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ON THE SOLVABILITY OF SOME FOURTH-ORDER EQUATIONS WITH FUNCTIONAL BOUNDARY CONDITIONS

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Abstract. In this paper it is considered a fourth order problem composed of a fully nonlinear differential equation and functional boundary conditions satisfying some monotone conditions.This functional dependence on u, u' and $u''$and generalizes several types of boundary conditions such as Sturm-Liouville, multipoint, maximum and/or minimum arguments, or nonlocal. The main theorem is an existence and location result as it provides not only the existence, but also some qualitative information about the solution.

1. Introduction.

In this work we consider the problem composed of the fully nonlinear fourth order equation

$$u^{(iv)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)) \tag{1}$$

with $x \in I := [0,1]$, where $f : I \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function, and the functional boundary conditions

$$\begin{gathered} L_0(u, u', u'', u(0)) = 0, \\ L_1(u, u', u'', u'(0)) = 0, \\ L_2(u, u', u'', u''(0), u'''(0)) = 0, \\ L_3(u, u', u'', u''(1), u'''(1)) = 0, \end{gathered} \tag{2}$$

where $L_0, L_1 : C(I)^3 \times \mathbb{R} \to \mathbb{R}$ and $L_2, L_3 : C(I)^3 \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying some monotonicity assumptions to be defined in the sequel.

These type of fourth order problems have been studied by several authors with different boundary conditions and several methods, see [4, 5, 7, 10, 11] and the references therein. The functional dependence covers several types of boundary conditions, such as separated, multi-point, nonlocal,...Therefore, the current result improves, somehow, the papers referred to above.

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The method used was suggested by [3], applied to third order problems and by [1] to fourth order, now improved with the functional dependence on every boundary conditions, L_0, L_1, L_2 and L_3. In this sense, this paper improves also [2].

In short, the keypoints of the arguments are: *a priori* estimates on the third derivative provided by a Nagumo-type condition ([10, 12]); an auxiliary and truncated problem, where the corresponding linear and homogeneous problem has only the trivial solution; an open and bounded set where the Leray-Schauder degree is well defined ([9]).

Lower and upper solutions technique allows us to obtain not only the existence of solutions but also to locate the solution and its first and second derivatives. In fact, this location part can be useful to get some information about the existent solution. Two examples: if lower and upper solutions are ordered and the lower function is nonnegative or strictly positive, the solution is nonnegative or strictly positive, respectively; if the second derivatives of lower and upper solutions have the same sign, the solution is not trivial and, moreover, it can not be a straight line (see Example 2 at last section).

2. **Definitions and auxiliary results.** In this section we define a Nagumo-type growth condition on the nonlinear part of the differential equation that will be an important tool to prove an *a priori* bound for the third derivative of the corresponding solutions.

In the following, $C^k([0,1])$ denotes the space of real valued functions with continuous i-derivative in $[0,1]$, for $i = 1, ..., k$, equipped with the norm

$$\|y\|_{C^k} = \max_{0\leq i\leq k} \left\{ \left| y^{(i)}(x) \right| : x \in [0,1] \right\}.$$

By $C([0,1])$ we denote the space of continuous functions with the norm

$$\|y\| = \max_{x\in[0,1]} |y(x)|.$$

Definition 2.1. Given a subset $E \subset [0,1] \times \mathbb{R}^4$, a continuos function $f : E \to \mathbb{R}$ is said to satisfy a Nagumo-type condition in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [k, +\infty[$, for some $k > 0$, such that

$$|f(x, y_0, y_1, y_2, y_3)| \leq h_E(|y_3|) \;\; \forall (x, y_0, y_1, y_2, y_3) \in E, \tag{3}$$

with

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt = +\infty. \tag{4}$$

Lemma 2.2. *[[10], Lemma 1] Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying Nagumo-type conditions (3) and (4) in*

$$E = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \;\; i = 0, 1, 2 \right\},$$

where $\gamma_i(x)$ and $\Gamma_i(x)$ are continuous functions such that, for $i = 0, 1, 2$,

$$\gamma_i(x) \leq \Gamma_i(x), \;\; \forall x \in [0,1].$$

Then for every $\rho > 0$ there is $R > 0$ such that every solution $u(x)$ of equation (1) satisfying

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \;\; \forall x \in [0,1], \tag{5}$$

for $i = 0, 1, 2$, satisfies $\|u'''\| < R$.

Remark 1. Observe that R depends only on the functions h_E, γ_2 and Γ_2 and not on the boundary conditions.

The following monotonicity assumptions on the boundary conditions will be considered:
(H_1) $L_0, L_1 : C([0,1])^3 \times \mathbb{R} \to \mathbb{R}$ are nondecreasing in all variables except the fourth one.
(H_2) $L_2 : C([0,1])^3 \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing in all variables, except the fourth one.
(H_3) $L_3 : C([0,1])^3 \times \mathbb{R}^2 \to \mathbb{R}$ is nondecreasing in the first, second and third variables and nonincreasing in the fifth one.

Definition 2.3. A function $\alpha \in C^4([0,1])$ is a lower solution of problem (1)-(2) if:

$$\alpha^{(iv)}(x) \geq f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)), \tag{6}$$

and

$$\begin{array}{c} L_0(\alpha, \alpha', \alpha'', \alpha(0)) \geq 0, \\ L_1(\alpha, \alpha', \alpha'', \alpha'(0)) \geq 0, \\ L_2(\alpha, \alpha', \alpha'', \alpha''(0), \alpha'''(0)) \geq 0, \\ L_3(\alpha, \alpha', \alpha'', \alpha''(1), \alpha'''(1)) \geq 0. \end{array} \tag{7}$$

The function $\beta \in C^4([0,1])$ is an upper solution of the problem (1)-(2) if the reversed inequalities hold.

3. **Existence and location result.**

The main theorem can be said to be an existence and location result as it provides the existence of a solution but also some strips where the solution and its first and second derivatives are located.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function. Suppose that there are lower and upper solutions of the problem (1)-(2), $\alpha(x)$ and $\beta(x)$, respectively, such that,*

$$\alpha(0) \leq \beta(0), \quad \alpha'(0) \leq \beta'(0), \quad \alpha''(x) \leq \beta''(x), \quad \forall x \in [0,1], \tag{8}$$

f satisfies Nagumo conditions (3) and (4) in

$$E_* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \alpha(x) \leq y_0 \leq \beta(x), \\ \alpha'(x) \leq y_1 \leq \beta'(x), \ \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\}$$

and

$$f(x, \alpha, \alpha', y_2, y_3) \geq f(x, y_0, y_1, y_2, y_3) \geq f(x, \beta, \beta', y_2, y_3), \tag{9}$$

for $\alpha(x) \leq y_0 \leq \beta(x)$, $\alpha'(x) \leq y_1 \leq \beta'(x)$, in $[0,1]$, and fixed $(x, y_2, y_3) \in [0,1] \times \mathbb{R}^2$.

If conditions $(H_1) - (H_3)$ hold, then problem (1)-(2) has at least one solution $u(x) \in C^4([0,1])$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \quad \forall x \in [0,1], \ \textit{for } i = 0, 1, 2.$$

Proof. Let us consider the continuous functions δ_i given by

$$\delta_i(x, y_i) = \left\{ \begin{array}{lll} \alpha^{(i)}(x) & \text{if} & y_i < \alpha^{(i)}(x), \\ y^{(i)} & \text{if} & \alpha^{(i)}(x) \leq y^{(i)} \leq \beta^{(i)}(x) \\ \beta^{(i)}(x) & \text{if} & y^{(i)} > \beta^{(i)}(x). \end{array} \right. , \quad i = 0, 1, 2, \tag{10}$$

For $\lambda \in [0,1]$, consider the homotopic equation

$$\begin{aligned} u^{(iv)}(x) &= \lambda\left(f\left(x, \delta_0\left(x, u(x)\right), \delta_1\left(x, u'(x)\right), \delta_2\left(x, u''(x)\right), u'''(x)\right)\right) \\ &\quad + u''(x) - \lambda\delta_2\left(x, u''(x)\right), \end{aligned} \tag{11}$$

and the boundary conditions

$$\begin{array}{c} u(0) = \lambda\delta_0\left(0, u(0) + L_0\left(u, u', u'', u(0)\right)\right), \\ u'(0) = \lambda\delta_1\left(0, u'(0) + L_1\left(u, u', u'', u'(0)\right)\right), \\ u''(0) = \lambda\delta_2\left(0, u''(0) + L_2\left(u, u', u'', u''(0), u'''(0)\right)\right), \\ u''(1) = \lambda\delta_2\left(1, u''(1) + L_3\left(u, u', u'', u''(1), u'''(1)\right)\right). \end{array} \tag{12}$$

Let $r_2 > 0$ be large enough, such that, for every $x \in [0,1]$,

$$\begin{array}{c} -r_2 < \alpha''(x) \le \beta''(x) < r_2, \\ f\left(x, \beta(x), \beta'(x), \beta''(x), 0\right) + r_2 - \beta''(x) > 0, \\ f\left(x, \alpha(x), \alpha'(x), \alpha''(x), 0\right) - r_2 - \alpha''(x) < 0. \end{array} \tag{13}$$

<u>*Step 1*</u> *For every solution $u(x)$ of the problem (11)-(12) we have*

$$|u''(x)| < r_2 \quad |u'(x)| < r_1 \quad |u(x)| < r_0, \forall x \in [0,1],$$

with $r_1 := r_2 + \max\{|\alpha'(0)|, |\beta'(0)|\}$ and $r_0 := r_1 + \max\{|\alpha(0)|, |\beta(0)|\}$, independentely of $\lambda \in [0,1]$.

By contradiction assume that first condition does not hold. Then, there is $\lambda \in [0,1]$, $x \in [0,1]$ and a solution $u(x)$ of (11)-(12) such that $|u''(x)| \ge r_2$. In the case $u''(x) \le -r_2$ define

$$\min_{x\in[0,1]} u''(x) := u''(x_0) \le -r_2 < 0.$$

If $x_0 \in]0,1[$ then $u'''(x_0) = 0$ and $u^{(iv)}(x_0) \ge 0$. Therefore by (9) and (13), for $\lambda \in [0,1]$, we obtain the following contradiction

$$\begin{aligned} 0 &\le u^{(iv)}(x_0) = \\ &= \lambda\left(f\left(x_0, \delta_0\left(x_0, u(x_0)\right), \delta_1\left(x_0, u'(x_0)\right), \delta_2\left(x_0, u''(x_0)\right), u''(x_0)\right)\right) \\ &+ u''(x_0) - \lambda\delta_2\left(x_0, u''(x_0)\right) \\ &= \lambda\left(f\left(x_0, \delta_0\left(x_0, u(x_0)\right), \delta_1\left(x_0, u'(x_0)\right), \alpha''(x_0), 0\right)\right) + u''(x_0) - \alpha''(x_0) \\ &\le \lambda\left(f\left(x_0, \alpha(x_0), \alpha'(x_0), \alpha''(x_0), 0\right)\right) + r_2 - \alpha''(x_0) < 0 \end{aligned}$$

If $x_0 = 0$ then

$$\min_{x\in[0,1]} u''(x) := u''(0) \le -r_2 < 0.$$

For $\lambda \in]0,1]$, by (12) and (10), the following contradiction is obtained

$$\begin{aligned} -r_2 &\ge u''(0) = \lambda\delta_2\left(0, u''(0) + L_2(u, u', u'', u''(0), u'''(0))\right) \\ &\ge \lambda\alpha''(0) > -r_2. \end{aligned}$$

The arguments for $x_0 = 1$, are similar and therefore $u''(x) < r_2$, $\forall x \in [0,1]$, $\forall\lambda \in [0,1]$.

The case $u''(x) \ge r_2$ is analogous and so

$$|u''(x)| < r_2, \ \forall x \in [0,1], \forall\lambda \in [0,1].$$

Integrating in $[0,x]$,

$$\begin{aligned} u'(x) &= \int_0^x u''(s)ds + u'(0) \\ &= \int_0^x u''(s)ds + \lambda\delta_1\left(0, u'(0) + L_1\left(u, u', u'', u'(0)\right)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} |u'(x)| &\leq \int_0^x |u''(s)|\, ds + |\lambda\delta_1\left(0, u'(0) + L_1\left(u, u', u'', u'(0)\right)\right)| \\ &< r_2 + \max\left\{|\alpha'(0)|, |\beta'(0)|\right\}. \end{aligned}$$

Similarly,

$$\begin{aligned} u(x) &= \int_0^x u'(s)ds + u(0) \\ &= \int_0^x u'(s)ds + \lambda\delta_0\left(0, u(0) + L_0\left(u, u', u'', u(0)\right)\right). \end{aligned}$$

Therefore,

$$|u(x)| < r_1 + \max\left\{|\alpha(0)|, |\beta(0)|\right\}, \forall x \in [0,1].$$

Step 2- There is $R > 0$ such that, every solution $u(x)$ of problem (11)-(12) satisfies

$$|u'''(x)| < R, \ \forall x \in [0,1],$$

independently of $\lambda \in [0,1]$.

In order to apply Lemma 2.2, define the set

$$E_R = \left\{(x, y_0, y_1, y_2, y_3) \in [a,b] \times \mathbb{R}^4 : -r_i \leq y_i \leq r_i, \ i = 0,1,2\right\},$$

with r_i, $i = 0,1,2$, given by Step 1, and, for $\lambda \in [0,1]$, the function $F_\lambda : E_R \to \mathbb{R}$ is given by

$$F_\lambda(x, y_0, y_1, y_2, y_3) = \lambda f\left(x, \delta_0(x, y_0), \delta_1(x, y_1), \delta_2(x, y_2), y_3\right) + y_2 - \lambda\delta_2(x, y_2).$$

Since f satisfies (3) in E_R,

$$\begin{aligned} &|F_\lambda(x, y_0, y_1, y_2, y_3)| \\ =&|\lambda f\left(x, \delta_0(x, y_0), \delta_1(x, y_1), \delta_2(x, y_2), y_3\right) + y_2 - \lambda\delta_2(x, y_2)| \\ \leq&|\lambda h_{E_R}(|y_3|)| + r_2 + |\lambda\alpha''(x)| \\ \leq& h_{E_R}(|y_3|) + 2r_2. \end{aligned}$$

So F_λ satisfies (3) with h_E replaced by $\bar{h}_{E_R}(x) := h_{E_R}(x) + 2r_2$ in E_R. For the integral condition, we have

$$\begin{aligned} \int_0^{+\infty} \frac{t}{\bar{h}_{E_R}(t)} dt &= \int_0^{+\infty} \frac{t}{h_{E_R}(t) + 2r_2} dt \\ &\geq \frac{1}{1 + \frac{2r_2}{k}} \int_0^{+\infty} \frac{t}{h_{E_R}(t)} dt = +\infty, \end{aligned}$$

and therefore (4) holds.

Applying Lemma 2.2 with $\gamma_i(x) = -r_i$, $\Gamma_i(x) = r_i$, $i = 0, 1, 2$, there exists $R > 0$ such that

$$|u'''(x)| < R, \ \forall x \in [a, b].$$

Observe that as r_2 and h_{E_R} do not depend on λ, so R does not depend on λ.

Step 3- *Problem (11)-(12)has at least a solution $u_1(x)$ for $\lambda = 1$.*

Define the operators

$$\mathcal{L} : C^4([0,1]) \subset C^3([0,1]) \to C([0,1]) \times \mathbb{R}^4.$$

given by

$$\mathcal{L}u = \left(u^{(iv)} - u'', u(0), u'(0), u''(0), u''(1)\right),$$

and $\mathcal{N}_\lambda : C^3([a,b]) \to C([a,b]) \times \mathbb{R}^4$, given by

$$\mathcal{N}_\lambda = \left(\begin{array}{c} \lambda f(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), u'''(x)) - \lambda\delta_2(x, u''(x)), \\ A_{0\lambda}, \ A_{1\lambda}, \ A_{2\lambda}, \ A_{3\lambda} \end{array} \right),$$

where

$$\begin{aligned} A_{0\lambda} : &= \lambda\delta_0(0, u(0) + L_0(u, u', u'', u(0))), \\ A_{1\lambda} : &= \lambda\delta_1(0, u'(0) + L_1(u, u', u'', u'(0))), \\ A_{2\lambda} : &= \lambda\delta_2(0, u''(0) + L_2(u, u', u'', u''(0), u'''(0))), \\ A_{3\lambda} : &= \lambda\delta_2(1, u''(1) + L_3(u, u', u'', u''(1), u'''(1))). \end{aligned}$$

As $\mathcal{L}^{-1}$ is compact it can be used to define completely continuous operator

$$\mathcal{T}_\lambda : \left(C^4([0,1]), \mathbb{R}\right) \to \left(C^4([0,1]), \mathbb{R}\right)$$

given by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

For r_i, $i = 0, 1, 2$ and R given by Steps 1 and 2, we consider the set

$$\Omega = \left\{ y \in C^3([0,1]) : \left\| y^{(i)} \right\| < r_i, \ i = 0, 1, 2, \ \|y'''\| < R \right\}.$$

Therefore, the degree $d(\mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$, and by the invariance under homotopy, $d(\mathcal{T}_0, \Omega, 0) = d(\mathcal{T}_1, \Omega, 0)$.

The equation $T_0(u) = u$ is equivalent to the homogeneous problem

$$\begin{cases} u^{(iv)}(x) - u''(x) = 0, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases}$$

which admits only the trivial solution. Then, by degree theory, $d(\mathcal{T}_0, \Omega, 0) = \pm 1$, and so the equation $u = \mathcal{T}_1(u)$ has at least one solution. That is, the problem consisting of the equation

$$\begin{aligned} u^{(iv)}(x) &= f(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), u'''(x)) \\ &\quad + u''(x) - \delta_2(x, u''(x)) \end{aligned} \tag{14}$$

and the boundary conditions

$$\begin{aligned} u(0) &= \delta_0(0, u(0) + L_0(u, u', u'', u(0))), \\ u'(0) &= \delta_1(0, u'(0) + L_1(u, u', u'', u'(0))), \\ u''(0) &= \delta_2(0, u''(0) + L_2(u, u', u'', u''(0), u'''(0))), \\ u''(1) &= \delta_2(1, u''(1) + L_3(u, u', u'', u''(1), u'''(1))), \end{aligned}$$

has at least one solution $u_1(x)$ in Ω.

Step 4 - The function $u_1(x)$ is a solution of the problem (1)-(2)

This function $u_1(x)$ will be a solution of the original problem (1)-(2) if

$$\alpha^{(i)}(x) \leq u_1^{(i)}(x) \leq \beta^{(i)}(x), \ i = 0,1,2, \ \forall x \in [0,1],$$

and

$$\begin{array}{c} \alpha(0) \leq u_1(0) + L_0(u_1, u_1', u_1'', u_1(0)) \leq \beta(0) \\ \alpha'(0) \leq u_1'(0) + L_1(u_1, u_1', u_1'', u_1'(0)) \leq \beta'(0) \\ \alpha''(0) \leq u_1''(0) + L_2(u_1, u_1', u_1'', u_1''(0), u_1'''(0)) \leq \beta''(0) \\ \alpha''(1) \leq u_1''(1) + L_3(u_1, u_1', u_1'', u_1''(1), u_1'''(1)) \leq \beta''(1) \end{array}$$

hold.

Suppose, by contradiction, that there is $x \in [0,1]$ such that $\alpha''(x) > u_1''(x)$ and define

$$\min_{x\in[0,1]} [u_1''(x) - \alpha''(x)] := u_1''(x_2) - \alpha''(x_2) < 0.$$

If $x_2 \in \left]0,1\right[$, then $u_1'''(x_2) = \alpha'''(x_2)$ and $u^{(iv)}(x_2) - \alpha^{(iv)}(x_2) \geq 0$ and the following contradiction is obtained, by (6)

$$\begin{array}{rcl} 0 & \leq & u_1^{(iv)}(x_2) - \alpha^{(iv)}(x_2) \\ & = & f(x_2, \delta_0(x_2, u_1(x_2)), \delta_1(x_2, u_1'(x_2)), \alpha''(x_2), \alpha'''(x_2)) \\ & & +u_1''(x_2) - \alpha''(x_2) - \alpha^{(iv)}(x_2) \\ & < & f(x_2, \alpha(x_2), \alpha'(x_2), \alpha''(x_2), \alpha'''(x_2)) - \alpha^{(iv)}(x_2) \leq 0 \end{array}$$

If $x_2 = 0$, then

$$\min_{x\in[0,1]} [u_1''(x) - \alpha''(x)] := u_1''(0) - \alpha''(0) < 0$$

and

$$\begin{array}{rcl} u_1''(0) & = & \delta_2(0, u_1''(0) + L_2(u_1, u_1', u_1'', u_1''(0), u_1'''(0))) \\ & \geq & \alpha''(0) > u_1''(0) \end{array}$$

The case $x_2 = 1$ follows similar arguments and, therefore $\alpha''(x) \leq u_1''(x)$, for every $x \in [0,1]$. Analogously it can be proved that $u_1''(x) \leq \beta''(x)$, and so

$$\alpha''(x) \leq u_1''(x) \leq \beta''(x), \ \forall x \in [0,1].$$

The inequalities

$$\alpha'(x) \leq u_1'(x) \leq \beta'(x), \ \alpha(x) \leq u_1(x) \leq \beta(x), \ \forall x \in [0,1],$$

are easily obtained by integration.

As the boundary conditions, assume that

$$u_1(0) + L_0(u_1, u_1', u_1'', u_1(0)) < \alpha(0). \tag{15}$$

By (10),

$$u_1(0) = \delta_0(0, u_1(0) + L_0(u_1, u_1', u_1'', u_1(0))) = \alpha(0)$$

and, by (8), $u_1'(0) \geq \alpha'(0)$ and $u_1''(0) \geq \alpha''(0)$. Therefore, by (H_1) and (7) this contradiction with (15) is achieved:

$$\begin{array}{rcl} u_1(0) + L_0(u_1, u_1', u_1'', u_1(0)) & = & \alpha(0) + L_0(u_1, u_1', u_1'', \alpha(0)) \\ & \geq & \alpha(0) + L_0(\alpha, \alpha', \alpha'', \alpha(0)) \geq \alpha(0). \end{array}$$

Analogously it is shown that $u_1(0) + L_0(u_1, u_1', u_1'', u_1(0), u_1'(0), u_1''(0)) \leq \beta(0)$.

Remaining inequalities can be proved by a similar technique. □

4. **Examples.** The first example deals with the fourth order equation

$$u^{(iv)}(x) = -u(x) - u'(x) + (u''(x))^3 + |u'''(x) + 1|^{\theta}, \tag{16}$$

where $\theta \in [0,2]$, and the functional boundary conditions

$$\begin{aligned} &\textstyle\int_0^1 u(s)ds + \max_{x\in[0,1]} u'(x) + u''(x_0) - ku(0) = 0 \\ &u(x_1) - \eta u'(0) = 0 \\ &\textstyle\int_0^1 u(s)ds - u''(0) = 0 \\ &u'''(1) + u''(1) = 0 \end{aligned} \tag{17}$$

with $k \geq 41/6$, $\eta \geq 2$ and $x_0, x_1 \in [0,1]$.

Functions $\alpha, \beta \in [0,1] \to \mathbb{R}$ given by

$$\alpha(x) = -x^2 - x - 1 \text{ and } \beta(x) = x^2 + x + 1$$

are, respectively, lower and upper solutions for (16)-(17), with

$$\begin{aligned} f(x, y_0, y_1, y_2, y_3) &= -y_0 - y_1 + y_2^3 + |y_3 + 1|^{\theta}, \\ L_0(z_1, z_2, z_3, z_4) &= \int_0^1 z_1 ds + \max_{x\in[0,1]} z_2 + z_3(x_0) - kz_4, \\ L_1(z_1, z_2, z_3, z_4) &= z_1(x_1) - \eta z_4, \\ L_2(z_1, z_2, z_3, z_4, z_5) &= \int_0^1 z_1 ds - z_4, \\ L_3(z_1, z_2, z_3, z_4, z_5) &= -z_4 - z_5. \end{aligned}$$

As the continuous function f verifies (3) and (4) for

$$\varphi_{E_*}(y_3) = 14 + |y_3 + 1|^{\theta},$$

with $\theta \in [0,2]$, in

$$E_* = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \begin{array}{c} -x^2 - x - 1 \leq y_0 \leq x^2 + x + 1 \\ -2x - 1 \leq y_1 \leq 2x + 1 \\ -2 \leq y_2 \leq 2 \end{array} \right\}$$

then, by Theorem 3.1, there is a solution $u(x)$ for problem (16)- (17) such that

$$\begin{aligned} -x^2 - x - 1 &\leq u(x) \leq x^2 + x + 1, \\ -2x - 1 &\leq u'(x) \leq 2x + 1, \\ -2 &\leq u''(x) \leq 2, \qquad \forall x \in [0,1]. \end{aligned}$$

Second example considers the fourth order multipoint problem

$$\left\{\begin{array}{c} u^{(iv)}(x) = -0.1(u(x))^3 - 0.1|u''^{0.01u'(x)} + 20\sqrt[3]{|u'''(x)|} \\ \\ \sum\limits_{n=1}^{+\infty} a_n^0 u(x_n) + \sum\limits_{n=1}^{+\infty} b_n^0 u'(x_n) + \sum\limits_{n=1}^{+\infty} c_n^0 u''(x_n) - ku(0) = 0 \\ \sum\limits_{n=1}^{+\infty} a_n^1 u(\widehat{x}_n) + \sum\limits_{n=1}^{+\infty} b_n^1 u'(\widehat{x}_n) + \sum\limits_{n=1}^{+\infty} c_n^1 u''(\widehat{x}_n) - \eta u'(0) = 0 \\ u''(0) + 2u'''(0) = 0 \\ u''(1) = 2 \end{array}\right. , \tag{18}$$

where $\sum_{n=1}^{+\infty} a_n^i, \sum_{n=1}^{+\infty} b_n^i, \sum_{n=1}^{+\infty} c_n^i$, for $i = 0, 1$, are positive convergent series to a^i, b^i and c^i, respectively, $x_n, \widehat{x}_n \in [0,1]$, $k \geq 7a^0 + 8b^0 + 8c^0$ and $\eta \geq \frac{1}{3}(7a^1 + 8b^1 + 8c^1)$.

The functions $\alpha, \beta \in [0,1] \to \mathbb{R}$ given by

$$\alpha(x) = x^2 \text{ and } \beta(x) = -x^3 + 4x^2 + 3x + 1$$

are, respectively, lower and upper solutions of (18) with

$$\begin{aligned} f(x, y_0, y_1, y_2, y_3) &= -0.1(y_0)^3 - 0.1|y_2 - 2|e^{0.01y_1} + 20\sqrt[3]{|y_3|} \\ L_0(z_1, z_2, z_3, z_4) &= \sum_{n=1}^{+\infty} a_n^0 z_1(x_n) + \sum_{n=1}^{+\infty} b_n^0 z_2(x_n) + \sum_{n=1}^{+\infty} c_n^0 z_3(x_n) - kz_4 \\ L_1(z_1, z_2, z_3, z_4) &= \sum_{n=1}^{+\infty} a_n^1 z_1(\widehat{x}_n) + \sum_{n=1}^{+\infty} b_n^1 z_2(\widehat{x}_n) + \sum_{n=1}^{+\infty} c_n^1 z_3(\widehat{x}_n) - \eta z_4 \\ L_2(z_1, z_2, z_3, z_4, z_5) &= z_4 + 2z_5 \\ L_3(z_1, z_2, z_3, z_4, z_5) &= z_4 - 2. \end{aligned}$$

As the continuous function f verifies (3) and (4) for

$$\varphi_{E_*}(y_3) = 34.3 + 0.6\ e^{0.08} + 20\sqrt[3]{|y_3|}$$

in

$$E_* = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \begin{array}{c} x^2 \leq y_0 \leq -x^3 + 4x^2 + 3x + 1 \\ 2x \leq y_1 \leq -3x^2 + 8x + 3 \\ 2 \leq y_2 \leq -6x + 8 \end{array} \right\}$$

then, by Theorem 3.1, there is a solution $u(x)$ of problem (18) such that, for every $x \in [0,1]$,

$$x^2 \leq u(x) \leq -x^3 + 4x^2 + 3x + 1, \tag{19}$$

$$2x \leq u'(x) \leq -3x^2 + 8x + 3$$

$$2 \leq u''(x) \leq -6x + 8. \tag{20}$$

Remark that this solution u is nonnegative, by (19). Moreover, by (20), u is not a trivial solution, neither can be a straight line.

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