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COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. Compressible, stationary Navier-Stokes (N-S) equations are considered. The shape sensitivity analysis is performed in the case of small perturbations of the so-called *approximate solutions*. The proposed method of shape sensitivity analysis is general, and can be used to establish the well-posedness for distributed and boundary control problems as well as for inverse problems in the case of the state equations in the form of compressible N-S equations.

1. **Preliminaries.**

The shape optimization for compressible N-S equations is a field of active research, e.g. in aerodynamics. The main difficulty in analysis of such optimization problems is the mathematical modeling, i.e., the lack of the existence results for inhomogeneous boundary value problems in bounded domains [10]. The authors already proved the existence of an optimal shape for drag minimisation in three spatial dimensions under the Mosco convergence of admissible domains and assuming that the family of admissible domains is nonempty [9]. This is the general result on the compactness of the set of solutions to N-S equations for the admissible family of obstacles, we refer the reader to [8]-[13] for further details. The shape differentiability of solutions to N-S equations with respect to boundary perturbations is shown in [11], and leads to the optimality system for the shape optimisation problem under considerations.

1.1. **Function spaces.**

In this paragraph we assemble some technical results which are used throughout of the paper. Function spaces play a central role, and we recall some notations, fundamental definitions and properties, which are classical. The proofs of some results given here can be found, e.g. in [11]. For our applications we need the results in three spatial dimensions, therefore, the space dimension stands $d = 3$ in the paragraph on the embedding theorems.

Let Ω be the whole space $\mathbb{R}^3$ or a bounded domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class C^1. For an integer $l \geq 0$ and for an exponent $r \in [1, \infty)$, we denote by $H^{l,r}(\Omega)$ the Sobolev space endowed with the norm $\|u\|_{H^{l,r}(\Omega)} = \sup_{|\alpha|\leq l} \|\partial^\alpha u\|_{L^r(\Omega)}$. For

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real $0 < s < 1$, the fractional Sobolev space $H^{s,r}(\Omega)$ is obtained by the interpolation between $L^r(\Omega)$ and $H^{1,r}(\Omega)$, and consists of all measurable functions with the finite norm

$$\|u\|_{H^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega},$$

where

$$|u|^r_{s,r,\Omega} = \int\limits_{\Omega\times\Omega} |x-y|^{-d-rs}|u(x)-u(y)|^r\,dxdy. \tag{1}$$

In the general case, the Sobolev space $H^{l+s,r}(\Omega)$ is defined as the space of measurable functions with the finite norm $\|u\|_{H^{l+s,r}(\Omega)} = \sup_{|\alpha|\le l}\|\partial^\alpha u\|_{H^{s,r}(\Omega)}$. For $0 < s < 1$, the Sobolev space $H^{s,r}(\Omega)$ is, in fact the interpolation space $[L^r(\Omega), H^{1,r}(\Omega)]_{s,r}$.

Furthermore, the notation $H^{l,r}_0(\Omega)$, with an integer l, stands for the closed subspace of the space $H^{l,r}(\Omega)$ of all functions $u \in L^r(\Omega)$ which being extended by zero outside of Ω belong to $H^{l,r}(\mathbb{R}^3)$.

Denote by $\mathcal{H}^{0,r}_0(\Omega)$ and $\mathcal{H}^{1,r}_0(\Omega)$ the subspaces of $L^r(\mathbb{R}^3)$ and $H^{1,r}(\mathbb{R}^3)$, respectively, of all functions vanishing outside of Ω. Obviously $\mathcal{H}^{1,r}_0(\Omega)$ and $H^{1,r}_0(\Omega)$ are isomorphic topologically and algebraically and we can identify them. However, we need the interpolation spaces $\mathcal{H}^{s,r}_0(\Omega)$ for non-integers, in particular for $s = 1/r$.

Definition 1.1. For all $0 < s \le 1$ and $1 < r < \infty$, we denote by $\mathcal{H}^{s,r}_0(\Omega)$ the interpolation space $[\mathcal{H}^{0,r}_0(\Omega), \mathcal{H}^{1,r}_0(\Omega)]_{s,r}$ endowed with one of two equivalent norms [11] defined by interpolation method.

For an arbitrary bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary, we introduce the Banach spaces

$$X^{s,r} = H^{s,r}(\Omega)\cap H^{1,2}(\Omega),\ Y^{s,r} = H^{s+1,r}(\Omega)\cap H^{2,2}(\Omega),\ Z^{s,r} = \mathcal{H}^{s-1,r}(\Omega)\cap L^2(\Omega)$$

equipped with the norms

$$\|u\|_{X^{s,r}} = \|u\|_{H^{s,r}(\Omega)} + \|u\|_{H^{1,2}(\Omega)}, \quad \|u\|_{Y^{s,r}} = \|u\|_{H^{1+s,r}(\Omega)} + \|u\|_{H^{2,2}(\Omega)},$$
$$\|u\|_{Z^{s,r}} = \|u\|_{\mathcal{H}^{s-1,r}(\Omega)} + \|u\|_{L^2(\Omega)}.$$

It can be easily seen that the embeddings $Y^{s,r} \hookrightarrow X^{s,r} \hookrightarrow Z^{s,r}$ are compact and for $sr > 3$, each of the spaces $X^{s,r}$ and $Y^{s,r}$ is a commutative Banach algebra.

2. **Shape optimisation for Navier-Stokes equations.** We present an exemple of shape optimization in aerodynamics. Mathematical analysis of the drag minimization problem for compressible N-S equations can be found e.g., in [9] on the domain continuity of solutions, and in [11] on the shape differentiability of the drag functional.

Mathematical model in the form of N-S equations. We assume that the viscous gas occupies the double-connected domain $\Omega = B\backslash S$, where $B \subset \mathbb{R}^3$, B is a hold-all domain with the smooth boundary $\Sigma = \partial B$, and $S \subset B$ is a compact obstacle. Furthermore, we assume that the velocity of the gas coincides with a given vector field $\mathbf{U} \in C^\infty(\mathbb{R}^3)^3$ on the surface Σ. In this framework, the boundary of the flow domain Ω is divided into the three subsets, inlet Σ_{in}, outgoing set Σ_{out} and the characteristic set Σ_0. In its turn the compact $\Gamma = \Sigma_0 \cap \Sigma$ splits the surface Σ into three disjoint parts $\Sigma = \Sigma_{\text{in}} \cup \Sigma_{\text{out}} \cup \Gamma$. The problem is to find the velocity field $\mathbf{u}$

and the gas density ϱ satisfying the following equations along with the boundary conditions

$$\Delta \mathbf{u} + \lambda \nabla \mathrm{div}\, \mathbf{u} = R\varrho \mathbf{u} \cdot \nabla \mathbf{u} + \frac{R}{\epsilon^2} \nabla p(\varrho) \text{ in } \Omega, \quad \mathrm{div}\,(\varrho \mathbf{u}) = 0 \text{ in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S, \quad \varrho = \varrho_0 \text{ on } \Sigma_{\mathrm{in}}, \tag{3}$$

where the pressure $p = p(\varrho)$ is a smooth, strictly monotone function of the density, ϵ is the Mach number, R is the Reynolds number, λ is the viscosity ratio, and ϱ_0 is a positive constant.

Drag minimization. One of the main applications of the theory of compressible viscous flows is the optimal shape design in aerodynamics. The classical sample is the problem of the minimization of the drag of airfoil travelling in atmosphere with uniform speed $\mathbf{U}_\infty$. Recall that in our framework the hydro-dynamical force acting on the body S is defined by the formula,

$$\mathbf{J}(S) = -\int_{\partial S} (\nabla \mathbf{u} + (\nabla \mathbf{u})^* + (\lambda - 1)\mathrm{div}\, \mathbf{u}\mathbf{I} - \frac{R}{\epsilon^2} p\mathbf{I}) \cdot \mathbf{n} dS \; .$$

In a frame attached to the moving body the drag is the component of $\mathbf{J}$ parallel to $\mathbf{U}_\infty$,

$$J_D(S) = \mathbf{U}_\infty \cdot \mathbf{J}(S), \tag{4}$$

and the lift is the component of $\mathbf{J}$ in the direction orthogonal to $\mathbf{U}_\infty$. For the fixed data, the drag can be regarded as a functional depending on the shape of the obstacle S. The minimization of the drag and the maximization of the lift are between shape optimization problems of some practical importance.

2.1. **Shape sensitivity analysis.** We start with description of our framework for shape sensitivity analysis, or more general, for well-posedness of compressible NSE. To this end we choose the vector field $\mathbf{T} \in C^2(\mathbb{R}^3)^3$ vanishing in the vicinity of Σ, and define the mapping

$$y = x + \varepsilon \mathbf{T}(x), \tag{5}$$

which describes the perturbation of the shape of the obstacle. We refer the reader to [14] for more general framework and results in shape optimization. For small ε, the mapping $x \to y$ takes diffeomorphically the flow region Ω onto $\Omega_\varepsilon = B \setminus S_\varepsilon$, where the perturbed obstacle $S_\varepsilon = y(S)$. Let $(\bar{\mathbf{u}}_\varepsilon, \bar{\varrho}_\varepsilon)$ be solutions to problem (2) in Ω_ε. After substituting $(\bar{\mathbf{u}}_\varepsilon, \bar{\varrho}_\varepsilon)$ into the formulae for $\mathbf{J}$, the drag becomes the function of the parameter ε. Our aim is, in fact, to prove that this function is well-defined and differentiable at $\varepsilon = 0$. This leads to the first order shape sensitivity analysis for solutions to compressible N-S equations. It is convenient to reduce such an analysis to the analysis of dependence of solutions with respect to the coefficients of the governing equations. To this end, we introduce the functions $\mathbf{u}_\varepsilon(x)$ and $\varrho_\varepsilon(x)$ defined in the unperturbed domain Ω by the formulae

$$\mathbf{u}_\varepsilon(x) = \mathbf{N}\bar{\mathbf{u}}_\varepsilon(x + \varepsilon \mathbf{T}(x)), \quad \varrho_\varepsilon(x) = \bar{\varrho}_\varepsilon(x + \varepsilon \mathbf{T}(x)),$$

where

$$\mathbf{N}(x) = [\det\, (\mathbf{I} + \varepsilon \mathbf{T}'(x))(\mathbf{I} + \varepsilon \mathbf{T}'(x))]^{-1}. \tag{6}$$

is the adjugate matrix of the Jacobi matrix $\mathbf{I} + \varepsilon \mathbf{T}'$. Furthermore, we also use the notation $\mathfrak{g}(x) = \sqrt{\det \mathbf{N}}$. It is easily to see that the matrices $\mathbf{N}(x)$ depends analytically upon the small parameter ε and

$$\mathbf{N} = \mathbf{I} + \varepsilon \mathbf{D}(x) + \varepsilon^2 \mathbf{D}_1(\varepsilon, x), \tag{7}$$

where $\mathbf{D} = \operatorname{div} \mathbf{T}\mathbf{I} - \mathbf{T}'$. Calculations show that for $\mathbf{u}_\varepsilon, \varrho_\varepsilon$, the following boundary value problem is obtained

$$\Delta \mathbf{u}_\varepsilon + \nabla\Big(\lambda \mathfrak{g}^{-1} \operatorname{div} \mathbf{u}_\varepsilon - \frac{R}{\epsilon^2} p(\varrho_\varepsilon)\Big) = \mathscr{A}\mathbf{u}_\varepsilon + R\mathscr{B}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \text{ in } \Omega, \tag{8a}$$

$$\operatorname{div}\big(\varrho_\varepsilon \mathbf{u}_\varepsilon\big) = 0 \text{ in } \Omega, \tag{8b}$$

$$\mathbf{u}_\varepsilon = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_\varepsilon = 0 \text{ on } \partial S, \tag{8c}$$

$$\varrho_\varepsilon = \varrho_0 \text{ on } \Sigma_{\text{in}}. \tag{8d}$$

Here, the linear operator $\mathscr{A}$ and the nonlinear mapping $\mathscr{B}$ are defined in terms of $\mathbf{N}$,

$$\begin{aligned} \mathscr{A}(\mathbf{u}) &= \Delta \mathbf{u} - \mathbf{N}^{-1} \operatorname{div}\big(\mathfrak{g}^{-1}\mathbf{N}\mathbf{N}^*\nabla(\mathbf{N}^{-1}\mathbf{u})\big), \\ \mathscr{B}(\varrho, \mathbf{u}, \mathbf{w}) &= \varrho(\mathbf{N}^*)^{-1}\Big(\mathbf{u}\,\nabla\big(\mathbf{N}^{-1}\mathbf{w}\big)\Big). \end{aligned} \tag{9}$$

2.2. **Transport to the fixed domain by the change of variables.** In this section we derive equations (8). We will write $\mathbf{u}(y)$ and $\varrho(y)$, $y \in \Omega$, and set

$$y = x + \varepsilon \mathbf{T}(x), \quad \mathbf{M}(x) = \mathbf{I} + \varepsilon \mathbf{T}'(x), \quad \tilde{\mathbf{u}}(x) = \mathbf{u}(y(x)), \quad \varrho_\varepsilon(x) = \varrho(y(x)).$$

Thus we get $\mathbf{u}_\varepsilon = N\tilde{\mathbf{u}}$. The Jacobi matrix $\mathbf{M}$ is connected with the matrix $\mathbf{N}$ by the relations

$$\det\, \mathbf{M} = (\det\, N)^{1/2} \equiv \mathfrak{g}, \quad \mathbf{M} = \mathfrak{g}\mathbf{N}^{-1} \tag{10}$$

For any function $\phi \in C^1(\Omega)$ we have $\nabla_y \phi = (\mathbf{M}^*)^{-1}\nabla_x \tilde{\phi}$, where $\tilde{\phi}(x) = \phi(y(x))$. It follows from this that the identities

$$\int_{\tilde{\Omega}} (\operatorname{div}_y \mathbf{u})(y(x))\, \tilde{\phi}(x) \det\, \mathbf{M}\, dx = \int_{\Omega} (\operatorname{div}_y \mathbf{u})(y)\phi(y) \det\ dy = -\int_{\Omega} \mathbf{u}\cdot\nabla_y \phi\, dy =$$

$$-\int_{\tilde{\Omega}} \tilde{\mathbf{u}}\cdot(\mathbf{M}^*)^{-1}\nabla_x\tilde{\phi}(x) \det\, \mathbf{M}\, dx = \int_{\tilde{\Omega}} \operatorname{div}_x\big((\det\, \mathbf{M})\, \mathbf{M}^{-1}\tilde{\mathbf{u}}\big)\, \tilde{\phi}(x)\, dx$$

hold true for all $\phi \in C_0^\infty(\Omega)$. On the other hand, by virtue of (10) we have $(\det\, \mathbf{M})\, \mathbf{M}^{-1}\tilde{\mathbf{u}} = \mathbf{u}_\varepsilon(x)$. This leads to the equalities

$$\begin{aligned} (\operatorname{div}_y \mathbf{u})(y(x)) &= \mathfrak{g}^{-1}\operatorname{div}_x\big(\mathbf{N}\, \tilde{\mathbf{u}}(x)\big) \equiv \mathfrak{g}^{-1}\operatorname{div}_x \mathbf{u}_\varepsilon(x), \\ \operatorname{div}_y(\varrho\mathbf{u})(y(x)) &= \mathfrak{g}^{-1}\operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon), \end{aligned} \tag{11}$$

which imply the modified mass balance equation (8b). From (11) and the identity $(\mathbf{M}^*)^{-1} = \mathfrak{g}^{-1}\mathbf{N}^*$ we obtain

$$\nabla\Big(\lambda \operatorname{div}\mathbf{u} - \frac{R}{\epsilon^2}p(\varrho)\Big) = \mathfrak{g}^{-1}\mathbf{N}^*\nabla\Big(\lambda\mathfrak{g}^{-1} \operatorname{div} \mathbf{u}_\varepsilon - \frac{R}{\epsilon^2}p(\varrho_\varepsilon)\Big). \tag{12}$$

Combining (11) with the identity $\Delta = \operatorname{div}\nabla$ we obtain

$$\begin{aligned} \Delta \mathbf{u}(y) &= \mathfrak{g}^{-1} \operatorname{div}\big(\mathbf{N}(\mathbf{M}^*)^{-1}\nabla\tilde{\mathbf{u}}\big) = \\ \mathfrak{g}^{-1} \operatorname{div}\big(\mathfrak{g}^{-1}\mathbf{N}\mathbf{N}^*\nabla(\mathbf{N}^{-1}\mathbf{u}_\varepsilon)\big) &= \mathfrak{g}^{-1}\mathbf{N}^*\Big(\Delta\mathbf{u}_\varepsilon - \mathscr{A}(\mathbf{u}_\varepsilon)\Big) \end{aligned} \tag{13}$$

Nex note that the components $(\mathbf{u}\nabla\mathbf{u})_i$ of the vector $\mathbf{u}\nabla\mathbf{u}$ satisfy the equalities

$$(\mathbf{u}\nabla\mathbf{u})_i = \mathbf{u}\cdot\nabla_y u_i = \tilde{\mathbf{u}}\cdot\big((\mathbf{M}^*)^{-1}\nabla\tilde{u}_i\big) = \mathfrak{g}^{-1}\mathbf{N}\tilde{\mathbf{u}}\cdot\nabla\tilde{u}_i = \mathfrak{g}^{-1}\mathbf{u}_\varepsilon\cdot\nabla\big(\mathbf{N}^{-1}\mathbf{u}_\varepsilon\big)_i$$

This gives

$$\varrho\mathbf{u}\nabla\mathbf{u} = \mathbf{g}^{-1}\mathbf{N}^*\mathscr{B}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon). \tag{14}$$

Substituting (12)-(14) into mass balance equation and multiplying both sides of the resulting equality by $\mathfrak{g}(\mathbf{N}^*)^{-1}$ we obtain modified equation (8a).
Before formulation of main results we write the governing equation in more transparent form using the change of unknown functions proposed by M. Padula. To do so we introduce *the effective viscous pressure*

$$q = \frac{R}{\epsilon^2}\, p(\varrho) - \lambda \mathfrak{g}^{-1} \operatorname{div} \mathbf{u},$$

and rewrite equations (8) in the equivalent form

$$\Delta \mathbf{u} - \nabla q = \mathscr{A}(\mathbf{u}) + R\mathscr{B}(\varrho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega, \tag{15a}$$

$$\operatorname{div} \mathbf{u} = a\sigma_0 p(\varrho) - \frac{\mathfrak{g} q}{\lambda} \text{ in } \Omega, \tag{15b}$$

$$\mathbf{u} \cdot \nabla \varrho + \mathfrak{g}\sigma_0 p(\varrho)\, \varrho = \frac{\mathfrak{g} q}{\lambda} \varrho \text{ in } \Omega, \tag{15c}$$

$$\mathbf{u} = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u} = 0 \text{ on } \partial S, \tag{15d}$$

$$\varrho = \varrho_0 \text{ on } \Sigma_{\text{in}}. \tag{15e}$$

where $\sigma_0 = R/(\lambda\epsilon^2)$. In the new variables $(\mathbf{u}, q, \varrho)$ the expression for the force $\mathbf{J}$ reads

$$\mathbf{J} = -\int_\Omega \big[\mathfrak{g}^{-1}\big(\mathbf{N}^*\nabla(\mathbf{N}^{-1}\mathbf{u}) + \nabla(\mathbf{N}^{-1}\mathbf{u})^*\mathbf{N} - \operatorname{div}\mathbf{u}\big) - q - R\varrho\mathbf{u}\otimes\mathbf{u}\big]\mathbf{N}^*\nabla\eta\, dx. \tag{16}$$

where $\eta \in C^\infty(\Omega)$ is an arbitrary function, which is equal to 1 in an open neighborhood of the obstacle S and 0 in a vicinity of Σ. The value of $\mathbf{J}$ is independent of the choice of the function η.

3. **Perturbations of the approximate solutions.** We assume that $\lambda \gg 1$ and $R \ll 1$, which corresponds to almost incompressible flow with low Reynolds number. In such a case, the *approximate solutions* to problem (15) can be chosen in the form $(\varrho_0, \mathbf{u}_0, q_0)$, where ϱ_0 is a constant in boundary condition (15e), and $(\mathbf{u}_0, q_0)$ is a solution to the boundary value problem for the Stokes equations,

$$\begin{gathered} \Delta \mathbf{u}_0 - \nabla q_0 = 0, \quad \operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \\ \mathbf{u}_0 = \mathbf{U} \text{ on } \Sigma, \quad \mathbf{u}_0 = 0 \text{ on } \partial S, \quad \Pi q_0 = q_0 \;. \end{gathered} \tag{17}$$

In our notations Π is the projector,

$$\Pi u = u - \frac{1}{\operatorname{meas} \Omega} \int_\Omega u\, dx.$$

Equations (17) can be obtained as the limit of equations (15) for the passage $\lambda \to \infty$, $R \to 0$. It follows from the standard elliptic theory that for the boundary $\partial\Omega \in C^\infty$, we have $(\mathbf{u}_0, q_0) \in C^\infty(\Omega)$. We look for solutions to problem (15) in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}, \quad \varrho = \varrho_0 + \varphi, \quad q = q_0 + \lambda\sigma_0 p(\varrho_0) + \pi + \lambda m, \tag{18}$$

with the unknowns functions $\vartheta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. Substituting (18) into (15) we obtain the following boundary problem for ϑ,

$$\begin{aligned} \Delta \mathbf{v} - \nabla \pi &= \mathscr{A}(\mathbf{u}) + R\mathscr{B}(\varrho, \mathbf{u}, \mathbf{u}) \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} &= \mathfrak{g}\Big(\frac{\sigma}{\varrho_0}\varphi - \Psi[\vartheta] - m\Big) \text{ in } \Omega, \\ \mathbf{u}\cdot\nabla\varphi + \sigma\varphi &= \Psi_1[\vartheta] + m\mathfrak{g}\varrho \text{ in } \Omega, \\ \mathbf{v} = 0 \text{ on } \partial\Omega, \quad \varphi &= 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi, \end{aligned} \tag{19a}$$

where

$$\Psi_1[\vartheta] = \mathfrak{g}\Big(\varrho\Psi[\vartheta] - \frac{\sigma}{\varrho_0}\varphi^2\Big) + \sigma\varphi(1-\mathfrak{g}),\ \Psi[\vartheta] = \frac{q_0 + \pi}{\lambda} - \frac{\sigma}{p'(\varrho_0)\varrho_0} H(\varphi),$$
$$\sigma = \sigma_0 p'(\varrho_0)\varrho_0, \quad H(\varphi) = p(\varrho_0 + \varphi) - p(\varrho_0) - p\prime(\varrho_0)\varphi,$$

the vector field $\mathbf{u}$ and the function ϱ are given by (18). Finally, we specify the constant m. In our framework, in contrast to the case of homogeneous boundary problem, the solution to such a problem is not trivial. Note that, since $\operatorname{div} \mathbf{v}$ is of the null mean value, the right-hand side of equation $(19a)_3$ must satisfy the compatibility condition

$$m\int_\Omega \mathfrak{g}\, dx = \int_\Omega \mathfrak{g}\big(\frac{\sigma}{\varrho_0}\varphi - \Psi[\vartheta]\big)\, dx,$$

which formally determines m. This choice of m leads to essential mathematical difficulties. Tho make this issue clear note that in the simplest case $\mathfrak{g} = 1$ we have $m = \varrho_0^{-1}\sigma(\mathbf{I} - \Pi)\varphi + O(|\vartheta|^2, \lambda^{-1})$, and the principal linear part of the governing equations (19a) becomes

$$\begin{pmatrix} \Delta & -\nabla & 0 \\ \operatorname{div} & 0 & -\frac{\sigma}{\varrho_0} \\ 0 & 0 & \mathbf{u}\nabla + \sigma \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \pi \\ \varphi \end{pmatrix} + \begin{pmatrix} 0 \\ m \\ -m\varrho_0 \end{pmatrix} \sim \begin{pmatrix} \Delta\mathbf{v} - \nabla\pi \\ \operatorname{div}\mathbf{v} - \frac{\sigma}{\varrho_0}\Pi\varphi \\ \mathbf{u}\nabla\varphi + \sigma\Pi\varphi \end{pmatrix}$$

Hence, the question of solvability of the linearized equations derived for (19) can be reduced to the question of solvability of the boundary value problem for nonlocal transport equation

$$\mathbf{u}\nabla\varphi + \sigma\Pi\varphi = f\ ,$$

which is very difficult because of the loss of maximum principle. In fact, this question is concerned with the problem of the control of the total gas mass in compressible flows. Recall that the absence of the mass control is the main obstacle for proving the global solvability of inhomogeneous boundary problems for compressible N-S equations, we refer to [5] for discussion. In order to cope with this difficulty we write the compatibility condition in a sophisticated form, which allows us to control the total mass of the gas. To this end we introduce the auxiliary function ζ satisfying the equations

$$-\operatorname{div}(\mathbf{u}\zeta) + \sigma\zeta = \sigma\mathfrak{g} \text{ in } \Omega, \quad \zeta = 0 \text{ on } \Sigma_{\text{out}}, \tag{19b}$$

and fix the constant m as follows

$$m = \varkappa\int\limits_\Omega (\varrho_0^{-1}\Psi_1[\vartheta]\zeta - \mathfrak{g}\Psi[\vartheta])\, dx, \quad \varkappa = \Big(\int\limits_\Omega \mathfrak{g}(1 - \zeta - \varrho_0^{-1}\zeta\varphi)\, dx\Big)^{-1}. \tag{19c}$$

In this way the auxiliary function ζ becomes an integral part of the solution to problem (19).

3.1. **Existence and uniqueness theory.** Denote by E the closed subspace of the Banach space $Y^{s,r}(\Omega)^3 \times X^{s,r}(\Omega)^2$ in the following form

$$E = \{\vartheta = (\mathbf{v}, \pi, \varphi) \ : \ \mathbf{v} = 0 \text{ on } \partial\Omega, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi\}, \tag{20}$$

and denote by $\mathcal{B}_\tau \subset E$ the closed ball of radius τ centered at 0. Next, note that for $sr > 3$, elements of the ball $\mathcal{B}_\tau$ satisfy the inequality

$$\|\mathbf{v}\|_{C^1(\Omega)} + \|\pi\|_{C(\Omega)} + \|\varphi\|_{C(\Omega)} \le c_e(r, s, \Omega)\|\vartheta\|_E \le c_e\tau, \tag{21}$$

where the norm in E is defined by

$$\|\vartheta\|_E = \|\mathbf{v}\|_{Y^{s,r}(\Omega)} + \|\pi\|_{X^{s,r}(\Omega)} + \|\varphi\|_{X^{s,r}(\Omega)} \ .$$

The existence result is established under some assumptions. We assume that the surface Σ and given vector field $\mathbf{U}$ satisfy emergent field conditions: The set Γ is a a closed C^∞ one-dimensional manifold. Moreover, there is a positive constant c such that $\mathbf{U} \cdot \nabla(\mathbf{U} \cdot \mathbf{n}) > c > 0$ on Γ.

Furthermore, let σ^*, τ^* be sufficiently small constants.

Theorem 3.1. *Let positive numbers r, s, σ satisfy the inequalities*

$$1/2 < s \le 1, \quad 1 < r < 3/(2s-1), \quad sr > 3, \sigma > \sigma^*. \tag{22}$$

Then there exists $\tau_0 \in (0, \tau^*]$, depending only on $\mathbf{U}, \Omega, r, s, \sigma$, such that for all

$$\tau \in (0, \tau_0], \quad \lambda^{-1}, R \in (0, \tau^2], \quad \|\mathbf{N} - \mathbf{I}\|_{C^2(\Omega)} \le \tau^2, \tag{23}$$

problem (19), with $\mathbf{u}_0$ given by (17), has a unique solution $\vartheta \in B_\tau$. Moreover, the auxiliary function ζ and the constants $\varkappa, m$ admit the estimates

$$\|\zeta\|_{X^{s,r}} + |\varkappa| \le c, \quad |m| \le c\tau < 1, \tag{24}$$

where the constant c depends only on $\mathbf{U}, \Omega, r, s$ and σ.

3.2. **Material derivatives of solutions.** Theorem 3.1 guarantees the existence and uniqueness of solutions to problem (19) for all $\mathbf{N}$ close to the identity matrix $\mathbf{I}$. The totality of such solutions can be regarded as the mapping from $\mathbf{N}$ to the solution of the N-S equations. The natural question is the smoothness properties of this mapping, in particular its differentiability. With application to shape optimization problems in mind, we consider the particular case where the matrices $\mathbf{N}$ depend on the small parameter ε and have representation (7). We assume that C^1 norms of the matrix-valued functions $\mathbf{D}$ and $\mathbf{D}_1(\varepsilon)$ in (7) have a majorant independent of ε. By virtue of Theorem 3.1, there are the positive constants ε_0 and τ such that for all sufficiently small R, λ^{-1} and $\varepsilon \in [0, \varepsilon_0]$, problem (19) with $\mathbf{N} = \mathbf{N}(\varepsilon)$ has a unique solution $\vartheta(\varepsilon) = (\mathbf{v}(\varepsilon), \pi(\varepsilon), \varphi(\varepsilon))$, $\zeta(\varepsilon), m(\varepsilon)$, which admits the estimate

$$\|\vartheta(\varepsilon)\|_E + |m(\varepsilon)| \le c\tau, \quad \|\zeta(\varepsilon)\|_{X^{s,r}} \le c, \tag{25}$$

where the constant c is independent of ε, and the Banach space E is defined by (20). Denote the solution for $\varepsilon = 0$ by $(\vartheta(0), m(0), \zeta(0))$ by (ϑ, m, ζ), and define the finite differences with respect to ε

$$(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon) = \varepsilon^{-1}(\vartheta - \vartheta(\varepsilon)), \quad \xi_\varepsilon = \varepsilon^{-1}(\zeta - \zeta(\varepsilon)), \quad n_\varepsilon = \varepsilon^{-1}(m - m(\varepsilon)).$$

Formal calculations shows that the limit $(\mathbf{w}, \omega, \psi, \xi, n) = \lim_{\varepsilon\to 0}(\mathbf{w}_\varepsilon, \omega_\varepsilon, \psi_\varepsilon, \xi_\varepsilon, n_\varepsilon)$ is a solution to linearized equations

$$\begin{aligned}
\Delta\mathbf{w} - \nabla\omega = R\,\mathscr{C}_0(\mathbf{w},\psi) + \mathscr{D}_0(\mathbf{D}) \text{ in } \Omega,\\
\operatorname{div}\mathbf{w} = b^0_{21}\,\psi - b^0_{22}\,\omega + b^0_{23}\,n + b^0_{30}\,\mathfrak{d} \text{ in } \Omega,\\
\mathbf{u}\nabla\psi + \sigma\psi = -\mathbf{w}\cdot\nabla\varphi + b^0_{11}\,\psi + b^0_{12}\,\omega + b^0_{13}\,n + b^0_{10}\,\mathfrak{d} \text{ in } \Omega,\\
-\operatorname{div}(\mathbf{u}\xi) + \sigma\xi = \operatorname{div}(\zeta\mathbf{w}) + \sigma\mathfrak{d} \text{ in } \Omega,\\
\mathbf{w} = 0 \text{ on } \partial\Omega, \quad \psi = 0 \text{ on } \Sigma_{\text{in}}, \qquad \xi = 0 \text{ on } \Sigma_{\text{out}},\\
\omega - \Pi\omega = 0, \quad n = \varkappa\int\limits_\Omega \Big(b^0_{31}\,\psi + b^0_{32}\,\omega + b^0_{34}\,\xi + b^0_{30}\,\mathfrak{d}\Big)\,dx,
\end{aligned} \tag{26}$$

where $\mathfrak{d} = 1/2\,\mathrm{Tr}\,\mathbf{D}$, the variable coefficients b^0_{ij} and the operators $\mathscr{C}_0$, $\mathscr{D}_0$, are defined by the formulae

$$\begin{aligned}
b^0_{11} = \Psi[\vartheta] - \varrho H'(\varphi) + m - \frac{2\sigma}{\varrho_0}\varphi,\ \ b^0_{12} = \lambda^{-1}\varrho,\ \ b^0_{13} = \varrho,\\
b^0_{10} = \varrho\Psi[\vartheta] - \frac{\sigma}{\varrho_0}\varphi^2 - \sigma\varphi + m\varrho,\ \ b^0_{21} = \frac{\sigma}{\varrho_0}\psi_0 + H'(\varphi),\\
b^0_{22} = -\lambda^{-1},\ \ b^0_{23} = -1,\ \ b^0_{20} = \sigma\varphi\varrho_0^{-1} - \Psi[\vartheta] - m,\\
b^0_{31} = \varrho_0^{-1}\zeta\Big(\Psi[\vartheta] - \varrho H'(\varphi) - \frac{2\sigma}{\varrho_0}\varphi\Big) - H'(\varphi) + m\varrho_0^{-1}\zeta,\\
b^0_{32} = (\lambda\varrho_0)^{-1}\varrho\zeta b^0_{12} + \lambda^{-1},\ \ b^0_{34} = \varrho_0^{-1}\Psi_1[\vartheta] + m(1 + \varrho_0^{-1}\varphi)\\
b^0_{30} = \varrho_0^{-1}\zeta(\mathfrak{d}_0 - m\varrho) + \Psi[\vartheta] - m(1 - \zeta - \varrho_0^{-1}\zeta\varphi),
\end{aligned} \tag{27}$$

$$\mathscr{C}_0(\psi,\mathbf{w}) = R\psi\mathbf{u}\nabla\mathbf{u} + R\varrho\mathbf{w}\nabla\mathbf{u}, + R\varrho\mathbf{u}\nabla\mathbf{w}, \tag{28}$$

$$\begin{aligned}
\mathscr{D}_0(\mathbf{D}) = R\mathbf{u}\nabla(\mathbf{D}\mathbf{u}) + R\mathbf{D}^*(\mathbf{u}\nabla\mathbf{u}) + \\
\operatorname{div}\big((\mathbf{D}+\mathbf{D}^*)\nabla\mathbf{u} - \frac{1}{2}\mathrm{Tr}\ \mathbf{D}\nabla\mathbf{u}\big) - \mathbf{D}\Delta\mathbf{u} - \Delta(\mathbf{D}\mathbf{u}).
\end{aligned} \tag{29}$$

Finally, we obtain the existence of the weak material derivatives for the approximate solutions of N-S equations.

Theorem 3.2. *Under our assumptions,*

$$\begin{aligned}
\mathbf{w}_\varepsilon \to \mathbf{w} \ \textit{weakly in}\ \mathcal{H}^{1-s,r'}_0(\Omega), \quad n_\varepsilon \to n \text{ in } \mathbb{R},\\
\psi_\varepsilon \to \psi, \quad \omega_\varepsilon \to \omega, \quad \xi_\varepsilon \to \xi\ (*)\textit{-weakly in}\ \mathbb{H}^{-s,r'}(\Omega)\ \textit{as}\ \varepsilon \to 0,
\end{aligned} \tag{30}$$

where the limits, vector field $\mathbf{w}$*, functionals* ψ, ω, ξ*, and the constant* n *are given by the weak solution to problem* (26)*.*

3.3. **Shape derivative of the drag functional.**

Theorem 3.3. *There exists the shape derivative of the drag functional in the following form*

$$\frac{d}{d\varepsilon}J_D(\mathcal{S}_\varepsilon)\Big|_{\varepsilon=0} = L_e(\mathbf{T}) + L_u(\mathbf{w},\omega,\psi).$$

The first form L_e is called the geometrical part of the shape gradient. It depends on the transformation T and the solution to the state equation. We can show that for the strictly convex hold-all domain and for the constant vector fields on the boundary, the form is null.

The second form L_u depends on the material derivatives of solutions to the state equation. The adjoint state is introduced in order to eliminate the material derivatives and derive the standard form of the shape derivative of the drag functional.

$$L_e(\mathbf{T}) = \int_\Omega \operatorname{div} \mathbf{T}(\nabla \mathbf{u} + \nabla \mathbf{u}^* - \operatorname{div} \mathbf{u}\mathbf{I})\mathbf{U}_\infty \, dx -$$
$$\int_\Omega \big[\nabla \mathbf{u} + \nabla \mathbf{u}^* - \operatorname{div} \mathbf{u} - q\mathbf{I} - R\varrho \mathbf{u} \otimes \mathbf{u}\big]\mathbf{D}\nabla\eta \cdot \mathbf{U}_\infty \, dx -$$
$$\int_\Omega \big[\mathbf{D}^*\nabla \mathbf{u} + \nabla \mathbf{u}^*\mathbf{D} + \nabla(\mathbf{D}\mathbf{u}) + \nabla(\mathbf{D}\mathbf{u})^*\big]\nabla\eta \cdot \mathbf{U}_\infty \, dx$$

and

$$L_u(\mathbf{w}, \omega, \psi) = \int_\Omega \mathbf{w}\big[\Delta\eta \mathbf{U}_\infty + R\varrho(\mathbf{u} \cdot \nabla\eta)\mathbf{U}_\infty + R\varrho(\mathbf{u} \cdot \mathbf{U}_\infty)\nabla\eta\big] \, dx$$
$$+ \big\langle \omega, \nabla\eta \cdot \mathbf{U}_\infty \big\rangle + R\big\langle \psi, (\mathbf{u} \cdot \nabla\eta)(\mathbf{u} \cdot \mathbf{U}_\infty)\big\rangle.$$

3.4. **Boundary shape gradient.** Assume the following form of the trace on the boundary of obstacle for the mapping $\mathbf{T}$

$$\mathbf{T}(\omega) = f(\omega)\mathbf{n}(\omega) \text{ for } \omega \in \partial S.$$

It is reasonable to eliminate η and $\mathbf{T}$ from formulae for the shape gradient and the adjoint state equations and reformulate the expression for forms L_e and L_u in terms of the normal shift $f(\omega)$ only.

Theorem 3.4. *If the deformed surface ∂S_ε results from the mapping $\mathbf{I} + \varepsilon \mathbf{T}$ with $f \in C^\infty(\partial S)$, then*

$$L_e(\mathbf{T}) = 0, \tag{31}$$

and

$$L_u(\mathbf{w}, \psi, \omega) = \int_{\partial S} f(\omega)\big[\big(b_{10}\varsigma + b_{20}^0 g + \sigma\upsilon + \varkappa b_{30}\big) + (\partial_n \mathbf{h} \cdot \mathbf{n})(\partial_n \mathbf{u} \cdot \mathbf{n})\big] \tag{32}$$

with the adjoint state variables $(\mathbf{h}, g, \varsigma, \upsilon)$ satisfying the following equations and boundary conditions

$$(\mathbf{h}, g, \varsigma, \upsilon, l) = (\mathbf{H}, G, Z, 0, 0) + (\tilde{\mathbf{h}}, \tilde{g}, \tilde{\varsigma}, \tilde{\upsilon}, \tilde{l}), \tag{33}$$

$$\Delta \mathbf{H} - \nabla G - R\varrho\ (\mathbf{H}\nabla\mathbf{u} + \mathbf{u}\nabla\mathbf{H}) = 0, \tag{34a}$$
$$\operatorname{div} \mathbf{H} + \lambda^{-1}\Pi G = 0, \tag{34b}$$
$$-\operatorname{div}(\mathbf{u}Z) + \sigma Z = R(\mathbf{u}\nabla\mathbf{u}) \cdot \mathbf{H} + b_{21}G + b_{11}Z, \tag{34c}$$
$$\mathbf{H} = 0 \ \textit{on}\ \Sigma, \quad \mathbf{H} = -\mathbf{U} \ \textit{on}\ \partial S, \quad Z = 0 \text{ on } \Sigma_{\text{out}}. \tag{34d}$$

$$\Delta\tilde{\mathbf{h}} - \nabla\tilde{g} - R\varrho\ (\tilde{\mathbf{h}}\nabla\mathbf{u} + \mathbf{u}\nabla\tilde{\mathbf{h}}) + (Z + \tilde{\varsigma})\nabla\varphi + \zeta\nabla\tilde{\upsilon} = 0 \tag{35a}$$
$$\operatorname{div}\tilde{\mathbf{h}} - \Pi(b_{12}(Z + \tilde{\varsigma}) + \lambda^{-1}\Pi\tilde{g} = 0 \tag{35b}$$
$$-\operatorname{div}(\mathbf{u}\tilde{\varsigma}) + \sigma\tilde{\varsigma} = R(\mathbf{u}\nabla\mathbf{u}) \cdot (\tilde{\mathbf{h}}) + b_{21}\tilde{g} + b_{11}\tilde{\varsigma} + \varkappa b_{13}\tilde{l} \tag{35c}$$
$$\mathbf{u}\nabla\tilde{\upsilon} + \sigma\tilde{\upsilon} - \kappa b_{34}\tilde{l} = 0 \tag{35d}$$

$$\int_\Omega b_{13}(Z+\tilde{\varsigma})\,dx + \tilde{l} = 0 \tag{35e}$$

$$\tilde{\mathbf{h}} = 0 \text{ on } \partial\Omega, \quad \tilde{\varsigma} = 0 \text{ on } \Sigma_{\text{out}}, \quad \tilde{\upsilon} = 0 \text{ on } \Sigma_{\text{in}}. \tag{35f}$$

In addition, there exists σ_0 *with the following properties. For any* $\sigma > \sigma_0$*, there are* λ_0 *and* R_0*, depending only on* σ*,* ∂S *and* $\mathbf{U}$*, so that for every* $R < R_0$ *and* $\lambda > \lambda_0$ *the adjoint state equations have a solution* $(\mathbf{h}, g, \varsigma, \upsilon, l) \in W^{2,2}(\Omega) \times (W^{1,2}(\Omega))^2 \times \mathbb{R}$*. The boundary shape gradient is independent on the function* η *and depends only on the restriction of the transformation field to the boundary.*

The result is new, and its proof is based on the singular limits of volume integrals if the supports of the transformation fields converge to the boundary ∂S. The full proof of the result is given in a forthcoming paper.

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