasilinear parabolic equation have been given.

(b) It has been shown in [26] by means of examples that conditions (i)–(iii) in Theorem 3.1 are also necessary in order to get convergence of solutions towards equilibria $u_\infty \in \mathcal{E}$.

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