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ON THE ONE-DIMENSIONAL VERSION OF THE DYNAMICAL MARGUERRE-VLASOV SYSTEM WITH THERMAL EFFECTS

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Abstract. A one dimensional version of the dynamic Marguerre-Vlasov system in the presence of thermal effects is considered. The system depends on a parameter $\mathcal{E} > 0$ in a singular way as $\mathcal{E} \to 0$. Our interest is twofold: 1) To find the limit system as $\mathcal{E} \to 0$ and 2) To study the asymptotic behavior as $t \to +\infty$ of the total energy $E_{\mathcal{E}}(t)$ and compare it with the total energy of the limit system.

1. **Introduction.** Nonlinear shell theory could be considered as a generalization of the Plateau problem which studies surfaces assuming that the density of the potential energy of deformation is essentially proportional to the change in the area of the element. Good references on the subject are the books by Ph. Ciarlet [2] and I.I. Vorovich [8].

In this article we consider a one-dimensional version of the dynamical Marguerre-Vlasov system which describes the vibrations of shallow shells (see [6] or [2]). Thermal effects are present in the model. The complete model read as follows: Let $\mathcal{E} > 0$ and $0 \leq \alpha \leq 1$. Let us denote by $u = u^{\mathcal{E}}$, $w = w^{\mathcal{E}}$ and $\theta = \theta^{\mathcal{E}}$ the solution of the coupled system of equations

$$\begin{cases} \mathcal{E} u_{tt} = \mu_0 \left[u_x + \frac{1}{2} w_x^2 + K_1(x) w \right]_x - \mathcal{E}^\alpha u_t \\ w_{tt} + w_{xxxx} - w_{xxtt} = [f(u,w)]_x - g(u,w) - \theta_{xx} \\ \theta_t - \theta_{xx} - w_{xxt} = 0 \end{cases} \tag{1}$$

in $\Omega \times (0, +\infty)$, where $\Omega = \{0 < x < L\}$ with t denoting the (positive) time variable. The functions $u(x,t)$ and $w(x,t)$ represent respectively, the longitudinal and tranversal displacement of the beam at the point x and time t, $\theta(x,t)$ denotes the heat flux at x at time t

$$f(u,w) = \mu_0 \left[w_x \left(u_x + \frac{1}{2} w_x^2 + K_1(x) w \right) \right]$$

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and

$$g(u,w) = \mu_0 K_1(x)\left[u_x + \tfrac{1}{2}w_x^2 + K_1(x)w\right].$$

Finally, $K_1(x)$ represents the curvature of the beam at the point x and μ_0 is a positive constant related to the elastic modulus. We complement system (1) with boundary conditions

$$\begin{cases} u = 0, w = 0, \theta = 0 \text{ at } x = 0, L \text{ for any } t > 0 \\ w_x = 0 \text{ at } x = 0, L \text{ for any } t > 0 \end{cases} \tag{2}$$

and initial conditions

$$(u, u_t)|_{t=0} = (u_0, u_1), \quad (w, w_t)|_{t=0} = (w_0, w_1), \quad \theta(x, 0) = \theta_0(x) \tag{3}$$

for any $x \in \Omega$. This article is devoted to analyzing the following questions: 1) We investigate the "proximity" as $\mathcal{E} \to 0$ of the components $w^{\mathcal{E}}$ and $\theta^{\mathcal{E}}$ in (1) to the solution of a beam equation with thermal effects and 2) We also investigate the uniform (with respect to $\mathcal{E} \to 0$) rate of decay of the total energy of (1)–(3) as $t \to +\infty$ and compare it with the total energy of the limit system.

Quite recently, some authors considered the one-dimensional version of Marguerre-Vlasov system with either interior mechanical damping [6] or boundary damping [3]. Since thermal effects are a quite natural dissipative phenomenon in the modelling for the vibrations of shells, then, the conclusion of this article may be of interest in the subject. Related results on the analysis of singular limits for plates or beams were considered by G. Perla Menzala, A. Pazoto and E. Zuazua [5].

2. **Existence and uniqueness.**

We briefly describe the functional spaces for the solutions using well known techniques of semigroup theory. Let

$$X = H_0^1(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

with the norm $||\cdot||_X$ given by

$$||(u, v, w, z, \theta)||_X^2 = \mu_0||u_x||^2 + \mathcal{E}||v||^2 + ||w_{xx}||^2 + ||z||^2 + ||z_x||^2 + ||\theta||^2$$

where $||\cdot||$ denotes the norm in $L^2(\Omega)$. Let us rewritte (1)–(3) as

$$\begin{cases} BU_t = AU + N(U) \\ U(0) = U_0 = (u_0, u_1, w_0, w_1, \theta_0) \in X \end{cases}$$

where

$$B(u, v, w, z, \theta) = (u, \mathcal{E}v, w, z - z_{xx}, \theta),$$
$$A(u, v, w, z, \theta) = (v, \mu_0 u_{xx}, z, -w_{xxxx} - \theta_{xx}, z_{xx} + \theta_{xx}).$$

It is not difficult to prove that the operator $\tilde{A} = B^{-1}A$ with domain

$$\mathcal{D}(\tilde{A}) = (H^2 \cap H_0^1) \times H_0^1 \times (H^3 \cap H_0^2) \times H_0^2(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$$

is dissipative, maximal and $\mathcal{D}(\tilde{A})$ is dense in X. Using Lumer-Phillips'theorem it follows that $\tilde{A}$ is the infinitesimal generator of a C_0 semigroup in X. Next, we consider the map $B^{-1}N(U)$.

Lemma 2.1. *$B^{-1}N(\cdot)$ is locally Lipschitz continuous in X provided $K_1 \in H^1(\Omega)$.*

Proof. Let $U=(u,v,w,z,\theta)$ and $V=(\tilde{u},\tilde{v},\tilde{w},\tilde{z},\tilde{\theta})$ in X. Direct calculation shows that

$$B^{-1}[N(U)-N(V)]=(0,T_1,0,T_2,T_3)$$

where

$$T_1=\mathcal{E}^{-1}\left\{\mu_0\left[\tfrac{1}{2}w_x^2+K_1w\right]_x-\left(\tfrac{1}{2}\tilde{w}_x^2+K_1\tilde{w}\right)_x-\mathcal{E}^{\alpha}(v-\tilde{v})\right\}$$

$$T_2=\left(I-\frac{d^2}{dx^2}\right)^{-1}\Big\{[f(u,w)]_x-[f(\tilde{u},\tilde{w})]_x-g(u,w)+g(\tilde{u},\tilde{w}) -\left(I-\frac{\partial^2}{\partial x^2}\right)(z-\tilde{z})-\frac{d^2}{dx^2}(\theta-\tilde{\theta})\Big\}$$

and $T_3=z_{xx}-\tilde{z}_{xx}$. In what follows we will denote by C a positive constant which may vary from line to line. Now, we want to estimate the norm

$$||B^{-1}[N(U)-N(V)||_X^2=\mathcal{E}||T_3||^2+||T_2||^2+||\frac{\partial}{\partial x}T_2||^2+||T_3||^2.$$

Using Poincare's inequality and the embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ we obtain

$$\begin{aligned}\mathcal{E}||T_1||^2 &\le 2\mathcal{E}^{-1}\mu_0^2\left\|\left[\frac{1}{2}w_x^2+K_1w-\frac{1}{2}\tilde{w}_x^2-K_1\tilde{w}\right]_x\right\|^2+2\mathcal{E}^{2\alpha-1}||v-\tilde{v}||^2\\ &\le \mathcal{E}^{-1}C[||K_1||_{H^1}^2+||w_{xx}||^2+||\tilde{w}_{xx}||^2]||w_{xx}-\tilde{w}_{xx}||^2+2\mathcal{E}^{2\alpha-1}||v-\tilde{v}||^2\\ &\le C[1+||U||_X^2+||V||_X^2]||U-V||_X^2.\end{aligned}$$

Now, the operator $\left(I-\frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial}{\partial x}: L^2(\Omega)\mapsto H_0^1(\Omega)$ is bounded, therefore

$$\begin{aligned}&\left\|\left(I-\frac{\partial^2}{\partial x^2}\right)^{-1}\frac{\partial}{\partial x}[f(u,w)-f(\tilde{u},\tilde{w})]\right\|_{H_0^1(\Omega)}^2\\ &\le C||f(u,w)-f(\tilde{u},\tilde{w})||^2\\ &\le C||a+b||^2\\ &\le C\{(1+||w_{xx}||^2+||\tilde{w}_{xx}||^2)^2||w_{xx}-\tilde{w}_{xx}||^2+||u_x||^2||w_{xx}-\tilde{w}_{xx}||^2\\ &\quad+||\tilde{w}_{xx}||^2||u_x-\tilde{u}_x||^2\}\\ &\le C\{(1+||U||_X^2+||V||_X^2)^2||U-V||_x^2.\end{aligned}$$

where

$$a=(u_x+\tfrac{1}{2}w_x^2+K_1w)(w_x-\tilde{w}_x)$$

and

$$b=\tilde{w}_x\left[(u_x-\tilde{u}_x)+\tfrac{1}{2}(w_x^2-\tilde{w}_x^2)+K_1(w-\tilde{w})\right]$$

Similarly, since

$$\left(I-\frac{\partial^2}{\partial x^2}\right)^{-1}: L^2(\Omega)\mapsto H^2(\Omega)\cap H_0^1(\Omega)$$

is bounded, then

$$\left\| \left(I - \frac{\partial^2}{\partial x^2} \right)^{-1} [g(u,w) - g(\tilde{u},\tilde{w})] \right\|^2_{H_0^1} \le C||g(u,w) - g(\tilde{u},\tilde{w})||^2$$
$$\le C(1 + ||U||_X^2 + ||V||_X^2)||U - V||_X^2.$$

Clearly, $||z - \tilde{z}||^2_{H_0^1(\Omega)} \le ||U - V||_X^2$.

Thus

$$||B^{-1}[N(U) - N(V)]||_X \le C(1 + ||U||_X^2 + ||V||_X^2)||U - V||_X$$

which proves Lemma 2.1. □

As a consequence of Lemma 2.1 we obtain local existence of a unique solution with finite energy. Global existence follows considering the total energy of (1)

$$E_{\mathcal{E}}(t) = \frac{1}{2}\int_0^L \left[\mathcal{E}u_t^2 + \mu_0 \left(u_x + \frac{1}{2}w_x^2 + K_1 w \right)^2 + w_t^2 + w_{xx}^2 + w_{xt}^2 + \theta^2 \right] dx. \quad (4)$$

Easy calculation using the boundary conditions (2) proves

$$\frac{d}{dt}E_{\mathcal{E}}(t) = -\int_0^L (\mathcal{E}^\alpha u_t^2 + \theta_x^2)dx. \quad (5)$$

Clearly (5) gives us the apriori estimates needed to prove global existence. Uniqueness follows as usual by using Gronwall's inequality.

Theorem 2.2. *Let $\mathcal{E} > 0$, $\alpha \ge 0$, $K_1 \in H^1(\Omega)$ and $(u_0, u_1, w_0, w_1, \theta_0) \in X$. Then, problem (1)–(3) has a unique global mild solution $(u^{\mathcal{E}}, u_t^{\mathcal{E}}, w_t^{\mathcal{E}}, \theta^{\mathcal{E}})$ which belongs to $C([0, +\infty); X)$*

3. **The asymptotic limit.**

Let $\mathcal{E} > 0$ and $0 < \alpha \le 1$ and consider the solution $\{u^{\mathcal{E}}, w^{\mathcal{E}}, \theta^{\mathcal{E}}\}$ as in Theorem 2.2 Again, we write $u = u^{\mathcal{E}}$, $w = w^{\mathcal{E}}$, $\theta = \theta^{\mathcal{E}}$. We want to find the limit of w and θ as $\mathcal{E} \to 0$. Also we want to write explicity the equations that the limit system satisfies.

Theorem 3.1. *Let $\mathcal{E} > 0$, $0 < \alpha \le 1$. Then, there exist functions $Z = Z(x,t)$ and $\phi = \phi(x,t)$ such that $w^{\mathcal{E}} \rightharpoonup Z$, $\theta^{\mathcal{E}} \rightharpoonup \phi$ as $\mathcal{E} \to 0$ and $\{Z, \phi\}$ is the solution of the coupled model*

$$\begin{cases} Z_{tt} + Z_{xxxx} - Z_{xxtt} = H(t)Z_{xx} - K_1(x)H(t) - \phi_{xx} \\ \phi_t - \phi_{xx} - Z_{xxt} = 0 \end{cases} \quad (6)$$

with boundary conditions

$$\begin{cases} Z = 0, \quad Z_x = 0 \quad at\ x = 0, L \quad \forall\ t > 0 \\ \phi = 0 \quad at\ x = 0, L, \quad \forall\ t > 0 \end{cases} \quad (7)$$

and initial conditions

$$Z(x,0) = w_0(x), \quad Z_t(x,0) = w_1(x), \quad \phi(x,0) = \theta_0(x), \quad \forall\ x \in \Omega. \quad (8)$$

Here

$$H(t) = \tfrac{\mu_0}{2L}\textstyle\int_0^L (Z_x^2 + 2K_1(x)Z)dx.$$

Proof. Due to (5) we have $E_{\mathcal{E}}(t) \le E_{\mathcal{E}}(0) \le C$, $\forall\ 0 < \mathcal{E} < 1$, the constant C does not depend on $\mathcal{E}$. This give us bounds on all terms in $E_{\mathcal{E}}(t)$:

$$\{\sqrt{\mathcal{E}}u_t\} \text{ is bounded in } L^\infty(0,+\infty;L^2(\Omega)), \quad \{u_x + \tfrac{1}{2}(w_x)^2 + K_1 w\},$$

$\{w_t\}$, $\{w_{xx}\}$, $\{w_{xt}\}$ and $\{\theta\}$ are bounded in $L^\infty(0,\infty;L^2(\Omega))$. From (5) we also have that $\{\mathcal{E}^{\alpha/2}u_t\}$ and $\{\theta_x\}$ are bounded in $L^2(0,+\infty;L^2(\Omega))$. Thus, we can extract subsequences (which we still denote with the same symbols) to guarantee existence of functions $\xi(x,t)$, $\eta(x,t)$, $Z(x,t)$ and $\phi(x,t)$ (independent of $\mathcal{E}$) such that

$$\sqrt{\mathcal{E}}u_t \rightharpoonup \xi \text{ weak-}* \text{ in } L^\infty(0,\infty;L^2(\Omega)) \tag{9}$$

$$u_x + \frac{1}{2}(w_x)^2 + K_1 w \rightharpoonup \eta \text{ weak-}* \text{ in } L^\infty(0,\infty;L^2(\Omega)) \tag{10}$$

$$w \rightharpoonup Z \text{ weak-}* \text{ in } L^\infty(0,\infty;H_0^2(\Omega)) \cap W^{1,\infty}(0,\infty;H_0^1(\Omega)) \tag{11}$$

$$\theta \rightharpoonup \phi \text{ weak-}* \text{ in } L^\infty(0,+\infty;L^2(\Omega)) \cap L^2(0,\infty;H_0^1(\Omega)) \tag{12}$$

as $\mathcal{E} \to 0$. Convergence (9) up to (12) are sufficient to pass to the limit in the linear terms of the last two equations of (1). Remains to identify the limit of the nonlinear terms $[w_x(u_x + \frac{1}{2}w_x^2 + K_1(x)w)]_x$ and $[\frac{1}{2}w_x^2]$ as $\mathcal{E} \to 0$. Due to (5) we know that $\{w\}$ is uniformly bounded in $L^\infty(0,\infty;H_0^2(\Omega)) \cap W^{1,\infty}(0,\infty;H_0^1(\Omega))$. Therefore, we can use the Aubin-Lions compactness criteria to obtain a subsequence of $\{w\}$ such that $w \to Z$ strong in $L^\infty(0,T;H^{2-\delta}(\Omega))$ as $\mathcal{E} \to 0$ for $\delta > 0$ and $T > 0$. Combining this informaton with (10) we deduce

$$w_x(u_x + \tfrac{1}{2}w_x^2 + K_1 w) \rightharpoonup Z_x\eta \text{ weak in } L^2(0,T;L^2(\Omega))$$

as $\mathcal{E} \to 0$. We claim that η is independent of x, actually η is given by

$$\eta(t) = \frac{1}{2L}\int_0^L Z_x^2 dx + \frac{1}{L}\int_0^L K_1(x)Z\,dx. \tag{13}$$

In fact using the embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ we can verify that

$$\textstyle\int_0^L w_x^4 dx \le C||w_{xx}||^4_{L^2(\Omega)}.$$

Consequently

$$\begin{aligned}\int_0^L u_x^2 dx &= \int_0^L [(u_x + \frac{1}{2}w_x^2 + K_1 w) - (\frac{1}{2}w_x^2 + K_1 w)]^2 dx \\ &\le C[E_{\mathcal{E}}(0) + \int_0^L w_x^4 dx + \int_0^L (K_1 w)^2 dx] \\ &\le C[E_{\mathcal{E}}(0) + ||w_{xx}||^4_{L^2} + ||K_1||^2_{H^1}\int_0^L w_{xx}^2 dx]\end{aligned}$$

and

$$\textstyle\int_0^T\int_0^L u_x^2 dxdt \le CTE_{\mathcal{E}}(0).$$

Thus, we can extract a subsequence of $\{u_x\}_{\mathcal{E}>0}$ such that $u_x \rightharpoonup \rho$ weakly in $L^2(0,T;L^2(\Omega))$ as $\mathcal{E} \to 0$ for some $\rho \in L^2(0,T;L^2(\Omega))$. This implies

$$u_x + \tfrac{1}{2}w_x^2 + K_1 w \rightharpoonup \rho + \tfrac{1}{2}Z_x^2 + K_1 Z = \eta$$

as $\mathcal{E} \to 0$, weakly in $L^2(0,T;L^2(\Omega))$. We also know that $\mathcal{E}^\alpha u_t \rightharpoonup 0$ weakly in $H^{-1}(0,T;H_0^1(\Omega))$ as $\mathcal{E} \to 0$. Since $\mathcal{E}u_{tt} \rightharpoonup 0$ weak in $H^{-1}(0,T;L^2(\Omega))$ as $\mathcal{E} \to 0$, then it follows that

$$\eta_x = [\rho + \tfrac{1}{2}Z_x^2 + K_1 Z]_x = 0$$

due to the first equation in (1). Consequently, η is independent of x. Next, we integrate $\eta = \rho + \frac{1}{2}Z_x^2 + K_1 Z$ in x over the interval $[0, L]$ which gives (13) because $\int_0^L \rho dx = \lim_{\mathcal{E}\to 0} \int_0^L u_x^{\mathcal{E}} dx = 0$ which proves our claim. As a consequence we have

$$[w_x(u_x + \tfrac{1}{2}w_x^2 + K_1 w)]_x \rightharpoonup \eta Z_{xx} = \left[\tfrac{1}{2L}\int_0^L Z_x^2 dx + \tfrac{1}{L}\int_0^L K_1 Z dx\right] Z_{xx}$$

as $\mathcal{E} \to 0$,

$$[f(u,w)]_x \rightharpoonup \mu_0 \left[\tfrac{1}{2L}\int_0^L Z_x^2 dx + \tfrac{1}{L}\int_0^L K_1 Z dx\right] Z_{xx}$$

and

$$g(u,w) \rightharpoonup \mu_0 K_1(x) \left[\tfrac{1}{2L}\int_0^L Z_x^2 dx + \tfrac{1}{L}\int_0^L K_1 Z dx\right]$$

as $\mathcal{E} \to 0$, which concludes the proof of Theorem 3.1. □

4. **Uniform stabilization.** Let $\mathcal{E} > 0$ and denote by $u = u^{\mathcal{E}}$, $w = w^{\mathcal{E}}$, $\theta^{\mathcal{E}} = \theta$. We want to find a uniform rate of decay of the total energy $E_{\mathcal{E}}(t)$ of problem (1)–(3). We use the Lyapunov functional

$$G_{\mathcal{E},\delta}(t) = E_{\mathcal{E}}(t) + \delta F_{\mathcal{E}}(t) + 2\delta I_{\mathcal{E}}(t), \quad \delta > 0 \tag{14}$$

where $E_{\mathcal{E}}(t)$ is given by (4)

$$F_{\mathcal{E}}(t) = \mathcal{E}\int_0^L u u_t dx + \int_0^L (w w_t + w_x w_{xt}) dx \tag{15}$$

and

$$I_{\mathcal{E}}(t) = \int_0^L \theta w_t dx - \frac{1}{2}\int_0^L \theta^2 dx + \int_0^L w_t(-\frac{d^2}{dx^2})^{-1}\theta dx. \tag{16}$$

Observe that in $F_{\mathcal{E}}(t)$ are quite standard expressions in order to build the Lyapunov functional. However, $I_{\mathcal{E}}(t)$ is not. In previous work in thermoelasticity for plates and beams, similar expressions for $I_{\mathcal{E}}(t)$ were considered by G. Avalos and I. Lasiecka [1] and G. Perla Menzala and E. Zuazua [4].

Lemma 4.1. *Let $H_{\mathcal{E}}(t) = F_{\mathcal{E}}(t) + 2I_{\mathcal{E}}(t)$. Then, there exists a positive constant C such that*

$$|H_{\mathcal{E}}(t)| \le C\{1 + \mathcal{E}[E_{\mathcal{E}}(0) + ||K_1||_\infty^2]\}E_{\mathcal{E}}(t) \tag{17}$$

for any $t \ge 0$.

Proof. In what follows C wil denote a positive constant which may vary from line to line, but it is independent of $\mathcal{E}$. From (15) and (16) for any $\gamma > 0$ we get the estimate

$$\begin{aligned} |H_{\mathcal{E}}(t)| \le{}& \frac{1}{2\gamma}\int_0^L \mathcal{E}u_t^2 dx + \frac{\mathcal{E}\gamma}{2}\int_0^L u^2 dx + \frac{C\gamma}{2}\int_0^L w_{xx}^2 dx \\ &+ \frac{1}{2\gamma}\int_0^L w_t^2 dx + \frac{C\gamma}{2}\int_0^L w_{xx}^2 dx + \frac{1}{2\gamma}\int_0^L w_{xt}^2 dx \\ &+ \frac{1}{\gamma}\int_0^L w_t^2 dx + \gamma\int_0^L \theta^2 dx + \frac{1}{2}\int_0^L \theta^2 dx + \frac{1}{\gamma}\int_0^L w_t^2 dx \\ &+ \gamma\int_0^L \left[\left(-\frac{d^2}{dx^2}\right)^{-1}\theta\right]^2 dx. \end{aligned} \tag{18}$$

Since $\left(-\frac{d^2}{dx^2}\right)^{-1} : L^2(\Omega) \mapsto H^2(\Omega)\cap H_0^1(\Omega)$ is a bounded operator and $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, where $\Omega=(0,L)$ then we know

$$\left\|\left(-\frac{d^2}{dx^2}\right)^{-1}\theta\right\|_{L^2}^2 \le C\left\|\left(-\frac{d^2}{dx^2}\right)^{-1}\theta\right\|_{H_0^1}^2 \le C\|\theta\|_{L^2}^2. \tag{19}$$

Using (19) together with the inequalities

$$\int_0^L w_x^4 dx \le C||w_{xx}||_{L^2}^4 \tag{20}$$

$$\int_0^L u_x^2 dx \le C\left[E_{\mathcal{E}}(0)+||w_{xx}||_{L^2}^4+||k_1||_{H^1}^2\int_0^L w_{xx}^2 dx\right] \tag{21}$$

we used already during the proof of Theorem 3.1 we deduce from (18) the estimate

$$\begin{aligned}|H_{\mathcal{E}}(t) \le C &\left[\int_0^L\left\{\mathcal{E}u_t^2+\mathcal{E}(u_x+\frac{1}{2}w_x^2+k_1w)^2\right\}dx\right]\\ &+(E_{\mathcal{E}}(0)+||k_1||_{L^\infty}^2)\int_0^L w_{xx}^2dx + C\int_0^L w_{xx}^2dx\\ &+C\int_0^L w_t^2 dx+\frac{1}{2}\int_0^L w_{xt}^2dx+(2+C)\int_0^L\theta^2dx\\ &\le C[1+\mathcal{E}\{E_{\mathcal{E}}(0)+||k_1||_\infty^2\}]E_{\mathcal{E}}(t).\end{aligned}$$

Here $C=C(\gamma)>0$. This proves Lemma 4.1. □

Lemma 4.2. *Under the assumptions of Theorem 2.2 then the estimate*

$$\begin{aligned}\frac{dH_{\mathcal{E}}}{dt} \le (1+\frac{\mathcal{E}^{\alpha-1}}{2\gamma})&\int_0^L\mathcal{E}u_t^2dx\\ &+\left[-4\mu_0+\frac{\mathcal{E}^\alpha\gamma C}{2}+2\gamma C||k_1||_\infty^2\right]\int_0^L(u_x+\frac{1}{2}w_x^2+k_1w)^2dx\\ &+(\gamma-1)\int_0^L w_t^2dx-\int_0^L w_{xt}^2dx\\ &+\left[\frac{\gamma}{2}-1+\frac{\mathcal{E}^\alpha\gamma C}{2}\big(E_{\mathcal{E}}(0)+||k_1||_\infty^2\big)+2\gamma C\big(1+E_{\mathcal{E}}(0)\big)\right]\int_0^L w_{xx}^2dx\\ &+(\frac{2}{\gamma}+2\gamma C)\int_0^L\theta^2dx+2\int_0^L\theta_x^2dx\end{aligned} \tag{22}$$

holds for any $t\ge 0$ and any $\gamma>0$.

Proof. Using equations (1) we obtain the identities

$$\begin{aligned}
\frac{d}{dt}\int_0^L \mathcal{E}uu_t dx &= \int_0^L \mathcal{E}u_t^2 dx - \int_0^L \mu_0(u_x + \frac{1}{2}w_x^2 + k_1 w)u_x dx \\
&\quad - \mathcal{E}^\alpha \int_0^L uu_t dx \\
\frac{d}{dt}\int_0^L ww_t dx &= \int_0^L w_t^2 dx - \int_0^L w_{xx}^2 dx + \int_0^L ww_{xxtt} dx \\
&\quad - \int_0^L \mu_0(u_x + \frac{1}{2}w_x^2 + k_1 w)w_x^2 dx \\
&\quad - \int_0^L \mu_0(u_x + \frac{1}{2}w_x^2 + k_1 w)k_1 w dx - \int_0^L w\theta_{xx} dx \\
\frac{d}{dt}\int_0^L w_x w_{xt} dx &= \int_0^L w_{xt}^2 dx - \int_0^L ww_{xxtt} dx.
\end{aligned}$$

The above identities imply

$$\begin{aligned}
\frac{dF_\mathcal{E}}{dt} \leq &- 4\mu_0 \int_0^L (u_x + \frac{1}{2}w_x^2 + k_1 w)^2 dx - \mathcal{E}^\alpha \int_0^L uu_t dx \\
&+ \int_0^L \mathcal{E}u_t^2 dx - \int_0^L w_{xx}^2 dx + \int_0^L w_t^2 dx + \int_0^L w_{xt}^2 dx \\
&- \int_0^L w\theta_{xx} dx.
\end{aligned} \tag{23}$$

Similarly

$$\begin{aligned}
\frac{d}{dt}\int_0^L w_t\theta dx &= \int_0^L w_{tt}\theta dx - \int_0^L w_{xt}\theta_x dx - \int_0^L w_{xt}^2 dx \\
\frac{d}{dt}\int_0^L -\frac{1}{2}\theta^2 dx &= \int_0^L \theta_x^2 dx + \int_0^L w_{xt}\theta_x dx \\
\frac{d}{dt}\int_0^L w_t(-\frac{d^2}{dx^2})^{-1}\theta dx &= \int_0^L w_{tt}(-\frac{d^2}{dx^2})^{-1}\theta dx - \int_0^L w_t\theta dx - \int_0^L w_t^2 dx.
\end{aligned}$$

The above identities imply

$$\frac{dI_\mathcal{E}}{dt}(t) = -\int_0^L (w_t^2 + w_{xt}^2 + w_t\theta)dx + \int_0^L (\theta_x^2 + w_{tt}\theta + w_{tt}(-\frac{d^2}{dx^2})^{-1}\theta]dx \tag{24}$$

From (23) and (24) we deduce, for any $\gamma > 0$, the estimate

$$\begin{aligned}\frac{dH_{\mathcal{E}}}{dt} \leq & \int_0^L (\mathcal{E}u_t^2 + 2\theta_x^2)dx \\ & - \int_0^L \{w_t^2 + w_{xx}^2 + w_{xt}^2 + 4\mu_0(u_x + \frac{1}{2}w_x^2 + k_1 w)^2\}dx \\ & + \frac{1}{2\gamma}\int_0^L \theta^2 dx + \gamma\int_0^L w_t^2 dx + \frac{1}{\gamma}\int_0^L \theta^2 dx + \gamma\int_0^L w_{tt}^2 dx \\ & + \frac{1}{\gamma}\int_0^L \theta^2 dx + \gamma\int_0^L w_{tt}^2 dx + \frac{1}{\gamma}\int_0^L ([-\frac{d^2}{dx^2}]^{-1}\theta)^2 dx \\ & + \frac{\gamma}{2}\int_0^L w_{xx}^2 dx + \mathcal{E}^\alpha\left(\frac{\gamma}{2}\int_0^L u^2 dx + \frac{1}{2\gamma}\int_0^L u_t^2 dx\right).\end{aligned} \tag{25}$$

We need to obtain estimates for the terms $\int_0^L u^2 dx$ and $\int_0^L w_{tt}^2 dx$. Using Poincar's inequality and (20) we deduce

$$\begin{aligned}\int_0^L u^2 dx \leq & C\int_0^L u_x^2 dx \\ & \leq 2C\int_0^L (u_x + \frac{1}{2}w_x^2 + k_1 w)^2 dx + 2C\int_0^L (\frac{1}{2}w_x^2 + k_1 w)^2 dx \\ & \leq C\Big[\int_0^L (u_x + \frac{1}{2}w_x^2 + k_1 w)^2 dx + \Big(\int_0^L w_{xx}^2 dx\Big)^2 \\ & \quad + ||k_1||_{H^1}^2 \int_0^L w_{xx}^2 dx\Big].\end{aligned} \tag{26}$$

Now, using (1) we can write

$$w_{tt} = \left(I - \tfrac{d^2}{dx^2}\right)^{-1}\left[-\tfrac{d^4 w}{dx^4} + [f(u,w)]_x + g(u,w) - \theta_{xx}\right].$$

Since $\left(I - \frac{d^2}{dx^2}\right)^{-1} : H^{-2}(\Omega) \mapsto L^2(\Omega)$ is bounded we obtain

$$\begin{aligned}||w_{tt}||_{L^2} \leq & C|| - w_{xxxx} + [f(u,w)]_x - g(u,w) - \theta_{xx}||_{H^{-2}(\Omega)} \\ & \leq C\{||w_{xxxx}||_{H^{-2}(\Omega)} + ||[f(u,w)]_x||_{H^{-2}(\Omega)} + ||g(u,w)||_{H^{-2}(\Omega)} \\ & \quad + ||\theta_{xx}||_{H^{-2})\Omega)}\} \\ & \leq C\{||w_{xx}||_{L^2} + ||f(u,w)||_{L^2} + ||g(u,w)||_{L^2} + ||\theta||_{L^2}\} \\ ||w_{tt}||_{L^2}^2 \leq & C\Big\{\int_0^L w_{xx}^2 dx + \left(\int_0^L w_{xx}^2 dx\right)\left(\int_0^L (u_x + \frac{1}{2}w_x^2 + k_1 w)^2 dx\right) \\ & + ||k_1||_{H^1}^2 \int_0^L (u_x + \frac{1}{2}w_x^2 + k_1 w)^2 dx + \int_0^L \theta^2 dx\Big\}.\end{aligned} \tag{27}$$

Using estimates (24) and (27) we deduce from (25)

$$\begin{aligned}\frac{dH_{\mathcal{E}}}{dt} &\leq \{1+\frac{\mathcal{E}^{\alpha-1}}{2\gamma}\}\int_0^L \mathcal{E}u_t^2dx + (\gamma-1)\int_0^L w_t^2dx \\ &- \int_0^L w_{xt}^2dx + \{\frac{\mathcal{E}^\alpha\gamma C}{2} - 4\mu_0 + 2\gamma C||k_1||^2_{H^1(\Omega)}\}\int_0^L \left(u_x + \frac{1}{2}w_x^2 + k_1 w\right)^2dx \\ &+ \{\frac{\gamma}{2} - 1 + \frac{\mathcal{E}^\alpha\gamma}{2}\}(E_{\mathcal{E}}(0) + ||k_1||^2_{H^1(\Omega)}) + 2\gamma C(1+E_{\mathcal{E}}(0))\int_0^L w_{xx}^2dx \\ &+ \{\frac{2}{\gamma} + 2\gamma C\}\int_0^L \theta^2dx + 2\int_0^L \theta_x^2dx\end{aligned}$$

which proves Lemma 4.2. ☐

Lemma 4.3. *Under the assumptions of Theorem 2.2, let $\delta > 0$, $\mathcal{E} > 0$ and consider*

$$G_{\mathcal{E},\delta}(t) = E_{\mathcal{E}}(t) + \delta H_{\mathcal{E}}(t).$$

Then, there exists a positive constant $C = C(\delta)$ such that

$$\frac{dG_{\mathcal{E},\delta}(t)}{dt} \leq -C(\delta)E_{\mathcal{E}}(t) \quad \text{for any } t \geq 0 \tag{28}$$

Proof. Using Lemma 4.2 we deduce, for any $\gamma > 0$

$$\begin{aligned}\frac{dG_{\mathcal{E},\delta}(t)}{dt} \leq &- \{\mathcal{E}^{\alpha-1} - \delta[1+\frac{\mathcal{E}^{\alpha-1}}{2\gamma}]\}\int_0^L \mathcal{E}u_t^2dx \\ &- \delta(1-\gamma)\int_0^L w_t^2dx - \delta\int_0^L w_{xt}^2dx \\ &- \delta\{4\mu_0 - C\gamma(1+||k_1||^2_{H^1})\}\int_0^L (u_x + \frac{1}{2}w_x^2 + k_1w)^2dx \\ &- \delta\{1 - C\gamma[1+E_{\mathcal{E}}(0) + \mathcal{E}^\alpha(E_{\mathcal{E}}(0) + ||k_1||^2_{H^1})\}\int_0^L w_{xx}^2dx \\ &+ (2\delta-1)\int_0^L \theta_x^2dx + C\delta(\frac{1}{\gamma}+\gamma)\int_0^L \theta^2dx\end{aligned} \tag{29}$$

Let $K = 1 + E_{\mathcal{E}}(0) + \mathcal{E}^\alpha(E_{\mathcal{E}}(0) + ||k_1||^2_{H^1})$. Obviously $K > 1$ for any $\mathcal{E} > 0$ thus, if we take $\gamma = \lambda K^{-1}$ then $\delta C(\frac{1}{\gamma}+\gamma) \leq \delta C(\frac{K}{\lambda}+\lambda)$. Using the above choices and Poincar's inequality we deduce from (29)

$$\begin{aligned}\frac{dG_{\mathcal{E},\delta}(t)}{dt} \leq &- \left\{\mathcal{E}^{\alpha-1} - \delta\left[1+\frac{\mathcal{E}^{\alpha-1}}{2\gamma}\right]\right\}\int_0^L \mathcal{E}u_t^2dx \\ &- \delta\{1-\lambda K^{-1}\}\int_0^L w_t^2dx - \delta\int_0^L w_{xt}^2dx \\ &- \delta\{4\mu_0 - C\lambda K^{-1}(1+||k_1||^2_{H^1})\}\int_0^L (u_x + \frac{1}{2}w_x^2 + k_1w)^2dx \\ &- \delta\{1-C\lambda\}\int_0^L w_{xx}^2dx \\ &- \left[\frac{1}{C} - 2\delta C - \delta C\left(\frac{K}{\lambda}+\lambda\right)\right]\int_0^L \theta^2dx\end{aligned} \tag{30}$$

Now, we choose λ and δ as follows:

a) $1 - C\lambda > 0$

b) $1-\lambda K^{-1}>0$
c) $4\mu_0 - C\lambda K^{-1}(1+||k_1||^2_{H^1})>0$
d) $\mathcal{E}^{\alpha-1}-\delta\left[1+\frac{\mathcal{E}^{\alpha-1}}{2\gamma}\right]>0$
e) $C^{-1}-2\delta C-\delta C\left(\frac{K}{\lambda}+\lambda\right)>0.$

Clearly a), b) and c) hold provided $0<\lambda<\min\left\{C^{-1},1,\frac{4\mu_0}{C[1+||k_1||^2_{H^1}]}\right\}$ because $K>1$. Now let us keep λ fixed.

Observe that for $0<\mathcal{E}<1$ then $K=K(\mathcal{E})\le C_0$ where C_0 is a positive constant which depends only on the initial data (independent of $\mathcal{E}$). Thus, $2+\frac{K}{\lambda}+\lambda\le 2+\frac{C_0}{\lambda}+\lambda$ and in order to verify e) is enough to choose $\delta>0$ such that

$$\delta<\frac{C^{-1}}{C[2+\frac{C_0}{\lambda}+\lambda]}\le\frac{C^{-1}}{C[2+\frac{K}{\lambda}+\lambda]}.$$

Similarly, since $K\le C_0$ then $2\lambda\mathcal{E}^{1-\alpha}+K\le 2\lambda+C_0$ for any $0<\mathcal{E}<1$. Consequently, in order to satisfy d) is sufficient to choose $\delta>0$ such that

$$\delta<\frac{2\lambda}{C_0+2\lambda}\le\frac{2\lambda}{2\lambda\mathcal{E}^{1-\alpha}+K}=\frac{2\gamma\mathcal{E}^{\alpha-1}}{2\gamma+\mathcal{E}^{\alpha-1}}.$$

Thus, considering

$$\delta<\min\left\{\frac{C^{-1}}{C[2+\frac{C_0}{\lambda}+\lambda]},\frac{2\lambda}{C_0+2\lambda}\right\}.$$

with that choice (30) is bounded by $-C(\delta)E_{\mathcal{E}}(t)$ where $C(\delta)>0$ is independent of $0<\mathcal{E}<1$. This proves Lemma 4.3. □

Theorem 4.4. *Under the assumptions of Theorem 2.2. There exist positive constants C and β such that*

$$E_{\mathcal{E}}(t)\le CE_{\mathcal{E}}(0)\exp\left(\frac{-\beta t}{1+\mathcal{E}^\alpha[E_{\mathcal{E}}(0)+||K_1||^2_{H^1}]}\right)\tag{31}$$

for any $t\ge0$.

Proof. It follows directly from Lemmas 4.1 and 4.3. □

Remark 1. If $\alpha>0$ and $\mathcal{E}\to0$ we obtain from the conclusion of theorem 4.4

$$\overline{\lim_{\mathcal{E}\to0}}E_{\mathcal{E}}(t)\le C\overline{\lim_{\mathcal{E}\to0}}E_{\mathcal{E}}(0)\exp(-\beta t).\tag{32}$$

The total energy of the limit system (6), (8) is given by

$$\mathcal{L}(t)=\frac12\int_0^L(Z_t^2+Z_{xx}^2+Z_{xt}^2+\phi^2)dx$$
$$+\frac{\mu_0}{L}\left(\frac12\int_0^L Z_x^2dx+\int_0^L K_xZdx\right)^2$$

and satisfies $\frac{d\mathcal{L}}{dt}=-\int_0^L\phi_x^2dx$. It is not difficult to prove that $\mathcal{L}(t)$ decays exponentially as $t\to+\infty$, which is agreement with (32). The case $\alpha=0$ is special (see [6], Section 5).

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