DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **592–601**

GLOBAL SOLUTIONS TO THE CAUCHY PROBLEM FOR THE WEAKLY COUPLED SYSTEM OF DAMPED WAVE EQUATIONS

Takashi Narazaki

Hiratsuka, Kanagawa, 259-1292, Japan

Abstract. In this paper we study the Cauchy problem for the weakly coupled system of damped wave equations. Recently Sun and Wang [12] have shown the existence and nonexistence of the Cauchy problem for the weakly coupled system of damped wave equations, provided that the space dimension $n = 1, 3$. In this paper we will generalize their existence result to the case where $n = 1, 2.3$, and we improve time decay estimates when $n = 3$. Moreover, the Cauchy problem with slow decaying initial data is treated.

1. Introduction.

In this paper we are concerned with global existence and asymptotic behavior of solutions to the Cauchy problem for the weakly coupled system of the following damped wave equations

$$\begin{aligned} \partial_t^2 u - \Delta u + \partial_t u = f(v), \quad t > 0, x \in \mathbb{R}^n, \\ \partial_t^2 v - \Delta v + \partial_t v = g(u), \quad t > 0, x \in \mathbb{R}^n, \\ u(0,x) = \varphi_0(x), \quad u_t(0,x) = \varphi_1(x), \quad x \in \mathbb{R}^n, \\ v(0,x) = \psi_0(x), \quad v_t(0,x) = \psi_1(x), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $n = 1, 2, 3$, $\partial_t = \partial/\partial t$, Δ is the Laplace operator in $\mathbb{R}^n$, nonlinear terms f and g are of class C^1, and they satisfy that $|f(v)| \le C|v|^{\sigma_1}, |f'(v)| \le C|v|^{\sigma_1 - 1})$ when $|v| \le 1$, $|g(u)| \le C|u|^{\sigma_2}, |g'(u)| \le C|u|^{\sigma_2 - 1}$ when $|u| \le 1$ for some constants $\sigma_1 \ge 1$ and $\sigma_2 \ge 1$. Here and after we denote by $\partial_j = \partial/\partial x_j$ $(j = 1, \cdots, n)$, $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for a multi-index of non negative integers, $\mathcal{F}h(\xi) = \hat{h}(\xi)$ is the Fourier transformation of h with respect to x and $\mathcal{F}^{-1}\varrho(x) = \check{\varrho}(x)$ is the inverse Fourier transformation. We use standard function spaces $L^q := L^q(\mathbb{R}^n)$ and

$$W^{1,q} := W^{1,q}(\mathbb{R}^n) := \{h \in L^q | \partial_j h \in L^q, j = 1, \cdots, n.\}$$

$$\|h\|_{1,q} := \|h\|_{W^{1,q}} := \|h\|_q + \sum_{j=1}^n \|\partial_j h\|_q ,$$

where $\|h\|_q$ denote the usual L^q norm $\|h\|_{L^q}$.

Recently, Sun and Wang [12] have shown that problem (1) admits global solution in time when $n = 1$ or 3, $f(v) = |v|^{\sigma_1}$, $g(u) = |u|^{\sigma_2}$, provided that initial data $(\varphi_0, \varphi_1) \in (W^{1,1} \cap W^{1,\infty}) \times (L^1 \cap L^\infty)$ and $(\psi_0, \psi_1) \in (W^{1,1} \cap W^{1,\infty}) \times (L^1 \cap L^\infty)$ are small, and (σ_1, σ_2) satisfies that

$$\max\left(\frac{\sigma_1 + 1}{\sigma_1\sigma_2 - 1}, \frac{\sigma_2 + 1}{\sigma_1\sigma_2 - 1}\right) < \frac{n}{2}, \ \sigma_1 \ge 1, \ \sigma_2 \ge 1, \ \sigma_1\sigma_2 > 1. \tag{2}$$

2000 *Mathematics Subject Classification.* Primary: 35L55, 35L15, 35L70.
Key words and phrases. Damped wave equation, Weakly coupled system, Cauchy problem.

They also have shown the decay estimates of the above solution. Moreover, they have shown that any non-negative and non-trivial solutions to (1) replaced with $(f(v), g(u))$ by $(|v|^{\sigma_1}, |u|^{\sigma_2})$, blow up in finite time, when condition (2) does not hold. Condition (2) was proposed by Escobedo-Herrero [1] to the system of heat equations for $n \in \{1, 2, \cdots\}$

$$u_t - \Delta u = |v|^{\sigma_1}, \quad v_t - \Delta v = |u|^{\sigma_2}, \quad t > 0, x \in \mathbb{R}^n. \tag{3}$$

However, no global existence theorem is given when $n = 2$ in [12], and L^∞ estimates of the solutions of (1) seems to be rather complicated when $n = 3$.

In this paper, we will show the sufficient condition for that the Cauchy problem (1) admits global solutions when $n \in \{1, 2, 3\}$, provided that the initial data are sufficiently small in $E_p \times E_p$ for some $p \in [1, \infty)$, where

$$E_p := \left(W^{1,p} \cap W^{1,\infty}\right) \times (L^p \cap L^\infty),$$

$$\|(u_0, u_1)\|_{E_p} := \|u_0\|_{1,p} + \|u_0\|_{1,\infty} + \|u_1\|_p + \|u_1\|_\infty .$$

Moreover, we show the asymptotic behavior of the above solutions.

Several authors have studied the Cauchy problem to the corresponding single equation

$$\begin{aligned} &\partial_t^2 U - \Delta U + \partial_t U = f(U), \quad t > 0,\ x \in \mathbb{R}^n, \\ &U(0, x) = U_0(x), \quad \partial_t U(0, x) = U_1(x) \qquad x \in \mathbb{R}^n. \end{aligned} \tag{4}$$

It is known that $\sigma_F := 1 + 2/n$ is the critical exponent to (4). When $\sigma_1 > \sigma_F := 1 + 2/n$ and initial data $(U_0, U_1) \in E_1$ are sufficiently small, then (4) admits a unique global solution, and the solution behaves as that to the corresponding heat equation when $t \to \infty$. On the other hand, when $1 < \sigma_1 \le \sigma_F$ and $f(U) = |U|^{\sigma_1}$, then any non-trivial, non-negative solutions of Problem (4) blow-up in finite time ([4], [6], [8], [11], [13] and [14]). Recently Narazaki and Nishihara [10] have studied the case where initial data decay slowly, namely, $(U_0, U_1) \in E_p$ $(1 < p < \infty)$, and they have obtained the conjecture that the critical exponent of (4) is $1 + 2p/n$ (see also [5]). We note that Problem (4) may admit global solution in time when initial data are sign-changing and $f(U) = |U|^{\sigma_1 - 1}U$, even if $\sigma \le \sigma_F$, (See [9]).

To state the result, we need some notations. We set a weight function

$$Q_d(x) = \begin{cases} \prod_{j=1}^d \sqrt{1 + x_j^2}, & d = 1, \cdots, n, \\ 1, & d = 0. \end{cases}$$

Let $d \in \{1, \cdots, n\}$. A function $h(x)$ is said to be odd with respect to $(x_1, \cdots, x_d)$ if

$$h(\cdots, -x_j, \cdots) = -h(\cdots, x_j, \cdots)$$

for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $j \in \{1, \cdots, d\}$.

Our theorem is as follows.

Main Theorem. *Let $n \in \{1, 2, 3\}$, $d \in \{0, 1, \cdots, n\}$, $p \in [1, \infty)$ be fixed. Assume that*

$$\max\left(\frac{\sigma_1 + 1}{\sigma_1\sigma_2 - 1}, \frac{\sigma_2 + 1}{\sigma_1\sigma_2 - 1}\right) < \frac{n}{2p} + \frac{d}{2},\ \sigma_1 \ge 1,\ \sigma_2 \ge 1,\ \sigma_1\sigma_2 > 1.$$

When $d \geq 1$, assume that the initial data φ_j, ψ_j $(j = 0, 1)$ are odd with respect to $(x_1, \cdots, x_d)$, and $f(v)$, $g(u)$ are odd. Then there exists a positive constant ϵ such that if

$$\|(Q_d\varphi_0, Q_d\varphi_1)\|_{E_p} + \|(Q_d\psi_0, Q_d\psi_1)\|_{E_p} \leq \epsilon,$$

(1) admits a unique global solution in time

$$(u, v) \in C([0, \infty); L^p \times L^p) \cap L^\infty([0, \infty); L^\infty \times L^\infty),$$

and it satisfies the following estimates

$$\|u(t)\|_{L^q} + (1+t)^{-d/2}\|Q_d u(t)\|_{L^q} \leq C\epsilon(1+t)^{n/2q-(\sigma_1+1)/(\sigma_1\sigma_2-1)},$$
$$\|v(t)\|_{L^q} + (1+t)^{-d/2}\|Q_d v(t)\|_{L^q} \leq C\epsilon(1+t)^{n/2q-(\sigma_2+1)/(\sigma_1\sigma_2-1)}$$

$\forall q \in [p, \infty]$.

Remark 1. (1) When $\sigma_1 = \sigma_2$, the condition in Theorem implies that $\sigma_1 = \sigma_2 > \sigma_{d,p} := 1 + 2p/(n + dp)$. Note that $\sigma_{d,p}$ is equal to the critical exponent proposed in [10] when $d = 0$.
(2) Let consider the case where $\min(\sigma_1, \sigma_2) > \sigma_{d,p}$. Then, by the arguments in [9] and [10], one can prove that the decay order of the solutions (u, v) of in (1) is same as one of solutions of the corresponding linear problem, and it is better than one obtained in Theorem. Note that this decay order is same as one of solutions of the corresponding linear problem, and is better than one obtained in Theorem.

For the proof of Theorem, we need the next lemma.

Lemma 1.1. *Let μ_1, μ_2 be constants.*

(1) *If $\mu_1 < 1$, then*

$$\int_{t/2}^{t} (1+t-\tau)^{-\mu_1}(1+\tau)^{-\mu_2}\, d\tau \leq C(1+t)^{1-\mu_1-\mu_2} \quad \forall t > 0.$$

(2) *If $\mu_2 < 1$, then*

$$\int_{0}^{t/2} (1+t-\tau)^{-\mu_1}(1+\tau)^{-\mu_2}\, d\tau \leq C(1+t)^{1-\mu_1-\mu_2} \quad \forall t > 0.$$

(3) *If δ is a positive constant, then*

$$\int_{0}^{t} e^{-\delta(t-\tau)}(1+\tau)^{-\mu_1}\, d\tau \leq C(1+t)^{-\mu_1} \quad \forall t > 0.$$

2. **Linear estimate.** In this section we show estimates for the solutions of the following linear damped wave equation when $n = 1$, 2 or 3.

$$\partial_t^2 U - \Delta U + \partial_t U = 0, \quad U(0, x) = U_0(x), \quad \partial_t U(0, x) = U_1(x) \tag{5}$$

The solution of (5) is given by the following equation:

$$U(t, x) = S_0(t)U_0(x) + S_1(t)\left(\frac{1}{2}U_0 + U_1\right)(x), \tag{6}$$

where

$$S_0(t)h := \mathcal{F}^{-1}\left(e^{-t/2}\cos t\sqrt{|\xi|^2 - (1/4)}\,\hat{h}(\xi)\right),$$
$$S_1(t)h := \mathcal{F}^{-1}\left(e^{-t/2}\frac{\sin t\sqrt{|\xi|^2 - 1/4}}{\sqrt{|\xi|^2 - 1/4}}\hat{h}(\xi)\right).$$

Next Theorem gives estimate of the low-frequency part of solution to (5).

Theorem 2.1. *(See [9]) Let $d \in \{0, \cdots, n\}$, $p \in [1, \infty)$. Assume that $Q_d U_j \in L^p$, $\hat{U}_j(\xi) = 0$ when $|\xi| \geq 2$ for $j = 0, 1$, and U_0, U_1 are odd with respect to $(x_1, \cdots, x_d)$ when $d \neq 0$. Let $U(t, x)$ be the solution of (5), then for any $q \in [p, \infty]$, $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^n$, $Q_d \partial_t^k \partial_x^\alpha U(t) \in L^q$ for $t \geq 0$, and the following estimates hold.*

$$\left\| Q_d^\theta \partial_t^k \partial_x^\alpha U(t, \cdot) \right\|_q \leq C(1+t)^{-n(1/p-1/q)/2-k-|\alpha|/2-(1-\theta)d/2} \sum_{j=0}^{1} \left\| Q_d U_j \right\|_p \quad \forall t > 0.$$

To estimate high-frequency part of solution to (5), we need several Lemmas.

Lemma 2.2. *Let $\chi(\xi) \in C^\infty$ be a radial cut-off function:*

$$\chi(\xi) = \begin{cases} 0, & |\xi| \leq 2/3, \\ 1, & |\xi| \geq 3/4, \end{cases} . \tag{7}$$

The functions $L_c(t, x)$ and $L_s(t, x)$ defined by

$$L_c(t, x) = \mathcal{F}^{-1} \left(\chi(\xi) \left(\cos t\sqrt{|\xi|^2 - 1/4} - \cos t|\xi| - \frac{t}{8} \frac{\sin t|\xi|}{|\xi|} \right) \right),$$

$$L_s(t, x) = \mathcal{F}^{-1} \left(\chi(\xi) \left(\frac{\sin t\sqrt{|\xi|^2 - 1/4}}{\sqrt{|\xi|^2 - 1/4}} - \frac{\sin t|\xi|}{|\xi|} \right) \right).$$

satisfy

$$\|Q_d L_c(t, \cdot)\|_1 + \|Q_d L_s(t, \cdot)\|_1 \leq C(1+t)^N, \quad \forall t \geq 0,$$

where N depends on n and d.

Proof. We set

$$\Theta_1(\xi) = |\xi| - \sqrt{|\xi|^2 - 1/4}, \quad \Theta_2(\xi) = \Theta_1(\xi) - \frac{1}{8|\xi|}.$$

Since

$$\begin{aligned} \hat{L}_c(t, \xi) &:= \mathcal{F} L_c(t, \cdot) \\ &= \chi \left(\cos t|\xi| \left(\cos t\Theta_1 - 1 \right) + \sin t|\xi| \left(\sin \frac{t}{8|\xi|} - \frac{t}{8|\xi|} \right) + \cos \frac{t}{8|\xi|} \sin t\Theta_2 \right), \end{aligned}$$

$$\begin{aligned} \hat{L}_s(t, \xi) &:= \mathcal{F} L_s(t, \cdot) \\ &= \chi \left(\sin t|\xi| \frac{\Theta_1}{|\xi|(|\xi| - \Theta_1)} + \frac{\sin t|\xi|}{\sqrt{|\xi|^2 - 1/4}} \left(\cos t\Theta_1 - 1 \right) - \cos t|\xi| \sin t\Theta_1 \right), \end{aligned}$$

direct calculations show that

$$\sum_{|\alpha| \leq m} \left(\left| \partial_\xi^\alpha \hat{L}_c(t, \xi) \right| + \left| \partial_\xi^\alpha \hat{L}_s(t, \xi) \right| \right) \leq C(1+t)^{m+3}(1+|\xi|)^{-2}$$

for any $m = 0, 1, 2, \cdots$. The above implies that

$$\sum_{|\alpha| \leq m} \left(\|x^\alpha L_c(t, \cdot)\|_2 + \|x^\alpha L_s(t, \cdot)\|_2 \right) \leq C(1+t)^{m+3}. \tag{8}$$

Since $(1 + |x|)^{-m} Q_d \in L^2$ when $m > n/2 + d$, and $Q_d(x) \leq C(1 + |x|)^d$, Hölder's inequality and (8) show the desired estimate:

$$\|Q_d L_c(t, \cdot)\|_1 + \|Q_d L_s(t, \cdot)\|_1 \leq C(1+t)^{m+3}$$

for $m > n/2 + d$. □

Let $\mathcal{I}_d$ be the set of all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_d)$ such that $\alpha_k \in \{0,1\}$ $(k = 1, \cdots, d)$ when $d > 0$. Then it follows that $0 < Q_d(x) \le \sum_{\alpha \in \mathcal{I}} |x^\alpha| \le cQ_d(x)$, therefore Young's inequality gives the next lemma.

Lemma 2.3. *Let constants $q_1, q_2, q_3 \in [1, \infty]$ satisfy that $1/q_3 = 1/q_1 + 1/q_2 - 1$. Assume that $Q_d h_1 \in L^{q_1}$ and $Q_d h_2 \in L^{q_2}$, then $Q_d(h_1 * h_2) \in L^{q_3}$ and*

$$\|Q_d(h_1 * h_2)\|_{q_3} \le C \|Q_d h_1\|_{q_2} \|Q_d h_1\|_{q_2}.$$

Theorem 2.4. *Let $n \in \{1,2,3\}$, $d \in \{0, \cdots, n\}$ and $q \in [1, \infty]$. Assume that $(U_0, U_1) \in W^{1,q} \times L^q$ satisfy $(Q_d U_0, Q_d U_1) \in W^{1,q} \times L^q$, and $\hat{U}_0(\xi) = \hat{U}_1(\xi) = 0$ when $|\xi| \le 1$. Let $U(t,x)$ be the solution of (5), then $Q_d U(t, \cdot) \in L^\infty([0,\infty); L^q)$, and, for any $\delta_0 < 1/2$, the following estimates hold:*

$$\|Q_d U(t,\cdot)\|_q \le Ce^{-\delta_0 t} \left(\|Q_d U_0\|_{1,q} + \|Q_d U_1\|_q \right) \quad t \ge 0,$$

where the above C depends only on δ_0.

Proof. Let χ, L_c and L_s be the functions defined in Lemma 2.2. Since $\hat{U}_j(\xi) = \chi(\xi)\hat{U}_j(\xi)$ $(j = 0, 1)$, then

$$\begin{aligned} S_0(t)U_0 &= e^{-t/2} \left(W_0(t)U_0 + \frac{t}{8} W_1(t)U_0 \right) + e^{-t/2} L_c(t) * U_0, \\ S_1(t)U_1 &= e^{-t/2} W_1(t)U_1 + e^{-t/2} L_s(t) * U_1, \end{aligned} \tag{9}$$

where

$$W_0(t) = \cos t\sqrt{-\Delta}, \qquad W_1(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}},$$

Lemmas 2.2–2.3 show that

$$\|Q_d L_c(t) * h_0\|_q + \|Q_d L_s(t) * h_1\|_q \le C(1+t)^N \left(\|Q_d h_0\|_q + \|Q_d h_1\|_q \right). \tag{10}$$

The solution formula to wave equation and Jensen's inequality give

$$\begin{aligned} \left\| Q_d \left(W_0(t)U_0 + \frac{t}{8} W_1(t)U_0 \right) \right\|_q &\le C(1+t)^{d+2} \|Q_d U_0\|_{1,q}, \\ \|Q_d W_1(t)U_1\|_q &\le C(1+t)^{d+1} \|Q_d U_1\|_q \end{aligned} \tag{11}$$

Theorem 2.4 is a direct consequence of (9)–(11). □

Remark 2. When $d = 0$ and $1 \le n \le 3$ the similar estimates to Theorems 2.1, 2.4 are shown by several authors. See [4], [6] and [11]. When $d \ge 1$, Narazaki [9] proved the similar estimates. without the restriction on space dimension that $n \le 3$. But neither estimates of L^1-norm nor L^∞-norm to solutions were given there.

3. **Nonlinear problem.**

We may assume that $\sigma_1 \le \sigma_2$ without loss of generality. Problem (1) is rewritten as the system of the following integral equations:

$$\begin{aligned} u(t,x) &= u_0(t,x) + \int_0^t S_1(t-\tau)|v|^{\sigma_1 - 1} v(\tau, \cdot)\, d\tau, \\ v(t,x) &= v_0(t,x) + \int_0^t S_1(t-\tau)|u|^{\sigma_2 - 1} u(\tau, \cdot)\, d\tau, \end{aligned} \tag{12}$$

where

$$
\begin{aligned}
u_0(t,x) &= S_0(t)\varphi_0 + S_1(t)\left(\frac{1}{2}\varphi_0 + \varphi_1\right), \\
v_0(t,x) &= S_0(t)\psi_0 + S_1(t)\left(\frac{1}{2}\psi_0 + \psi_1\right)
\end{aligned}
\tag{13}
$$

Let $\chi_l(\xi) \in C^\infty$ be a radial cut-off function:

$$
\chi_l(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2, \end{cases}
$$

and let $\chi_h(\xi) = 1 - \chi_l(\xi)$. For $w \in \mathcal{S}'(\mathbb{R}^n)$, we set

$$
w^{(l)}(x) = \mathcal{F}^{-1}(\chi_l \hat{w}), \quad w^{(h)}(x) = \mathcal{F}^{-1}(\chi_h \hat{w}). \tag{14}
$$

Let (u,v) be the solution of integral equations (12) then it follows that

$$
\begin{aligned}
u^{(\rho)}(t,x) &= u_0^{(\rho)}(t,x) + \int_0^t S_1(t-\tau) f^{(\rho)}(v)(\tau,\cdot)\,d\tau, \\
v^{(\rho)}(t,x) &= v_0^{(\rho)}(t,x) + \int_0^t S_1(t-\tau) g^{(\rho)}(u)(\tau,\cdot)\,d\tau,
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
f^{(\rho)}(v)(t,x) &:= \mathcal{F}^{-1}\left(\chi_\rho(\cdot)\mathcal{F}\left(|v|^{\sigma_1-1}v(t,\cdot)\right)\right), \\
g^{(\rho)}(u)(t,x) &:= \mathcal{F}^{-1}\left(\chi_\rho(\cdot)\mathcal{F}\left(|u|^{\sigma_2-1}u(t,\cdot)\right)\right)
\end{aligned}
$$

for $\rho = l, h$. Note that $u = u^{(l)} + u^{(h)}$, $v = v^{(l)} + v^{(h)}$ and integral equations (15) are equivalent to (12).

Let $X_{j,d}$ $(j = 1,2)$ be the set of all functions

$$
w \in C([0,\infty); L^p) \cap L^\infty([0,\infty); L^\infty)
$$

which are odd with respect to $(x_1, \cdots, x_d)$ when $d \neq 0$, and which satisfy

$$
Q_d w \in C([0,\infty); L^p) \cap L^\infty_{loc}([0,\infty); L^\infty), \quad \|w\|_{X_{jd}} < \infty
$$

where

$$
\begin{aligned}
\|w\|_{X_{jd}} = &\sup_{t>0}(1+t)^{(\sigma_j+1)/(\sigma_1\sigma_2-1)} \\
&\times \Big\{(1+t)^{-n/2p}\left(\left\|w^{(l)}(t)\right\|_p + (1+t)^{-d/2}\left\|Q_d w^{(l)}(t)\right\|_p\right) \\
&+ \left\|w^{(l)}(t)\right\|_\infty + (1+t)^{-d/2}\left\|Q_d w^{(l)}(t)\right\|_\infty \\
&+ (1+t)^{1-n/2p}\left(\left\|w^{(h)}(t)\right\|_p + (1+t)^{-d/2}\left\|Q_d w^{(h)}(t)\right\|_p\right) \\
&+ (1+t)\left(\left\|w^{(h)}(t)\right\|_\infty + (1+t)^{-d/2}\left\|Q_d w^{(h)}(t)\right\|_\infty\right)\Big\}
\end{aligned}
\tag{16}
$$

for $j = 1,2$, and let $Y_{j,d}$ $(j = 1,2)$ be the set of all functions $w \in X_{j,d}$ that satisfy

$$
Q_d w \in C([0,\infty); L^p) \cap L^\infty_{loc}([0,\infty); L^\infty), \quad \|w\|_{Y_{jd}} < \infty
$$

where

$$\begin{aligned}\|w\|_{Y_{jd}} = &\sup_{t>0}(1+t)^{(\sigma_j+1)/(\sigma_1\sigma_2-1)+1}\\ &\times\Big\{(1+t)^{-n/2p}\Big(\|w(t)\|_p+(1+t)^{-d/2}\|Q_dw(t)\|_p\Big)\\ &+\Big(\|w(t)\|_\infty+(1+t)^{-d/2}\|Q_dw\|_\infty\Big)\Big\}\end{aligned} \tag{17}$$

for $j=1,2$.

When initial data are odd with respect to $(x_1,\cdots,x_d)$ and $d\ge 1$, $u_0(t,\cdot)$ and $v_0(t,\cdot)$ are also odd with respect to $(x_1,\cdots,x_d)$. Theorems 2.1, 2.4 and (13) give the next lemma.

Lemma 3.1. *Let $n=1,2,3$, $p\in[1,\infty)$, $d\in\{0,1,2,3\}$. Assume that*

$$(Q_d\varphi_0,Q_d\varphi_1),(Q_d\psi_0,Q_d\psi_1)\in E_p.$$

Moreover assume that φ_0, ψ_0, φ_1 and ψ_1 are odd with respect to $(x_1,\cdots,x_d)$ when $d\neq 0$. Let u_0 and v_0 are functions that are defined by (13). Then $u_0\in X_{1,d}$, $v_0\in X_{2,d}$, and

$$\|u_0\|_{X_{1d}}\le C\|(Q_d\varphi_0,Q_d\varphi_1)\|_{E_p},\quad \|v_0\|_{X_{2d}}\le C\|(Q_d\psi_0,Q_d\psi_1)\|_{E_p}.$$

Lemma 3.2. *Assume that $u_k\in X_{1,d}$, $v_k\in X_{2,d}$ $(k=1,2)$, then $f(v_k)\in Y_{1d}$, $g(u_k)\in Y_{2d}$ $(k=1,2)$, and they satisfy the following estimates:*

$$\begin{aligned}\|f(v_1)-f(v_2)\|_{Y_{1d}}&\le C\Big(\|v_1\|_{X_{2d}}^{\sigma_1-1}+\|v_2\|_{X_{2d}}^{\sigma_1-1}\Big)\|v_1-v_2\|_{X_{2d}},\\ \|g(u_1)-g(u_2)\|_{Y_{2d}}&\le C\Big(\|u_1\|_{X_{1d}}^{\sigma_2-1}+\|u_2\|_{X_{1d}}^{\sigma_2-1}\Big)\|u_1-u_2\|_{X_{1d}}.\end{aligned} \tag{18}$$

Proof. Note that $f(v_k)$ and $g(u_k)$ $(k=1,2)$ are odd with respect to $(x_1,\cdots,x_d)$ when $d>0$. Since

$$\left\|Q_d^\theta\left(f(v_1)-f(v_2)\right)\right\|_q\le C\Big(\|v_1\|_\infty^{\sigma_1-1}+\|v_2\|_\infty^{\sigma_1-1}\Big)\left\|Q_d^\theta\left(v_1-v_2\right)\right\|_q$$

for $q=p,\infty$, $\theta=0,1$, (16) shows that

$$\begin{aligned}\left\|Q_d^\theta\left(f(v_1)-f(v_2)\right)\right\|_q\le\ &C(1+t)^{-(\sigma_1+1)/(\sigma_1\sigma_2-1)-1+n/2q+\theta d/2}\\ &\times\Big(\|v_1\|_{X_{2d}}^{\sigma_1-1}+\|v_2\|_{X_{2d}}^{\sigma_1-1}\Big)\|v_1-v_2\|_{X_{2d}}\end{aligned}$$

for $\theta=0,1$, $q=p,\infty$. The above estimate and (17) give the first estimate of (18). Repeating the argument we obtain the second estimate of (18). □

Lemma 3.3. *Let $j\in\{1,2\}$ be fixed. Assume that $w\in Y_{jd}$, then*

$$\int_0^t S_1(t-\tau)w(\tau)\,d\tau\in X_{jd}$$

and it satisfies

$$\left\|\int_0^t S_1(t-\tau)w(\tau)\,d\tau\right\|_{X_{jd}}\le C\|w\|_{Y_{jd}}.$$

Proof. Let $\theta\in\{0,1\}$ and $q\in\{p,\infty\}$ be arbitrarily fixed. Since $Q_d^\theta\check{\chi}_l\in L^1\cap L^\infty$, Lemma 2.3 and (17) show

$$\left\|Q_dw^{(l)}(\tau)\right\|_q\le C(1+\tau)^{n/2p-(\sigma_j+1)/(\sigma_1\sigma_2-1)+d/2-1}\|w\|_{Y_{jd}}.$$

The inequality

$$-\frac{n}{2p}+\frac{\sigma_j+1}{\sigma_1\sigma_2-1}-\frac{d}{2}+1<1,$$

Lemma 1.1, Theorem 2.1 and (17) show that

$$\begin{aligned}&\int_0^{t/2}\left\|Q_d^\theta S_1(t-\tau)w^{(l)}\right\|_q d\tau\\ \le&\int_0^{t/2}(1+t-\tau)^{-(n/2)(1/p-1/q)-(1-\theta)d/2}\left\|Q_d w^{(l)}(\tau)\right\|_p d\tau\\ \le& C(1+t)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)+\theta d/2}\|w\|_{Y_{jd}}.\end{aligned}\tag{19}$$

$$\begin{aligned}\int_{t/2}^t\left\|Q_d^\theta S_1(t-\tau)w^{(l)}\right\|_q d\tau &\le \int_{t/2}^t\left\|Q_d^\theta w(\tau)\right\|_q d\tau\\ &\le C\int_{t/2}^t(1+\tau)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)+\theta d/2-1}d\tau\,\|w\|_{Y_{jd}}\\ &\le C(1+t)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)+\theta d/2-1}\int_{t/2}^t d\tau\,\|w\|_{Y_{jd}}\\ &\le C(1+t)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)+\theta d/2}\|w\|_{Y_{jd}}.\end{aligned}\tag{20}$$

Hence, (19)–(20) shows that

$$\left\|Q_d^\theta\left(\int_0^t S_1(t-\tau)w(\tau)\,d\tau\right)^{(l)}\right\|_q \le C(1+t)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)+\theta d/2}\|w\|_{Y_{jd}}.\tag{21}$$

Lemma 1.1, Theorem 2.1 and (17) also show that

$$\begin{aligned}\left\|Q_d^\theta\left(\int_0^t S_1(t-\tau)w(\tau)\,d\tau\right)^{(h)}\right\|_q &\le \int_0^t\left\|Q_d^\theta S_1(t-\tau)w^{(h)}(\tau)\right\|_q d\tau\\ &\le C(1+t)^{n/2q-(\sigma_j+1)/(\sigma_1\sigma_2-1)+\theta d/2-1}\|w\|_{Y_{jd}}.\end{aligned}\tag{22}$$

(21)–(22) give the desired estimates. □

Now we prove our main theorem.

Proof. Let $X=X_{1,d}\times X_{2,d}$ be a Banach space equipped norm

$$\|(u,v)\|_X=\begin{cases}\|u\|_{X_{1d}}+\|v\|_{X_{2d}} & \min(\sigma_1,\sigma_2)>1,\\ \|u\|_{X_{1d}}+M\|v\|_{X_{2d}} & \min(\sigma_1,\sigma_2)=\sigma_1=1,\end{cases}$$

where M is a large constant determined latter. For positive constant η, we set

$$X(\eta)=\{(u,v)\in X|\,\|(u,v)\|_X\le\eta\}$$

For $(u,v)\in X$ let (U,V) be the solution of the following integral equations

$$\begin{aligned}U(t)&=u_0(t)+\int_0^t S_1(t-\tau)f(v(\tau))\,d\tau,\\ V(t)&=v_0(t)+\int_0^t S_1(t-\tau)g(u(\tau))\,d\tau.\end{aligned}\tag{23}$$

Define the mapping Φ by $\Phi(u,v) = (U,V)$ $(u,v) \in X$. Lemmas 3.1–3.3 show that Φ is a transformation in X. First consider the case where $\min(\sigma_1,\sigma_2) > 1$. If ϵ is sufficiently small, then, Lemma 3.1 shows that

$$\|(u_0,v_0)\|_X \le \frac{\eta}{2}. \tag{24}$$

Let $(u_j,v_j) \in X(\eta)$ $(j=1,2)$, and we set $(U_j,V_j) = \Phi(u_j,v_j)$ $(j=1,2)$. Then, Lemmas 3.2 and 3.3 show that

$$\begin{aligned}\|(U_1-U_2,V_1-V_2)\|_X &\le C\max(\eta^{\sigma_1-1},\eta^{\sigma_2-1})\,\|(u_1-u_2,v_1-v_2)\|_X\\ &\le \frac{1}{2}\,\|(u_1-u_2,v_1-v_2)\|_X\,,\end{aligned} \tag{25}$$

provided that

$$C\max(\eta^{\sigma_1-1},\eta^{\sigma_2-1})\eta \le 1/2. \tag{26}$$

(25) with $(u_1,v_1) = (u,v) \in X_\eta$, $(u_2,v_2) = (0,0)$ and $(U,V) = \Phi(u,v)$ shows that

$$\begin{aligned}\|(U,V)\|_X &\le \|(u_0,v_0)\|_X + \|(U-u_0,V-v_0)\|_X\\ &\le \|(u_0,v_0)\|_X + \frac{1}{2}\,\|(u,v)\|_X \le \eta.\end{aligned} \tag{27}$$

(25) and (27) show that Φ is a contraction map on $X(\eta)$, provided that η is small so that (26) is valid. Therefore Φ admits a unique fixed point $(u,v) \in X_\eta$, which is the desired solution of (1).

Now we consider the case where $\min(\sigma_1,\sigma_2) = \sigma_1 = 1$, then $\sigma_2 > 1$. Let $(u_j,v_j) \in X(\eta)$ $(j=1,2)$, and we set $(U_j,V_j) = \Phi(u_j,v_j)$ $(j=1,2)$. Then, Lemmas 3.1 and 3.3 again show that

$$\begin{aligned}\|(U_1-U_2,V_1-V_2\|_X &= \|U_1-U_2\|_{X_{1d}} + M\,\|V_1-V_2\|_{X_{2d}}\\ &\le C\left(\|v_1-v_2\|_{X_{2d}} + \eta^{\sigma_2-1}M\,\|u_1-u_2\|_{X_{2d}}\right)\\ &= CM^{-1}\left(\eta^{\sigma_2-1}M^2\,\|u_1-u_2\|_{X_{1d}} + M\,\|v_1-v_2\|_{X_{2d}}\right)\\ &= \frac{1}{2}\left(\|u_1-u_2\|_{X_{1d}} + M\,\|v_1-v_2\|_{X_{2d}}\right)\\ &\le \frac{1}{2}\,\|(u_1-u_2,v_1-v_2)\|_X\,,\end{aligned} \tag{28}$$

where we have chosen M and η so that

$$\frac{C}{M} = \frac{1}{2}, \quad \eta^{\sigma_2-1}M^2 = 1. \tag{29}$$

(28) with $(u_1,v_1) = (u,v) \in X_\eta$, $(u_2,v_2) = (0,0)$ and $(U,V) = \Phi(u,v)$ shows that

$$\begin{aligned}\|(U,V)\|_X &= \|U-u_0\|_{X_{1d}} + M\,\|V-v_0\|_{X_{2d}}\\ &\le \|u_0\|_{X_{1d}} + M\,\|v_0\|_{X_{2d}} + C\left(\|v_1\|_{X_{2d}} + \eta^{\sigma_2-1}M\,\|u_1\|_{X_{2d}}\right)\\ &\le \frac{\eta}{2} + CM^{-1}\left(\eta^{\sigma_2-1}M^2\,\|u_1\|_{X_{1d}} + M\,\|v_1\|_{X_{2d}}\right)\\ &= \frac{\eta}{2} + \frac{1}{2}\left(\|u_1\|_{X_{1d}} + M\,\|v_1\|_{X_{2d}}\right)\\ &\le \eta,\end{aligned} \tag{30}$$

provided that ϵ is sufficiently small so that

$$\|u_0\|_{X_{1d}} + M\,\|v_0\|_{X_{2d}} \le \frac{\eta}{2}.$$

$C(1+M)\epsilon = \eta/2$ and (29) is valid. (28) and (30) show that Φ is a contraction map on $X(\eta)$. Hence Φ admits a unique fixed point $(u,v) \in X(\eta)$ which is a desired solution of (1). □

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] M. Escobedo and M. Herrero, *A Boundedness and blow up for a semilinear reaction-diffusion system*, J. Differential Equations, **89**(1991), 176–202.

[2] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Tokyo Sect. IA Math., **13**(1966), 109–124.

[3] N. Hayashi, E. I. Kaikina and P. I. Naumkin, *Damped wave equation with super critical nonlinearities*, Differential Integral Equations, **17**(2004), 637–652.

[4] T. Hosono and T. Ogawa, *Large time behavior and* L^p*-*L^q *estimate of solutions of 2-dimensional nonlinear damped wave equations*, J. Differential Equations **203**(2004), 82–118.

[5] R. Ikehata, M. Ohta, *Critical exponents for semilinear dissipative wave equations in* $\boldsymbol{R}^N$, J. Math. Anal. Appl., **269**(2002), 87–97.

[6] P. Marcati and K. Nishihara, *The* L^p*-*L^q *estimates of solutions to one-dimensional damped wave equations and their application to the compressible flow through porous media*, J. Differential Equations **191**(2003), 445–469.

[7] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **12**(1976), 169–189.

[8] T. Narazaki, L^p*-*L^q *estimates for damped wave equations and their applications to semi-linear problem*, J. Math. Soc. Japan, **56**(2004), 585–626.

[9] T. Narazaki, L^q *estimates for damped wave equations with odd initial data*, Electron. J. Differential Equations 2005. No. 74, 17 pp. (electronic).

[10] T. Narazaki and K. Nishihara, *Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption*, J. Math. Anal. Appl. **338**(2008), 803–819.

[11] K. Nishihara, L^p*-*L^q *estimates of solutions to the damped wave equation in 3-dimensional space and their application*, Math. Z. **244**(2003), 631–649.

[12] F. Sun and M. Wang, *Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping*, Nonlinear Analysis, **66**(2007), 2889-2910.

[13] Y. Todorova and B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Differential Equations **174**(2001), 464–489.

[14] Qi S. Zhang, *A blow-up result for a nonlinear wave equation with damping: The critical case*, C. R. Acad. Sci. Paris, Sér. I, Math. **333**(2001), 109–114.

Received August 2008; revised August 2009.

E-mail address: narazaki@tokai-u.jp