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# A SOBOLEV SPACE APPROACH FOR GLOBAL SOLUTIONS TO CERTAIN SEMI-LINEAR HEAT EQUATIONS IN BOUNDED DOMAINS

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Abstract. We present a Sobolev space approach for semilinear heat equations $u_t = \Delta u + F\big(u(t,x)\big)$ for $t > 0$ on a bounded domain $\Omega \subset \mathbf{R}^n$. By proving that there exists a solution in the anisotropic Sobolev space $W_p^{1,2}\big(\mathbf{R}_+ \times \Omega\big)$, we can deduce more than just global existence in time. For example, both the solution and its time derivative are of class $L^p$, and the solution tends to zero in $L^\infty(\Omega)$ as $t \to \infty$. The main result shows that the existence of a solution in $W_p^{1,2}$ depends primarily on the existence of an appropriate *a priori* estimate on the $L^\infty$ norm of solutions as the initial data is deformed to zero.

1. **Introduction.** In this paper we consider the existence of strong solutions to semilinear parabolic partial differential equations of the following form:

$$\begin{aligned} u_t &= \Delta u + F\big(u(t,x)\big) \text{ in } \mathbf{R}_+ \times \Omega, \\ u(0,\,\cdot) &= g \text{ in } \Omega, \\ u(t,\,x) &= 0 \text{ on } \mathbf{R}_+ \times \partial\Omega. \end{aligned} \tag{1}$$

In the above, $\Omega$ is a bounded domain in $\mathbf{R}^n$ with $C^2$ boundary. The nonlinearity $F\colon \mathbf{R} \to \mathbf{R}$ is assumed to be $C^1$, and we also assume that $F(0) = F'(0) = 0$. The study of 1 when $F(\xi) = |\xi|^{\gamma-1}\,\xi$ goes back at least to Fujita [3], in which $\Omega = \mathbf{R}^n$. For small initial data, the effect of $F(u)$ is smallest when the exponent $\gamma$ is large. Indeed, Fujita showed that the exponent $\gamma$ has a critical value $\gamma^\star = 2/n$ such that for positive solutions:

- If $0 < \gamma < 2/n$, then all positive solutions blow up in finite time.
- If $\gamma > 2/n$, then there exist global positive solutions for small initial data.

Variants of this problem that focus on this critical exponent have received and continue to receive much attention, as can be seen by consulting the survey [6] and its sequel [2]. In the present paper, we are considering the case that $\Omega$ is bounded. This case is discussed briefly in [6, Section 1.3] (with reference to [8]), in which the question of critical exponent is seen to be resolved by virtue of the fact that the critical exponent becomes $\gamma^\star = 0$. That is, there are always global, small data solutions, as well as other solutions that blow up in finite time. The existence of global, small data solutions follows trivially from the stability of the zero solution;

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see for example Quittner and Souplet [12, page 75]. For the existence of blow-up solutions, see Kaplan [5].

Since it is known that there are global, small data solutions, it is reasonable to study the nature of the sets $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{D}$ of initial data such that the corresponding solutions are (respectively) global, bounded, or tend to zero as $t \to \infty$. The set $\mathcal{D}$ is the domain of attraction of the origin. For example, when $\Omega$ is bounded and $F$ is convex, it is well-known that these sets are convex and that $\mathcal{D}$ is an open neighborhood (in the $L^\infty$ topology) of zero. If $F$ is strictly convex, then any point of $\mathcal{B}$ that is not extremal lies in the interior of $\mathcal{B}$, and this interior coincides with $\mathcal{D}$. For these and other properties, see Lions [7], Ni, Sacks, and Tavantizis [10], and Quittner and Souplet [12]. In terms of this area of inquiry, we will show that for membership in $\mathcal{D}$, it is sufficient that there be appropriate *a priori* estimates on solutions as the initial data is smoothly deformed to zero. To show this, we will employ a degree theoretic argument in a Sobolev space setting.

In [9] the author used the degree theory [11] for Fredholm operators to find conditions such that 1 has a solution in $W_p^{1,2}(\mathbf{R}_+ \times \Omega)$. In [9] the emphasis is on nonautonomous problems $u_t = A(t)u + F(t,x,u) + f(t,x)$. Because of technicalities encountered in that (relatively general) setting, specific results were obtained only for problems with a dissipative nonlinear term. The special choices $A(t) = \Delta$, $F(t,x,u) = F(u)$ and $f = 0$ correspond to problem 1. As a result, the application of the Sobolev space approach to problem 1 is free of several of the technicalities that arise in [9]. On the other hand, the fact that $F$ is anti-dissipative introduces its own difficulties, particularly in the derivation of *a priori* bounds.

In this paper we apply the main results from [9] to problem 1. In Section 2 we present the functional and topological preliminaries necessary to state and prove the existence Theorem 3.1. For details and further backround, we refer the reader to [9] and the references therein. Section 3 consists of the statement and proof of Theorem 3.1, which could be viewed as a corollary or special case of Theorem 5.3 in [9]. For completeness, we include a proof that takes advantage of the special form of 1. Finally, in Section 4 we show that the hypotheses in Theorem 3.1 can effectively be reduced to the derivation of appropriate *a priori* bounds. We conclude with Theorem 4.9 (complemented by the simple examples in Section 5), which shows that everything reduces to finding small enough *a priori* bounds in $L^\infty$.

## 2. Functional and topological preliminaries.

Let $\mathbf{R}_+$ denote the positive half line. Let $p > n+1$, and denote by $\mathcal{X}$ the Banach space $L^p(\mathbf{R}_+ \times \Omega)$ with the usual norm $\|\cdot\|_p$. In the event that we need to consider a domain such as $\mathbf{R} \times \Omega$, we shall do so like this: $\mathcal{X}(\mathbf{R} \times \Omega)$. We seek solutions to problem 1 in the anisotropic Sobolev space $W_p^{1,2} = W_p^{1,2}(\mathbf{R}_+ \times \Omega) \subset \mathcal{X}$. This Banach space has the norm

$$\|u\|_{1,\,2;\,p} := \|u\|_p + \|u_t\|_p + \sum_{|\alpha| \le 2} \|D_x^\alpha u\|_p\,,$$

and as one result of the inequality $p > n+1$, the functions in $W_p^{1,2}$ are uniformly continuous (even Hölder continuous) up to the boundary of $\mathbf{R}_+ \times \Omega$. In particular, these functions are defined pointwise, and we take $\mathcal{W}$ to be the subset of all $u \in W_p^{1,2}$ such that $u(t,x) = 0$ when $x \in \partial\Omega$. We also will have need of the space $\mathcal{W}(0)$ of traces $u(0)$ of functions $u \in \mathcal{W}$ at $t = 0$. Because of the uniform continuity of the evaluation map, this space is a well defined Banach space when normed by the quotient of $\mathcal{W}$ by the kernel of the evaluation map. This trace space can be characterized by interpolation between $L^p(\Omega)$ and $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and can also be characterized by known Sobolev-Slobodeckii spaces; see for example Amann [1].

We take $F\colon \mathbf{R} \to \mathbf{R}$ to be a $C^1$ function such that $F(0) = F'(0) = 0$. We denote by $\mathbf{F}$ the Nemytskii operator associated to $F$. That is, $\mathbf{F}$ is the map that carries $u$ to the composition $F \circ u$. It follows that $\mathbf{F}$ is a $C^1$ mapping of $\mathcal{W}$ into $\mathcal{X}$, and that $D\mathbf{F}$ is the same as the Nemytskii operator associated to $F'$. Therefore, if we set

$$\Phi(u) := \big(u_t - \Delta u - \mathbf{F}(u),\, u(0)\big),$$

then $\Phi$ is a well-defined $C^1$ mapping from $\mathcal{W}$ into $\mathcal{X} \times \mathcal{W}(0)$. Moreover, the derivative of $\Phi$ is given by

$$D\Phi(u)v = \big(v_t - \Delta v - D\mathbf{F}(u)v,\, v(0)\big),$$

in which the multiplication $D\mathbf{F}(u) \cdot v$ is pointwise in $(t, x)$. Additionally, the operator $\Phi$ is Fredholm of index zero from $\mathcal{W}$ into $\mathcal{X} \times \mathcal{W}(0)$. This is because $D\Phi(u) - D\Phi(0)$ is compact for every $u \in \mathcal{W}$, and $D\Phi(0)v = \big(v_t - \Delta v,\, v(0)\big)$ is well known to be an isomorphism of Banach spaces.

We will use a degree theoretic argument to prove that 1 has a solution in $\mathcal{W}$. We ensure that the operator $\Phi$ has a nonzero degree at $(0, 0) \in \mathcal{X} \times \mathcal{W}(0)$ in a neighborhood of zero in $\mathcal{W}$, and then we follow a homotopy to $(0, g)$. However, the compactness that is needed for the classical Leray-Schauder degree is absent; this is because we are working with an unbounded time interval. Therefore, we use the degree of [11], which requires that we ensure that $\Phi$ is a $C^1$ map of Fredholm index zero, and that $\Phi$ is proper on the subsets of $\mathcal{W}$ that are closed and bounded. The latter condition represents a considerable (though surmountable) hurdle in the nonautonomous setting, but in the present situation the criterion is simple. It is necessary and sufficient that there be no nontrivial $u \in \mathcal{W}(\mathbf{R} \times \Omega)$ (so that $u$ vanishes on $\mathbf{R} \times \partial\Omega$) such that $u_t = \Delta u + F\big(u(t, x)\big)$. This follows from Theorem 4.9 in [9], in which $\omega(\Delta) = \{\Delta\}$ and $\omega(F) = \{F\}$ because these functions are $t$-independent.

For use in the homotopy part of the argument, we need to ensure that there are appropriate *a priori* bounds on the solutions of 1 as the data homotopes from 0 to $g$. In order that we may speak concisely but precisely about bounds with respect to different norms, let us introduce some terminology. Let $X$ be any Banach space such that the inclusion $\mathcal{W} \hookrightarrow X$ is continuous. (Usually $X = L^p$ or $X = L^\infty$ or $X = \mathcal{W}$.) Let $R \geq 0$, and let $B_X(0, R)$ denote the closed ball in $X$ of radius $R$, centered at the origin.

**Definition 2.1.** We say that $\Phi$ admits the *a priori* bound $R \geq 0$ in the norm of $X$ with respect to $g \in \mathcal{W}(0)$ if there is a $C^1$ path $h = h_s$ in $\mathcal{W}(0)$ such that $h_0 = 0$ and $h_1 = g$, and such that the $\Phi$-preimage of the set $\{0\} \times \big\{h_s \,\big|\, 0 \leq s \leq 1\big\}$ is contained in $B_X(0, R)$.

That is, as the data in 1 varies from 0 to $g$ along $h_s$, the set of $\mathcal{W}$-solutions to 1 is bounded in $X$-norm by $R$.

## 3. **Existence of strong solutions.**

**Theorem 3.1.** *Assume that $F$ is continuously differentiable and that $F(0) = F'(0) = 0$ and that $g \in \mathcal{W}(0)$. Assume that the following conditions are met.*

1. *There is no nontrivial $u \in \mathcal{W}$ such that $u_t = \Delta u + \mathbf{F}(u)$ and $u(0) = 0$. (Uniqueness for the homogeneous problem)*
2. *There is no nontrivial $u \in \mathcal{W}(\mathbf{R} \times \Omega)$ such that $u_t = \Delta u + \mathbf{F}(u)$. (Properness on the closed, bounded subsets)*

3. *The map* $\Phi$ *admits the* a priori *bound* $R \geq 0$ *in the norm of* $\mathcal{W}$ *with respect to* $g$.

*Then there exists* $u \in \mathcal{W}$ *such that* $u_t = \Delta u + \mathbf{F}(u)$ *and* $u(0) = g$.

*Proof.* Let $B$ be the open ball of radius $R + 1$ centered at the origin of $\mathcal{W}$. As discussed in Section 2, the operator $\Phi$ is a $C^1$ map of Fredholm index zero that is proper on the closed bounded subsets of $\mathcal{W}$. In particular, the restriction $\Phi\big|_{\bar{B}}$ is proper. All of this ensures that $\Phi$ is $B$-*admissible*, in the sense of Definition 4.1 of [11].

The *a priori* bound condition ensures that $(0,\, h_s) \in \big(\mathcal{X} \times \mathcal{W}(0) \setminus \Phi(\partial B)\big)$ for all $0 \leq s \leq 1$. As introduced following Corollary 5.5 of [11], the absolute degree $|d|\,\big(\Phi,\, B,\, (0,\, h_s)\big)$ is hence a well defined nonnegative integer for all $0 \leq s \leq 1$. Introduce the homotopy $H\colon [0,\, 1] \times \mathcal{W} \to \mathcal{X} \times \mathcal{W}(0)$ by

$$H(s,\, u) := \Phi(u) - (0,\, h_s).$$

We must verify that $H$ is a $B$-*admissible* homotopy, in the sense of Definition 4.2 of [11]. It is clear that $H$ is $C^1$. To see that $H$ is Fredholm of index one, note that $DH(s,\, u)$ is a rank one perturbation of the linear map $L(u) := (0,\, D\Phi(u))$ from $\mathbf{R} \times \mathcal{W}$ into $\mathcal{X} \times \mathcal{W}(0)$. The map $L$ has the same range and target space as $D\Phi(u)$, but $\ker L = \mathbf{R} \times \ker D\Phi(u)$. Thus, $L$ is Fredholm of index one because $D\Phi(u)$ is Fredholm of index zero. The compact perturbation $DH(s,\, u)$ of $L$ therefore enjoys the same Fredholm property and index.

The last requirement for $H$ to be a $B$-admissible homotopy is that $H_{[0,\,1]\times\bar{B}}$ should be proper. This readily follows from the properness of $\Phi\big|_{\bar{B}}$, taking into account the compactness of $[0,\, 1]$ and the continuity of the path $(0,\, h)$. It now follows from Theorem 5.1 of [11] that $|d|\,\big(H(s,\,\cdot),\, B,\, (0,\, 0)\big)$ is independent of $0 \leq s \leq 1$.

Now, assumption 1 means that $\Phi^{-1}((0,\, 0)) = \{0\}$, and so the fact that $D\Phi(0)$ is an isomorphism ensures that $(0,\, 0)$ is a regular point of $\Phi\big|_{B}$. Thus, according to the definition of the degree at regular values of $\Phi$, $|d|\,\big(\Phi,\, B,\, (0,\, 0)\big) = 1$. Therefore, $|d|\,\big(\Phi - (0,\, g),\, B,\, (0,\, 0)\big) = 1 \neq 0$ as well, since $H(0,\,\cdot) = \Phi$ and $H(1,\,\cdot) = \Phi - (0,\, g)$. It follows from the normalization property of the degree that $\Phi(u) - (0,\, g) = 0$ for some $u \in \mathcal{W}$. This completes the proof. ◻

## 4. ***A priori*** **bounds.**

We will now see that the hypotheses in Theorem 3.1 reduce to the problem of finding appropriate bounds on solutions. In fact, it is enough to find bounds in both $L^p$ and $L^\infty$. Moreover, if the bounds in $L^\infty$ are small enough relative to the diameter of $\Omega$, then the bound in $L^\infty$ will suffice. We remark that this last property follows the rule of thumb that small domains are more dissipative than large ones.

Let us first recall some terminology and results from [12].

**Definition 4.1.** Let $X$ denote a Banach space of functions on $\Omega$; we are most interested in the choices $X = L^\infty(\Omega)$ or $X = W_0^{1,p}(\Omega)$. Let $g \in X$ and $T \in (0,\, \infty]$. A function $u \in C\big([0, T),\, X\big)$ is called a *classical* $X$*-solution* of 1 if $u \in C^{1,2}\big((0, T) \times \Omega\big) \cap C\big((0, T) \times \overline{\Omega}\big)$, $u(0) = g$, and $u$ is a classical solution of 1 for $t \in (0, T)$. The exception to this definition is that if $X = L^\infty(\Omega)$ then instead of the condition $u \in C\big([0, T),\, X\big)$, we require $u \in C\big((0, T),\, X\big)$ and $\big\|u(t) - e^{-t\Delta} g\big\|_\infty \to 0$ as $t \to 0$, where $e^{-t\Delta}$ is the Dirichlet heat semigroup in $\Omega$.

**Definition 4.2.** We say that 1 is *well-posed in* $X$ if for each $g \in X$ there exists $T > 0$ and a unique classical $X$-solution in $[0, T)$.

**Remark 1.** It is well known that 1 is well-posed in $X = W_0^{1,p}(\Omega)$ (since $p > n$) and $X = L^\infty(\Omega)$. See [12, page 75].

**Definition 4.3.** The zero solution to 1 is said to be *asymptotically stable* in $X$ if there exists $\eta > 0$ such that for all $g \in X$ with $\|g\|_X \leq \eta$,

$$\lim_{t \to \infty} \|u(t)\|_X = 0.$$

**Definition 4.4.** We say that the zero solution is *exponentially asymptotically stable* in $X$ if there exist constants $\eta > 0$, $\mu > 0$, and $K \geq 1$ such that for all $g \in X$ with $\|g\|_X \leq \eta$,

$$\|u(t)\|_X \leq K \|g\|_X e^{-\mu t}, \qquad t > 0.$$

**Remark 2.** According to [12, Theorem 19.2], the zero solution is exponentially asymptotically stable in $L^\infty(\Omega)$ when $\Omega$ is bounded and $F$ is $C^1$ with $F(0) = F'(0) = 0$.

These concepts and properties have a direct bearing on the first two numbered hypotheses in Theorem 3.1, as shown in the following lemma:

**Lemma 4.5.** *Assume that $F$ is continuously differentiable and that $F(0) = F'(0) = 0$, so that the zero solution is exponentially asymptotically stable. Then*

1. *There is no nontrivial $u \in \mathcal{W}$ such that $u_t = \Delta u + \mathbf{F}(u)$ and $u(0) = 0$.*
2. *There is no nontrivial $u \in \mathcal{W}(\mathbf{R} \times \Omega)$ such that $u_t = \Delta u + \mathbf{F}(u)$.*

*Proof.* The first conclusion follows from the assumed exponential asymptotic stability of the zero solution; see Remark 2. Let $\eta$, $\mu$, and $K$ be constants that satisfy Definition 4.4. We prove the second assertion by contradiction; suppose that $u \in \mathcal{W}(\mathbf{R} \times \Omega)$ is a nontrivial function such that $u_t = \Delta u + \mathbf{F}(u)$. Then there is some $t_1 \in \mathbf{R}$ such that $u(t_1) \in X$ is not the zero function. We have that

$$\lim_{t \to -\infty} \|u(t)\|_X = 0$$

because $u \in W^{1,p}(\mathbf{R} \times \Omega)$ and $p > n + 1$. There is hence some $t_0 < t_1$ such that both $\|u(t_0)\|_X \leq \eta$ and

$$K \|u(t_0)\|_X e^{-\mu(t_1 - t_0)} < \|u(t_1)\|_X . \tag{2}$$

However, the function $v(t) = u(t + t_0)$ defines a classical $X$-solution to 1 with initial value $v(0) = u(t_0)$. Inequality 2 becomes

$$K \|v(0)\|_X e^{-\mu(t_1 - t_0)} < \|v(t_1 - t_0)\|_X ,$$

which contradicts the exponential asymptotic stability of the zero solution. □

We now turn to the question of *a priori* bounds. First, we show that everything comes to having bounds in both $L^p$ and $L^\infty$.

**Lemma 4.6.** *Suppose that $F$ is a $C^1$ function with $F(0) = F'(0) = 0$, and let $g \in \mathcal{W}(0)$. Suppose that the map $\Phi$ admits the* a priori *bound $R \geq 0$ in the norms of both $L^p(\mathbf{R}_+ \times \Omega)$ and $L^\infty(\mathbf{R}_+ \times \Omega)$ with respect to $g$, and suppose moreover that the two bounds are valid with respect to the same $C^1$ path $h = h_s$ that is mentioned in Definition 2.1. Then there exists $R_1 \geq 0$ such that $\Phi$ admits the* a priori *bound $R_1 \geq 0$ in the norm of $\mathcal{W}$ with respect to $g$.*

*Proof.* Suppose that $u \in \mathcal{W}$ with $\Phi(u) = (0, h_s)$ for some $0 \le s \le 1$. We need to estimate $\|u\|_{\mathcal{W}}$ in a way that depends only on the path $h$ and the assumed bound $R$ for $\|u\|_p$ and $\|u\|_\infty$. If $L$ is defined by $Lv := \big(v_t - \Delta v,\, v(0)\big)$, then for $u$ we have

$$Lu = \big(\mathbf{F}(u), h_s\big). \tag{3}$$

The linear operator $L$ is known to be an isomorphism of $\mathcal{W}$ onto $\mathcal{X} \times \mathcal{W}(0)$. Let $C = C(h) = \max \|h_s\|_{\mathcal{W}(0)}$ over $0 \le s \le 1$. We then apply $L^{-1}$ to equation 3 to obtain the estimate

$$\|u\|_{\mathcal{W}} \le \left\|L^{-1}\right\| \left(\|\mathbf{F}(u)\|_{\mathcal{X}} + C\right)$$

It remains to estimate $\|\mathbf{F}(u)\|_{\mathcal{X}}$. Observing that $\|u\|_\infty \le R$, we bring in $M \ge 0$ such that $|F'(\xi)| \le M$ whenever $|\xi| \le R$. It follows that for each $t \ge 0$ and $x \in \Omega$,

$$\begin{aligned} |F(u(t,x))| &\le \int_0^1 \left|F'\big(\tau u(t,x)\big)u(t,x)\right| \mathrm{d}\tau \\ &\le M\, |u(t,x)|\,. \end{aligned}$$

Observing that $\|u\|_p \le R$, we take $p^{\text{th}}$ powers and integrate over $\mathbf{R}_+ \times \Omega$. As a result,

$$\|\mathbf{F}(u)\|_p \le MR$$

Altogether, we have $\|u\|_{\mathcal{W}} \le R_1 := \left\|L^{-1}\right\| (MR + C)$, as desired. ☐

We would now like to show that one can often derive $L^p$-bounds from $L^\infty$-bounds. We begin by recalling the following Poincaré inequality (see [9, Lemma 6.4]).

**Lemma 4.7.** *There is a constant $C = C_{p,\Omega}$ such that*

$$\int_\Omega |u|^p \,\mathrm{d}x \le C_{p,\Omega} \int_\Omega |u|^{p-2} |\nabla u|^2 \,\mathrm{d}x$$

*whenever $u \in W_0^{1,p}(\Omega)$. Moreover, $C_{p,\Omega} \to 0$ as the diameter of $\Omega$ tends to zero.*

**Remark 3.** One can show that the choice $C(p,\Omega) = (p \cdot \rho(\Omega))^2$ works, where $\rho(\Omega)$ is the inradius of $\Omega$. (The inradius is the diameter of the largest ball that fits in $\Omega$.)

Before we prove the next lemma we remark that under the assumption that $F$ is a $C^1$ function with $F(0) = F'(0) = 0$, the function $|F(\xi)/\xi|$ is unbounded when $F$ is superlinear, but is still bounded on bounded subsets, and that

$$\lim_{\xi \to 0} \frac{F(\xi)}{\xi} = 0.$$

**Lemma 4.8.** *Suppose that $F$ is a $C^1$ function with $F(0) = F'(0) = 0$, and let $g \in \mathcal{W}(0)$. Suppose that the map $\Phi$ admits the* a priori *bound $R \ge 0$ in the norm of $L^\infty(\mathbf{R}_+ \times \Omega)$ with respect to $g$. Suppose also that*

$$\max_{|\xi| \le R} \left|\frac{F(\xi)}{\xi}\right| < (p-1) C_{p,\Omega}^{-1}. \tag{4}$$

*Then there exists $R_1 \ge 0$ such that $\Phi$ admits the* a priori *bound $R_1 \ge 0$ in the norm of $L^p(\mathbf{R}_+ \times \Omega)$ with respect to $g$.*

*Proof.* Once again, let $h = h_s$ be the $C^1$ path mentioned in Definition 2.1, and suppose that $u \in \mathcal{W}$ with $\Phi(u) = (0, h_s)$ for some $0 \le s \le 1$. Therefore,

$$u_t - \Delta u = F(u(t)).$$

We multiply by $u\,|u|^{p-2}$ and integrate over $\Omega$ to find that for all $t \geq 0$,

$$\int_\Omega \left( u_t u\,|u|^{p-2} - \Delta u u\,|u|^{p-2} \right) \mathrm{d}x = \int_\Omega F(u) u\,|u|^{p-2}\,\mathrm{d}x. \tag{5}$$

Because $\frac{\partial}{\partial t}\,|u|^p = p u_t u\,|u|^{p-2}$, we have

$$\int_\Omega u_t u\,|u|^{p-2}\,\mathrm{d}x = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|^p_{L^p(\Omega)}\,.$$

For the second term in 5, we integrate by parts and apply the Poincaré inequality:

$$\begin{aligned} -\int_\Omega \Delta u u\,|u|^{p-2}\,\mathrm{d}x &= (p-1)\int_\Omega |\nabla u|^2\,|u|^{p-2}\,\mathrm{d}x \\ &\geq (p-1)C^{-1}_{p,\,\Omega}\int_\Omega |u|^p\,\mathrm{d}x. \end{aligned}$$

Altogether from 5, we derive the estimate

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|^p_{L^p(\Omega)} + (p-1)C^{-1}_{p,\,\Omega}\,\|u(t)\|^p_{L^p(\Omega)} \leq \int_\Omega F(u)u\,|u|^{p-2}\,\mathrm{d}x.$$

To bound the right side, we use the assumed inequality 4. As a result,

$$\begin{aligned} \int_\Omega F(u)u\,|u|^{p-2}\,\mathrm{d}x &\leq \max_{|\xi|\leq R}\left|\frac{F(\xi)}{\xi}\right|\int_\Omega |u|^p\,\mathrm{d}x \\ &\leq \left((p-1)C^{-1}_{p,\,\Omega} - \epsilon\right)\|u(t)\|^p_{L^p(\Omega)} \end{aligned}$$

where $\epsilon > 0$ is the difference between the quantities related by inequality 4. We now have the estimate

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|^p_{L^p(\Omega)} + \epsilon\,\|u(t)\|^p_{L^p(\Omega)} \leq 0.$$

This inequality is then integrated over $t \geq 0$. This results in

$$p\epsilon\,\|u\|^p_{L^p(\mathbf{R}_+\times\Omega)} \leq \|u(0)\|^p_{L^p(\Omega)}\,.$$

Recall that $u(0)$ lies on a fixed $C^1$ path $h$ in $\mathcal{W}(0)$. Since $\mathcal{W}(0) \hookrightarrow L^p(\Omega)$, we may set

$$M := \max_{0\leq s\leq 1}\|h_s\|_{L^p(\Omega)}\,.$$

We conclude with the choice $R_1 = (p\epsilon)^{-1/p}M$. ☐

Here is one possible theorem that follows from Theorem 3.1, along with the above *a priori* bound results.

**Theorem 4.9.** *Assume that $F$ is continuously differentiable and that $F(0) = F'(0) = 0$. Assume that the map $\Phi$ admits the* a priori *bound $R \geq 0$ in the norm of $L^\infty(\mathbf{R}_+ \times \Omega)$ with respect to $g$. Suppose also that*

$$\max_{|\xi|\leq R}\left|\frac{F(\xi)}{\xi}\right| < (p-1)C^{-1}_{p,\Omega}. \tag{6}$$

*Then there exists $u \in \mathcal{W}$ such that $u_t = \Delta u + \mathbf{F}(u)$ and $u(0) = g$.*

*Proof.* According to [12, Theorem 19.2], the zero solution is exponentially asymptotically stable in $L^\infty(\Omega)$. Therefore, Lemma 4.5 implies that all but possibly the final hypothesis of Theorem 3.1 are met. The final assumption in Theorem 3.1 is met because of Lemmas 4.8 and 4.6. ☐

5. **Examples.** To demonstrate the applicability of Theorem 4.9, we recall a lemma of Holland [4, Lemma 1]. Let $\lambda > 0$ be the principle eigenvalue of the Laplacian operator in $\Omega$ with zero Dirichlet boundary conditions, and let $\phi$ be the associated $L^2$-normalized non-negative eigenfunction. We assume that $F$ is of class $C^2$ such that $F(0) = F'(0) = 0$, and that the initial data $g$ is of class $C^1$ and that $g$ vanishes on $\partial\Omega$. For each $r > 0$, define $U^*(r)$ as the infimum (possibly $\infty$) of all values $|\xi|$ such that $F(\xi) = r\xi$. (In particular, if $F(\xi) = |\xi|^{\gamma-1}\xi$, then we assume that $\gamma \geq 2$ and set $U^*(r) = r^{1/(\gamma-1)}$.) Under these assumptions, we have the following:

**Lemma 5.1** ([4], Lemma 1)**.** *Assume that there exist constants $C$, $r$ with $0 < r < \lambda$ such that*

$$g(x) < C\phi(x) < U^*(r).$$

*Then the solution to 1 exists for all $t > 0$ and satisfies*

$$|u(t,x)| < C\phi(x)e^{(r-\lambda)}t.$$

This lemma provides an estimate that is explicit enough to use with Theorem 4.9. Indeed, if $g$ is chosen in a way that satisfies the hypotheses of this lemma, then every point on the linear path that joins $g$ to the origin also satisfies the hypotheses of the Lemma.

In contrast, consider the possibility of using the following kind of estimate. We specialize to the problem

$$\begin{aligned} u_t &= \Delta u + |u|^{\gamma-1}u \text{ in } \mathbf{R}_+ \times \Omega, \\ u(0,\,\cdot) &= g \text{ in } \Omega, \\ u(t,\,x) &= 0 \text{ on } \mathbf{R}_+ \times \partial\Omega. \end{aligned} \tag{7}$$

Let $\gamma_S$ denote the usual critical Sobolev exponent. That is, if $n \leq 2$ then $\gamma_S = \infty$, and if $n > 2$ then $\gamma_S = (n+2)/(n-2)$. Assume that $1 < \gamma < \gamma_S$. As it is shown in [12, Section 22], there is the following *a priori* estimate on the global solutions of 7:

$$\sup_{t\geq 0} \|u(t)\|_\infty \leq R = R(\|g\|_\infty),$$

where $R$ is bounded on bounded sets. Moreover, the restriction $\gamma < \gamma_S$ is also necessary for such an *a priori* estimate, at least if $\Omega$ is star-shaped. In order to apply Theorem 4.9, notice that it suffices to ensure that

$$R^{\gamma-1} < \frac{p-1}{p^2\rho^2}. \tag{8}$$

Once again, $\rho$ stands for the inradius of $\Omega$; see Remark 3. We are working under the restriction $p > n+1$, so the right side of 8 is largest as $p$ approaches $n+1$.

It is our hope that the technique introduced in this paper can lead to results concerning variants of 1 in which the domain $\mathcal{D}$ of attraction of the origin is less well understood. For such problems, it would seem to be of interest if one were able to use a degree argument to relate membership in $\mathcal{D}$ to the existence of explicit *a priori* estimates on solutions.

Finally, we note that the contrapositive forms of Theorem 4.9 and Lemma 4.6 may be of interest in studying the behavior of solutions as the initial data leaves the basin of attraction of the origin. According to Theorem 4.9, if the solution to 1 with initial data $g$ is not in $\mathcal{W}$, then any $C^1$ path that joins $g$ to the origin will be such that the corresponding solutions must exit a particular ball in $L^\infty(\mathbf{R}_+ \times \Omega)$.

Moreover, even if the $L^\infty$ norm of solutions remains bounded, Lemma 4.6 implies that the set of $L^p\big(\mathbf{R}_+ \times \Omega\big)$ norms of solutions is unbounded.

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