DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **548–557**

STOCHASTIC INCLUSIONS WITH NON-CONTINUOUS SET-VALUED OPERATORS

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Abstract. In the paper we consider a stochastic integral inclusion with discontinuous multivalued right hand side, driven by a continuous semimartingale. Using selection properties and lower and upper solutions methods we demonstrate the existence of strong solutions for such inclusions. We extend some recent results both for deterministic differential inclusions and for stochastic differential equations for increasing operators.

1. **Introduction.** In general, investigating stochastically controlled dynamical systems by methods of multivalued analysis requires an appropriate kind of regularity of their multivalued structure. The properties of Lipschitz continuity, lower or upper semicontinuity, and maximal monotonicity are most often considered (see e.g. [2], [3], [4], [12], [13], [10], [11] and references therein). In this paper, we consider a stochastic inclusion for classes of increasing and "upper separated" set-valued functions (see [14]). Neither an increasing nor an upper separated multifunction need satisfy any of the classical continuity properties. The upper separatedness of a set-valued function F is necessary for the existence of a convex selection of F. As a consequence, using lower and upper solution method, we deduce the existence of solutions of stochastic inclusions with right-hand sides taken from these classes of multifunctions. Our technique involves combining selection procedures, the method of upper and lower solutions, stochastic comparison theorems and Amann's fixed point theorem. This enables us to investigate Itô stochastic inclusions for new classes of multivalued integrands. The work presented here extends results obtained recently both for differential inclusions (see e.g [1]) and stochastic equations (see e.g.[7], [8]).

We begin our considerations with some notations and auxiliary notions. Let $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathbf{F}_t\}_{t\geq 0}$ is an increasing and right continuous family of σ-subalgebras of $\mathbf{F}$ and $\mathbf{F}_0$ contains all P-null sets. Let $\mathcal{P}(\mathbf{F}_t)$ denote the smallest σ-algebra on $\mathbb{R}_+ \times \Omega$ with respect to which every continuous adapted process is measurable. A stochastic process x is said to be predictable if x is $\mathcal{P}(\mathbf{F}_t)$-measurable. In this case we will write $x \in \mathcal{P}(\mathbf{F}_t)$. One has $\mathcal{P}(\mathbf{F}_t) \subset \beta \otimes \mathbf{F}$, where β denotes the Borel σ-algebra on $\mathbb{R}_+$. Let $R = (R(t))_{t\geq 0}$ be a set-valued stochastic process with values in $Comp(\mathbb{R}^n)$, the space of all compact subsets of $\mathbb{R}^n$ considered with a Hausdorff metric $h(\cdot,\cdot)$, i.e., a family of $\mathbf{F}$ measurable set-valued mappings

2000 *Mathematics Subject Classification.* Primary 60H20, 90B05; Secondary 90C30.

Key words and phrases. Stochastic inclusion, upper lower solution, increasing and upper separated multifunctions.

$R(t) : \Omega \to Comp(\mathbb{R}^n)$, each $t \in \mathbb{R}_+$. We call R measurable if it is $\beta \otimes \mathbf{F}$ measurable in the sense of set-valued functions. Similarly, R is $\{\mathbf{F}_t\}_{t\geq 0}$-adapted if $R(t)$ is $\mathbf{F}_t$-measurable for each $t \in \mathbb{R}_+$. We call R predictable if R is $\mathcal{P}(\mathbf{F}_t)$ -measurable. Given a predictable set-valued process $R = (R_t)_{t\in\mathbb{R}_+}$ and a semimartingale Z let us denote

$$S(R,Z) := \{r \in \mathcal{P}(\mathbf{F}_t) : r_t \in R_t \text{ for each } t \in \mathbb{R}_+ \ a.e. \text{ and } r \text{ is } Z \text{ integrable}\}.$$

For conditions of Z-integrability see [16] Chapter IV. A predictable set-valued process R is said to be integrable with respect to a semimartingale Z (or simply Z-integrable) if $S(R,Z)$ is a nonempty set. Then, we define a set-valued Itô- type stochastic integral

$$\int R_t dZ_t := \left\{ \int r_t dZ_t : r \in S(R,Z) \right\}.$$

Consider an $\mathbf{F}_t$- adapted, continuous local martingale M and an $\mathbf{F}_t$- adapted continuous increasing process A. Let us also consider predictable set-valued functions $F, G : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to Cl(\mathbb{R})$, where $Cl(\mathbb{R})$ denotes the class of nonempty and closed subsets in $\mathbb{R}$.

Definition 1.1. By an *Itô* stochastic inclusion we mean the relation

$$\begin{aligned} X_t - X_s &\in \int_s^t F(u, X_u) dA_u + \int_0^t G(u, X_u) dM_u, \\ X_0 &= x_0. \end{aligned} \tag{1}$$

The above inclusion is well-defined if its right-hand side is nonempty.

Definition 1.2. A continuous semimartingale X defined on a filtered probability space $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\geq 0}, P)$ is said to be a strong solution (upper or lower solution) to the stochastic inclusion (1) if it satisfies the relation:

$$X_t = (\geq, \leq)\ X_0 + \int_0^t u_s dA_s + \int_0^t v_s dM_s,\ t \geq 0,$$

for some $\mathcal{F}_t$-adapted stochastic processes $u_t \in F(t, X_t)$, $v_t \in G(t, X_t)$ provided the Lebegue-Stieltjes integral and stochastic integral above exist.

2. **Increasing and upper separated set-valued maps and their selections.**

We describe below the classes of noncontinuous set-valued mappings appearing on the right-hand side of our inclusion (1). Let $\mathcal{X} = (\mathcal{X}, \preceq)$ be an ordered topological space, i.e. the order intervals $[a) := \{x \in \mathcal{X} : a \preceq x\}$ and $(a] := \{x \in \mathcal{X} : x \preceq a\}$ are closed sets in the space $\mathcal{X}$ for all $a \in \mathcal{X}$. We will use the following increasing set-valued mappings (see e.g.[1] and [6]).

Definition 2.1. A set-valued mapping $F : \mathcal{X} \to 2^{\mathcal{X}}$ is said to be increasing upward if for every $x, y \in \mathcal{X}, x \preceq y$ and $z \in F(x)$, there exists $w \in F(y)$ such that $z \preceq w$.

Definition 2.2. A set-valued mapping $F : \mathcal{X} \to 2^{\mathcal{X}}$ is said to be isotone increasing if for every $x, y \in \mathcal{X}, x \preceq y$ and $z \in F(x)$ we have $z \preceq w$ for every $w \in F(y)$.

Increasing set-valued mappings (isotone or upward) need not be upper or lower semicontinuous or even maximally monotone in the sense of multivalued analysis [9]. Isotone increasing set-valued mappings have been used in [1] to obtain existence results for ordinary differential inclusions via a lattice fixed point theorem and the method of upper and lower solutions. The notion of increasing upward multifunctions has been used recently in [6] in the context of their fixed points and further applications. For our purposes here we use the increasing upward set-valued functions. That is, such mapping will appear in the drift term of the stochastic inclusion (1). For set-valued random and increasing functions we have the following result.

Proposition 1. *Let $F : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to Comp(\mathbb{R})$ be a $\mathcal{P}(\mathbf{F}_t) \otimes \beta$ measurable set-valued function with compact subsets of $\mathbb{R}$, such that the mapping $F(t, \omega, \cdot)$ is increasing upward for every $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Then, the function*

$$f(t, \omega, x) := \sup F(t, \omega, x)$$

is a predictable and increasing selection of F.

Proof. Since, for every fixed $x \in X$, $F(\cdot, \cdot, x)$ is a predictable set-valued function, by Proposition 2.32 in [9] we conclude the predictability of the mapping $(t, \omega) \to \sup F(t, \omega, x)$, for every fixed $x \in \mathbb{R}$. The rest of the proof follows immediately by the properties of the multifunction F. □

Let G be a set-valued function from a Banach space X into nonempty subsets of $\bar{\mathbb{R}}$. We define upper and lower bounds of G by formulas

$$V_G : X \to \bar{\mathbb{R}}, \;\; V_G(x) = \sup\{a : a \in G(x)\},$$
$$W_G : X \to \bar{\mathbb{R}}, \;\; W_G(x) = \inf\{b : b \in G(x)\}.$$

Definition 2.3 ([14])**.** We say that G is upper separated if for every $x \in DomG$ and $\epsilon > 0$ there exists a hyperplane $H_{x,\epsilon}$ strongly separating a point $(x, W_G(x) - \epsilon)$ from the set $Epi(V_G) := \{(x, a) \in X \times \mathbb{R} : \; V_G(x) \le a\}$.

Theorem 2.4. *Let $G : \mathbb{R}_+ \times \Omega \times X \to Conv(\mathbb{R})$ be a proper $\mathcal{P}(\mathbf{F}_t) \otimes \beta(X)$ measurable set-valued function with closed and convex subsets of $\bar{\mathbb{R}}$. If $G(t, \omega, \cdot)$ is upper separated then it admits a predictable, convex and lower semicontinuous selection.*

Proof. By Proposition 2.32 in [9] we conclude the predictability of the mapping $(t, \omega) \to V_{G(t,\omega,\cdot)}(x)$, for every fixed $x \in X$. Similarly, one has the predictability for the mapping $(t, \omega) \to W_{G(t,\omega,\cdot)}(x)$, for every fixed $x \in X$. Fix $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Let $V^{**}_{G(t,\omega,\cdot)}$ denote the second conjugate function of $V_{G(t,\omega,\cdot)}$ i.e.

$$V^{**}_{G(t,\omega,\cdot)}(x) = \sup_{p \in X^*} \left\{ p(x) - \sup_{x \in X} (p(x) - V_{G(t,\omega,\cdot)}(x)) \right\}.$$

Thus, the mapping $(t, \omega) \to V^{**}_{G(t,\omega,\cdot)}$ is predictably measurable as well. We will prove that for every x, $V^{**}_{G(t,\omega,\cdot)}(x) \in G(t, \omega, x)$. Let $x \in DomV_{G(t,\omega,\cdot)}$ (i.e., such that $V_{G(t,\omega,\cdot)}(x) < \infty$) and $\epsilon > 0$. Since $G(t, \omega, \cdot)$ is upper separated, then there exists a continuous linear functional $x^*_{x,\epsilon}$ strictly separating $(x, W_{G(t,\omega,\cdot)}(x) - \epsilon)$ from the set $Epi(V_{G(t,\omega,\cdot)})$. Let $x^*_{x,\epsilon}$ be represented by the pair $(p, a) \in X^* \times \mathbb{R}$. Then there exists $\delta > 0$ such that for every $y \in DomV_{G(t,\omega,\cdot)}$ and each $b \ge 0$

$$(p, a)((y, V_{G(t,\omega,\cdot)}(y) + b)) \le (p, a)((x, W_{G(t,\omega,\cdot)}(x) - \epsilon)) - \delta.$$

Then we get

$$p(y) + aV_{G(t,\omega,\cdot)}(y) + ab \leq p(x) + aW_{G(t,\omega,\cdot)}(x) - a\epsilon - \delta.$$

Taking the supremum over b we deduce that $a \leq 0$. Assume first that $a < 0$. Then dividing by $-a$ and denoting $-p/a = q$ we get

$$q(y) - V_{G(t,\omega,\cdot)}(y) \leq q(x) - W_{G(t,\omega,\cdot)}(x) + \epsilon + \delta/a.$$

Taking the supremum over $y \in DomV_{G(t,\omega,\cdot)}$ we obtain

$$V^*_{G(t,\omega,\cdot)}(q) := \sup_{y \in DomV_{G(t,\omega,\cdot)}} (q(y) - V_{G(t,\omega,\cdot)}(y)) \leq q(x) - W_{G(t,\omega,\cdot)}(x) + \epsilon + \delta/a.$$

Then

$$\begin{aligned} &W_{G(t,\omega,\cdot)}(x) - \epsilon \leq q(x) - V^*_{G(t,\omega,\cdot)}(q) + \delta/a \leq q(x) - V^*_{G(t,\omega,\cdot)}(q) \\ &\leq \sup_q (q(x) - V^*_{G(t,\omega,\cdot)}(q)) = V^{**}_{G(t,\omega,\cdot)}(x). \end{aligned}$$

Letting ϵ converge to 0 we obtain

$$W_{G(t,\omega,\cdot)}(x) \leq V^{**}_{G(t,\omega,\cdot)}(x)$$

for every $x \in DomV_{G(t,\omega,\cdot)}$. Now assume $a = 0$. Then $x \notin DomV_{G(t,\omega,\cdot)}$. Indeed, for every $x \in DomV_{G(t,\omega,\cdot)}$, and for $y = x$, we get

$$p(x) + aV_{G(t,\omega,\cdot)}(x) \leq p(x) + aW_{G(t,\omega,\cdot)}(x) - a\epsilon - \delta$$

and therefore

$$a(V_{G(t,\omega,\cdot)}(x) - W_{G(t,\omega,\cdot)}(x) - \epsilon) \leq -\delta.$$

Hence, for each $x \in DomV_{G(t,\omega,\cdot)}$ a cannot be equal to 0. However, taking $a = 0$ and $x \notin DomV_{G(t,\omega,\cdot)}$ we get for every $y \in DomV_{G(t,\omega,\cdot)}$ that

$$p(y) \leq p(x) - \delta. \tag{2}$$

Let $r \in DomV^*_{G(t,\omega,\cdot)}$. By the definition of $V^*_{G(t,\omega,\cdot)}$ we deduce that

$$r(y) - V_{G(t,\omega,\cdot)}(y) \leq V^*_{G(t,\omega,\cdot)}(r)$$

Adding this to the inequality (2) multiplied by $n > 0$, we obtain

$$(np + r)(y) - V_{G(t,\omega,\cdot)}(y) \leq np(x) - n\delta + V^*_{G(t,\omega,\cdot)}(r)$$

Taking the supremum over $y \in DomV_{G(t,\omega,\cdot)}$ we have

$$V^*_{G(t,\omega,\cdot)}(np + r) \leq np(x) - n\delta + V^*_{G(t,\omega,\cdot)}(r).$$

Hence

$$r(x) + n\delta - V^*_{G(t,\omega,\cdot)}(r) \leq (np + r)(x) - V^*_{G(t,\omega,\cdot)}(np + r).$$

By the definition of $V^{**}_{G(t,\omega,\cdot)}$ we get

$$r(x) + n\delta - V^*_{G(t,\omega,\cdot)}(r) \leq V^{**}_{G(t,\omega,\cdot)}(x)$$

for every n and $x \notin DomV_{G(t,\omega,\cdot)}$. Taking $n \to \infty$ we deduce $V^{**}_{G(t,\omega,\cdot)}(x) = \infty$ and thus

$$W_{G(t,\omega,\cdot)}(x) \leq V^{**}_{G(t,\omega,\cdot)}(x)$$

for every $x \in X$. Since G admits closed convex values and

$$W_{G(t,\omega,\cdot)}(x) \leq V^{**}_{G(t,\omega,\cdot)}(x) \leq V_{G(t,\omega,\cdot)}(x),$$

then by Theorem 11.1 of [17], we deduce that $V^{**}_{G(t,\omega,\cdot)}$ is a predictable, proper lower semicontinuous and convex selection of G. This completes the proof. □

Corollary 1. *Let $G : \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to Conv(\mathbb{R})$ be a proper $\mathcal{P}(\mathbf{F}_t) \otimes \beta$ measurable set-valued function with closed and convex subsets of $\bar{\mathbb{R}}$. If $G(t, \omega, \cdot)$ is upper separated then it admits a predictable, convex and continuous selection.*

3. **Strong solutions.** Below we present a result concerning the existence of strong solutions of an Itô stochastic inclusion with predictable, increasing upward and upper separated set-valued functions F and G, respectively. The method of the proof is essentially based on the selection results given above, the method of upper and lower solutions to the inclusion (1) and Amann's fixed point theorem.

If we consider increasing upward and upper separated set valued mappings then, as pointed out above, they need not satisfy both a global Lipschitz and a linear growth condition. Hence there may be solutions of Itô stochastic inclusion that do not exist globally. In other words there may exist solutions which have finite time explosions. We recall that a random variable θ_X is an explosion time for a solution process X if X is a solution to the Itô inclusion on $[0, \theta_X)$, $X_{\theta_X} = +\infty$ P.1 on $\{\theta_X < \infty\}$ and $\theta_X = \lim S_n$, where

$$S_n := \inf\{t > 0 : |X_t| > n\}, \text{ for } n \geq 1.$$

The case $P\{\theta_X = \infty\} = 1$ refers to that of a nonexploding solution, while $P\{\theta_X < \infty\} > 0$ to an exploding one. Since we are dealing with upper and lower solutions for stochastic inclusions which can possess only local solutions (viz., solutions up to an explosion time), we will consider also local upper and lower solutions. We present now the main results of the paper.

Theorem 3.1. *Let $F : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to Comp(\mathbb{R})$ be a $\mathcal{P}(\mathbf{F}_t) \otimes \beta$ measurable set-valued function with compact subsets of $\mathbb{R}$, such that the mapping $F(t, \omega, \cdot)$ is increasing upward for every $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Let $G : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to Conv(\mathbb{R})$ be a proper, $\mathcal{P}(\mathbf{F}_t) \otimes \beta$ measurable set-valued function with closed and convex subsets of $\bar{\mathbb{R}}$, such that the mapping $G(t, \omega, \cdot)$ is upper separated, for every $(t, \omega) \in \mathbb{R}_+ \times \Omega$. If there exist upper and lower local sulutions U and L for stochastic inclusion:*

$$\begin{aligned} X_t - X_s &\in \int_s^t F(u, X_u) dA_u + \int_s^t G(u, X_u) dM_u, \\ X_0 &= x_0. \end{aligned}$$

such that $L_0 \leq x_0 \leq U_0$ a.e., then it admits at least one strong solution X (up to an explosion time) such that

$$P\{L_t \leq X_t \leq U_t, \textit{for every } t \in [0, \theta)\} = 1, \textit{where } \theta = \theta_L \wedge \theta_U.$$

Proof. By Proposition 1 and Corollary 1, there exist a predictable and increasing selection f, and a predictable, convex and continuous selection g for set-valued mappings F and G respectively. By Proposition 1.6 [15] the function $x \to g(t, \omega, x)$ is also locally Lipschitzian at any point, for $(t, \omega) \in \mathbb{R}_+ \times \Omega$. Hence the processes U and L are also upper and lower solutions for stochastic equation

$$X_t = x_0 + \int_0^t f(u, X_u) dA_u + \int_0^t g(u, X_u) dM_u. \tag{3}$$

Thus the study of existence of solutions to stochastic inclusion can be reduced to the existence of solutions to the stochastic equation above. The rest of the proof will follow directly from lemmas below. □

Lemma 3.2. *Let A be continuous, F_t- adapted incresing process and let M be continuous, F_t- adapted local maringale. Let $f,g:\mathbb{R}_+\times\Omega\times\mathbb{R}\to\mathbb{R}$ be a $\mathcal{P}(\mathbf{F}_t)\otimes\beta$ measurable mappings such that:*

a) $x\to f(t,\omega,x)$ is increasing and globally Lipschitzian (with a Lipschitz constant K,

b) $x\to g(t,\omega,x)$ is locally Lipschitzian, i.e. for each $N>0$, there exists a predictable process $G_N(t,\omega)$ such that

$$\begin{aligned} |g(t,x)-g(t,y)| &\le G_N(t)|x-y|,\\ \int_0^t G_N(s)d[M_s] &< \infty \text{ P.1} \end{aligned}$$

for all $t\in[0,N]$, and $x,y\in\mathbb{R}$ such that $|x|\le N, |y|\le N$.
If there exist upper and lower local sulutions U and L for stochastic equation

$$X_t = x_0 + \int_0^t f(u,X_u)dA_u + \int_0^t g(u,X_u)dM_u \tag{4}$$

such that $L_0\le x_0\le U_0$ a.e., then

$$P\{L_t\le X_t\le U_t, \text{for every } t\in[0,\theta)\}=1,$$

for an unique local strong solution X to the equation (4) and $\theta=\theta_L\wedge\theta_U$.

Proof. The existence of an unique local strong solution to equation (4) follows by Proposition 2.3 in [5]. We shall prove that

$$P\{X_t\le U_t, \text{for every } t\in[0,\theta_U)\}=1.$$

In a similar way one can prove that $P\{L_t\le X_t, t\in[0,\theta_L)\}=1$. For every $N\ge1$ we define a sequence of stopping times

$$T_N := \inf\{t\in[0,\theta_U): |X_t|\vee|U_t|\vee[M]_t\vee|A_t|>N\}\wedge N.$$

Then we have $T_N\nearrow\theta_U$, for $N\to+\infty$. Next we define a stopping time

$$\tau := \inf\{t\ge0: X_t-U_t>0\}.$$

It is enough to show that for each N

$$P\{\tau<T_N\}=0.$$

Indeed, if it is true then we will get that $P\{\tau<\theta_U\}=0$, which means $P\{X_t\le U_t$, for every $t\in[0,\theta_U)\}=1$. For every $q\in Q_+$ and $N\ge1$, let us define $\sigma_q^N := (\tau+q)\wedge T_N$ and $\Omega_q^N := \{\omega: X_{\sigma_q^N}-U_{\sigma_q^N}>0\}$. Let us note the following implication for every $q\in Q_+$ and $N\ge1$:

$$P\{\Omega_q^N\}=0 \Rightarrow P\{\tau<T_N\}=0.$$

Indeed, let us fix $N\ge1$ and suppose $P\{\Omega_q^N\}=0$, for arbitrary $q\in Q_+$. Then we have $X_{\sigma_q^N}-U_{\sigma_q^N}\le0$ a.e. on $\{\tau<T_N\}$. Thus, since X and U have continuous paths, we have also

$$X_{(\tau+t)\wedge T_N}-U_{(\tau+t)\wedge T_N}\le0 \text{ a.e.on } \{\tau<T_N\}$$

for every $t\ge0$. Hence, it follows that for a.e. $\omega\in\{\tau<T_N\}$ and $s\in[\tau(\omega),T_N(\omega)]$ one has $X_s-U_s\le0$. But since $X_0\le U_0$ and also $X_\tau\le U_\tau$ a.e., by the definition

of the stopping time τ we get $P\{\tau < T_N\} = 0$. So, it is enough to show now that $P\{\Omega_q^N\} = 0$. Let us fix $q \in Q_+$ and $N \geq 1$ and define:

$$\alpha_q^N := \sup\{t \in [0, \sigma_q^N) : X_t \leq U_t\}.$$

Since $X_\tau \leq U_\tau$ a.e. we observe that $\tau(\omega) \leq \alpha_q^N(\omega)$, for $\omega \in \{\tau < +\infty\}$. Consequently, by continuity of X and U we get the inequality:

$$X_{\alpha_q^N} \leq U_{\alpha_q^N} \text{ a.e. on } \{\tau < T_N\}.$$

By the definition of σ_q^N, α_q^N and Ω_q^N we have

$$\Omega_q^N := \{X_{\sigma_q^N} - U_{\sigma_q^N} > 0\} = \{\alpha_q^N < \sigma_q^N\}$$

and then $\Omega_q^N \in F_{\alpha_q^N}$. Hence, for $\omega \in \Omega_q^N$ and $t \in (\alpha_q^N(\omega), \sigma_q^N(\omega)]$ we get an inequality $X_t > U_t$. By assumption, the process U is an upper local solution of the equation (4). Consequently, for $\omega \in \Omega_q^N$ and $t \in (\alpha_q^N(\omega), \sigma_q^N(\omega)]$ we get the inequality:

$$X_t - U_t \leq \int_{\alpha_q^N}^t [f(s, X_s) - f(s, U_s)]dA_s + \int_{\alpha_q^N}^t [g(s, X_s) - g(s, U_s)]dM_s. \qquad (5)$$

Let us denote

$$V_t := \int_{\alpha_q^N}^t [f(s, X_s) - f(s, U_s)]dA_s + \int_{\alpha_q^N}^t [g(s, X_s) - g(s, U_s)]dM_s,$$

for simplicity. This process is a continuous semimartingale. Thus the inequality (5) can be rewritten as

$$[X_t - U_t]I_{\Omega_q^N} I_{(\alpha_q^N, \sigma_q^N]}(t) \leq V_t I_{\Omega_q^N} I_{(\alpha_q^N, \sigma_q^N]}(t). \qquad (6)$$

Let us take the semimartingale $V^+ := \max\{V, 0\}$. By the Tanaka formula (see e.g. [16]) we obtain the following equality:

$$V_t^+ I_{\Omega_q^N} = V_{\alpha_q^N}^+ I_{\Omega_q^N} + I_{\Omega_q^N} \int_{\alpha_q^N}^t I_{\{V_s > 0\}} dV_s + \frac{1}{2} I_{\Omega_q^N} \left[L_t^0(V) - L_{\alpha_q^N}^0(V)\right],$$

where $L_t^x(V)$ denotes a local time at the point x for the semimartingale V. Similarly as in Lemma 3.2. [5] one can prove that $L_t^0(V) - L_{\alpha_q^N}^0(V) = 0$, for $t \in (\alpha_q^N, \sigma_q^N]$ on Ω_q^N. Thus, because $V_{\alpha_q^N}^+ I_{\Omega_q^N} = 0$ and $\Omega_q^N \in F_{\alpha_q^N}$, the last equality has the form

$$V_t^+ I_{\Omega_q^N} = \int_{\alpha_q^N}^t I_{\{V_s > 0\}} I_{\Omega_q^N} [f(s, X_s) - f(s, U_s)]dA_s + N_t, \qquad (7)$$

where $N_t := \int_{\alpha_q^N}^t I_{\{V_s > 0\}} I_{\Omega_q^N} [g(s, X_s) - g(s, U_s)]dM_s$ is a continuous local martingale. By the assumption a) the mapping $x \to f(t, \omega, x)$ is increasing and globally Lipschitzian (with a Lipschitz constant K). Since, for $\omega \in \Omega_q^N$ and $t \in (\alpha_q^N(\omega), \sigma_q^N(\omega)]$ we have had the inequality $X_t > U_t$, we obtain

$$I_{\Omega_q^N}[f(s, X_s) - f(s, U_s)] \geq 0.$$

Thus, by (7) it follows that

$$V_t^+ I_{\Omega_q^N} \leq N_t + K \int_{\alpha_q^N}^t I_{\{V_s > 0\}} I_{\Omega_q^N} [X_s - U_s]dA_s.$$

Hence, using the inequality (6) one obtains the inequality

$$V_t^+ I_{\Omega_q^N} \leq N_t + \int_{\alpha_q^N}^t V_s^+ d\left(\int_{\alpha_q^N}^{\cdot} K dA_u\right)_s .$$

Let us denote $Z := \int_{\alpha_q^N}^{\cdot} K dA_u$. Now, using the Stochastic Gronwall inequality (cf e.g., Lemma 2, [18]) we obtain

$$I_{\Omega_q^N} V_{\sigma_q^N}^+ \exp\left(-Z_{\sigma_q^N}\right) \leq N_{\alpha_q^N} \exp\left(-Z_{\alpha_q^N}\right) + \int_{\alpha_q^N}^{\sigma_q^N} \exp\left(-Z_u\right) dN_u.$$

Thus, because $N_{\alpha_q^N} = 0$ we have the inequality

$$E\left(I_{\Omega_q^N} V_{\sigma_q^N}^+ \exp\left(-Z_{\sigma_q^N}\right)\right) \leq E \int_{\alpha_q^N}^{\sigma_q^N} \exp\left(-Z_u\right) dN_u = 0.$$

Consequently, using the inequality (6) once again we get

$$I_{\Omega_q^N}\left[X_{\sigma_q^N} - U_{\sigma_q^N}\right] \leq I_{\Omega_q^N} V_{\sigma_q^N}^+ = 0 \text{ a.e.}$$

Hence $X_{\sigma_q^N} \leq U_{\sigma_q^N}$ on Ω_q^N a.e. and finaly $P\{\Omega_q^N\} = 0$. This completes the proof of Lemma 3.2. ☐

Lemma 3.3. *Let A be continuous, F_t- adapted incresing process and let M be continuous, F_t- adapted local maringale. Let $f, g : \mathbb{R}_+ \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{P}(\mathbf{F}_t) \otimes \beta$ measurable mappings such that:*
a) $x \to f(t, \omega, x)$ is increasing,
b) $x \to g(t, \omega, x)$ is locally Lipschitzian.
If there exist upper and lower local sulutions U and L for the stochastic equation

$$X_t = x_0 + \int_0^t f(u, X_u) dA_u + \int_0^t g(u, X_u) dM_u \tag{8}$$

such that $L_0 \leq x_0 \leq U_0$ a.e., then there exists a local strong solution X to the equation (8) such that

$$P\{L_t \leq X_t \leq U_t, \text{for every } t \in [0, \theta)\} = 1, \text{ where } \theta = \theta_L \wedge \theta_U.$$

Proof. Let $\mathcal{X}$ be a space of F_t- adapted and continuous processes endowed with the order relation $\preceq$:

$$X \preceq Y \text{ if and only if } P\{X_t \leq Y_t, \text{ for every } t \geq 0\} = 1,$$

for $X, Y \in \mathcal{X}$. In the space $(\mathcal{X}, \preceq)$ we consider a subset

$$D := [L, U] := \{Z \in \mathcal{X} : P\{L_t \leq Z_t \leq U_t, t \in [0, \theta)\} = 1\},$$

where $\theta = \theta_L \wedge \theta_U$. Let $Z \in D$ be fixed. Then by Lemma 3.2 there exists a unique local strong solution X^* of the equation:

$$X_t^* = x_0 + \int_0^t f(u, Z_u) dA_u + \int_0^t g(u, X_u^*) dM_u \tag{9}$$

Let us introduce a single valued operator $\mathcal{S} : D \to \mathcal{X}$ as $\mathcal{S}(Z) = X^*$. Since the mapping $x \to f(t, \omega, x)$ is increasing and the process L is a lower solution for the equation (8), we obtain

$$L_t - L_s \leq \int_s^t f(u, Z_u) dA_u + \int_s^t g(u, L_u) dM_u.$$

Thus, the process L is also a lower local solution to equation (9). In a similar way one can show that the process U satisfies the inequality

$$U_t - U_s \geq \int_s^t f(u, Z_u) dA_u + \int_s^t g(u, U_u) dM_u,$$

which means that it is the upper local solution of (9). Hence, by Lemma 3.2 we obtain

$$P\{L_t \leq \mathcal{S}(Z)_t \leq U_t, t \in [0, \theta)\} = 1.$$

Since Z has been an arbitrary process from D, we conclude that $\mathcal{S} : D \to D$. In particular we have $L \preceq \mathcal{S}(L)$ and $\mathcal{S}(U) \preceq U$. To complete the proof it is enough to show that $\mathcal{S}$ has a fixed point. We shall apply Amann's fixed point theorem (see e.g.[19]). We show that $\mathcal{S}$ is an increasing mapping with respect to the order $\preceq$. Indeed, let us take $Z^1, Z^2 \in D$ and $Z^1 \preceq Z^2$. Let $X^i := \mathcal{S}(Z^i)$ i.e.

$$X_t^i = x_0 + \int_0^t f(u, Z_u^i) dA_u + \int_0^t g(u, X_u^i) dM_u, i = 1, 2.$$

By the monotone property of the mapping $x \to f(t, \omega, x)$ we have

$$X_t^1 - X_s^1 \leq \int_s^t f(u, Z_u^2) dA_u + \int_s^t g(u, X_u^1) dM_u.$$

Hence the process $X^1$is a local lower solution of the equation

$$X_t = x_0 + \int_0^t f(u, Z_u^2) dA_u + \int_s^t g(u, X_u) dM_u \tag{10}$$

which has a unique local strong solution $X^2 = \mathcal{S}(Z^2)$. On the other hand, since $Z^2 \preceq U$ and U is an upper solution to equation (8), by the properties of the mapping $x \to f(t, \omega, x)$ we arrive to the inequality:

$$U_t - U_s \geq \int_s^t f(u, Z_u^2) dA_u + \int_s^t g(u, U_u) dM_u.$$

This means that the process U is an upper local solution of the equation (10). Using

Lemma 3.2 once again we obtain

$$P\{L_t \leq \mathcal{S}(Z^1)_t \leq \mathcal{S}(Z^2)_t \leq U_t, t \in [0, \theta)\} = 1.$$

Finally, by Amann's theorem, there exist a fixed point of $\mathcal{S}$ in D. This completes the proof of Lemma 3.3 and consequently the proof of Theorem 3.1. □

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Received July 2008; revised March 2009.

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