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ON A SOLUTION WITH TRANSITION LAYERS FOR A BISTABLE REACTION-DIFFUSION EQUATION WITH SPATIALLY HETEROGENEOUS ENVIRONMENTS

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Abstract. In this paper we consider a boundary value problem for the following semilinear elliptic equation :$-\varepsilon^2\Delta u = h(|x|)^2(u-a(|x|))(1-u^2)$ in $B_1(0)$ with homogeneous Neumann boundary condition. The function a is a C^1 function satisfying $|a(r)| < 1$ for $r \in [0,1]$ and $a'(0) = 0$. The function h is a positive C^1 function satisfying $h'(0) = 0$. The nonlinear function in the equation is a typical example of the so-called *bistable* nonlinearity. Functions a and h in the nonlinearity represent spatial inhomogeneity. In particular, we consider the case where $a(r) = 0$ on some interval $I \subset (0,1)$. When $\varepsilon > 0$ is very small, there exist stationary solutions with sharp transition layers. We investigate asymptotic locations of transition layers of a global minimizer corresponding to an energy functional as $\varepsilon \to 0$. We use the variational procedure used in [4] with a few modifications prompted by the presence of the function h.

1. **Introduction and Main Results.** Reaction diffusion equations have been widely treated to study mechanisms of pattern formation for various phenomena, for example, phase transition, morphogenesis, population genetics and chemical reactions. In particular, some class of equations give rise to sharp transition layers when diffusion coefficients are very small and these transition layers are seen as interfaces between two phases. The dynamics of these interfaces and location of interfaces of stationary solutions are important to understand the mechanisms of pattern formation.

In this paper, we consider the stationary problem for the following scalar reaction-diffusion equation with inhomogeneity

$$(\mathrm{D}_\varepsilon)\begin{cases} u_t - \varepsilon^2\Delta u = f(|x|,u) & \text{in } B_1(0)\times(0,+\infty) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0)\times(0,+\infty) \\ u(x,0) = u_0(x) & \text{in } x\in B_1(0) \end{cases}$$

in which ε is a positive small parameter, $B_1(0)$ denotes the unit ball in $\mathbb{R}^N$ centered at the origin, $\nu = \nu(x)$ denotes the unit outer normal at $x \in \partial B_1(0)$ and the nonlinear function f is as follows:

$$f(r,u) = h(r)^2(u-a(r))(1-u^2).$$

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The function a in the nonlinear term f is a C^1 function on $[0,1]$ satisfying $-1 < a(r) < 1$ with $a'(0) = 0$ and the function h is a positive C^1 function on $[0,1]$ satisfying $h'(0) = 0$. These functions represents environmental effects. As stated later, we note that these environmental effect is important factor which affect dynamics of solutions and structure of stationary solutions in this problem. The nonlinear function f in this problem is a typical example of the so-called *bistable* nonlinearity, because we obtain a one-parameter family of ordinary differential equations parameterized in x by setting $\varepsilon = 0$ in (D_ε). For each x, the ordinary differential equation has two stable equilibria, $u = -1$ and $u = 1$, and solutions tend either to 1 or to -1 according as the initial value is greater than or less than $a(x)$. The value $a(x)$ here plays the role of a separatrix. Problem (D_ε) appears in various models such as population genetics, chemical reactor theory and phase transition phenomena. See [1] and the references therein. The readers can refer to [5] for more background material for this problem.

Since we are interested in the stationary problem, we consider the following problem:

$$(\mathrm{P}_\varepsilon) \begin{cases} -\varepsilon^2 \Delta u = f(|x|, u) & \text{in } B_1(0), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0). \end{cases}$$

We note that a function u is a solution to (P_ε) if and only if u is a critical point of following energy functional on $H^1(B_1(0))$

$$J_\varepsilon(u) = \int_{B_1(0)} \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 - F(|x|, u) \right\} dx,$$

where $F(r,u) = \int_{-1}^{u} f(r,s) ds$. We note that the function $-F(r,\cdot)$ takes local minima at $u = -1$ and $u = 1$. Let $A_- = \{x \in B_1(0) | a(|x|) < 0\}$ and $A_+ = \{x \in B_1(0) | a(|x|) > 0\}$ and we consider the case where $A_- \neq \emptyset$ and $A_+ \neq \emptyset$. We note that for $x \in A_-$, $-F(|x|,1) < -F(|x|,-1) = 0$ holds, for $x \in A_+$, $-F(|x|,1) > -F(|x|,-1) = 0$ holds and for $x \in B_1(0) \setminus (A_- \cup A_+)$, $-F(|x|,1) = -F(|x|,-1) = 0$ holds. In this sense, the nonlinear function f is called *unbalanced* bistable nonlinearity. On the other hand if $A_- = A_+ = \emptyset$ holds, then $-F(|x|,1) = -F(|x|,-1) = 0$ for $x \in B_1(0)$. In this case the nonlinear function f is called *balanced* bistable nonlinearity.

Now we state known results for reaction diffusion equations with unbalanced and balanced bistable nonlinearities. If the function h satisfies $h(r) \equiv 1$ and the function a satisfies $a(r) \not\equiv 0$ (unbalanced case), then the problem (P_ε) has been studied in [1], [4] and [9]. In this case, it is shown that there exist radially symmetric solutions which have transition layers near the set $\{x \in B_1(0) | a(|x|) = 0\}$. If the set $\{r \in [0,1] | a(r) = 0\}$ contains an interval I, then the problem to decide the location of transition layer on I is more delicate.

On the other hand, if the function h satisfies $h(r) \not\equiv 1$ and the function a satisfies $a(r) \equiv 0$(balanced case), then this problem (P_ε) becomes a stationary problem for well known Allen-Cahn equation with spatial inhomogeneity. It has been studied in [10] and [11] in the case where $N = 1$. In this case, it is shown that there exist stable solutions which have transition layers near prescribed local minimum points of h.

In this paper, we consider the case where the function a satisfies $a(r) \not\equiv 0$ with $a(r) = 0$ on some interval $I \subset (0,1)$. We show the function $H(r) := r^{N-1} h(r)$ has very important role to decide the location of a transition layer on I in this case.

We note that in [4] Dancer and Yan considered a problem similar to ours. They assume that $N \geq 2$, $h \equiv 1$ and the nonlinear term is $u(u - a(|x|))(1 - u)$. The function a in their nonlinearity satisfies $a(r) = 1/2$ on $[l_1, l_2] \subset (0, 1)$ and $a(r) < 1/2$ for $l_1 - r > 0$ small and $a(r) > 1/2$ for $r - l_2 > 0$ small. Then a global minimizer of a corresponding energy functional is radially symmetric and this minimizer has a transition layer near the inner boundary of the set $\{x \in B_1(0) : |x| \in [l_1, l_2]\}$ which is one of connected components of the set $\{x \in B_1(0)|a(|x|) = 1/2\}$. We note that the value l_1 is the minimum point of the function r^{N-1} on the interval $[l_1, l_2]$(see [4, Theorem 1.3]). In this sense, we can say that our results are natural extension of the results in [4]. We are going to follow throughout the variational procedure used in [4] with a few modifications prompted by the presence of the function h.

It is easy to see that the following minimization problem has a minimizer

$$\inf\{J_\varepsilon(u)|u \in H^1(B_1(0))\}. \tag{1}$$

In this paper, we will analyze the profile of the minimizer of (1). We note that we often use the notation $u(r)$ instead of $u(x)$ when we see this as a function of $r = |x|$ for a radially symmetric function u. Our main theorem is the following:

Theorem 1.1. *Assume that $A_- \neq \emptyset$ and $A_+ \neq \emptyset$ and let u_ε be a global minimizer of (1). Then u_ε is radially symmetric and*

$$u_\varepsilon \to \begin{cases} 1 & \text{, uniformly on any compact subset of } A_-, \\ -1 & \text{, uniformly on any compact subset of } A_+, \end{cases}$$

as $\varepsilon \to 0$. In particular, the global minimizer u_ε converges uniformly near the boundary of $B_1(0)$, that is, if $a(r) < 0$ on $[r_0, 1]$ for some $r_0 > 0$, then $u_\varepsilon \to 1$ uniformly on $\overline{B_1(0)} \setminus B_{r_0+\tau}(0)$ for any $\tau > 0$ small and if $a(r) > 0$ on $[r_0, 1]$ for some $r_0 > 0$, then $u_\varepsilon \to -1$ uniformly on $\overline{B_1(0)} \setminus B_{r_0+\tau}(0)$ for any $\tau > 0$ small. Moreover, for any $0 < r_1 \leq r_2 < 1$ with $a(r_i) = 0$, $i = 1, 2$, $a(r) \neq 0$ for $r_1 - r > 0$ small and for $r - r_2 > 0$ small, $a(r) = 0$ if $r \in [r_1, r_2]$, we have:

(i) *If $a(r) < 0$ for $r_1 - r > 0$ small and $a(r) > 0$ for $r - r_2 > 0$, then u_ε has exact one transition layer near the set $\{x \in B_1(0) : |x| \in [r_1, r_2]\}$ for sufficiently small $\varepsilon > 0$, that is, for any small $\eta > 0$ and for any small $\theta > 0$, there exists a positive number ε_0 which has the following properties: For any $\varepsilon \in (0, \varepsilon_0]$, there exist $t_{\varepsilon,1} < t_{\varepsilon,2}$ such that*
   (a) *The function u_ε satisfies*

$$\begin{cases} u_\varepsilon(r) > 1 - \eta & \text{for } r \in [r_1 - \theta, t_{\varepsilon,1}), \\ u_\varepsilon(t_{\varepsilon,1}) = 1 - \eta, & \\ u_\varepsilon(t_{\varepsilon,2}) = -1 + \eta, & \\ u_\varepsilon(r) < -1 + \eta, & \text{for } r \in (t_{\varepsilon,2}, r_2 + \theta]. \end{cases}$$

   (b) *The function $u_\varepsilon(r)$ is decreasing in $(t_{\varepsilon,1}, t_{\varepsilon,2})$*
   (c) *The inequality $0 < R_1 \leq \frac{t_{\varepsilon,2} - t_{\varepsilon,1}}{\varepsilon} \leq R_2$ holds, where R_1 and R_2 are two constants independent of $\varepsilon > 0$.*
   (d) *If $t_{\varepsilon_j,1},\ t_{\varepsilon_j,2} \to \bar{t}$ for some positive sequence $\{\varepsilon_j\}$ converging to zero as $j \to \infty$, then $\bar{t}$ satisfies $h(\bar{t})\bar{t}^{N-1} = \min_{s \in [r_1, r_2]} h(s)s^{N-1}$.*

(ii) *If $a(r) > 0$ for $r_1 - r > 0$ small and $a(r) < 0$ for $r - r_2 > 0$, then then u_ε has exact one transition layer near the set $\{x \in B_1(0) : |x| \in [r_1, r_2]\}$ for sufficiently small $\varepsilon > 0$, that is, for any small $\eta > 0$ and for any small $\theta > 0$, there exists a positive number ε_0 which has the following properties: For any $\varepsilon \in (0, \varepsilon_0]$, there exist $t_{\varepsilon,1} < t_{\varepsilon,2}$ such that*

(a) *The function* u_ε *satisfies*

$$\begin{cases} u_\varepsilon(r) < -1+\eta & \text{for } r \in [r_1-\theta, t_{\varepsilon,1}),\\ u_\varepsilon(t_{\varepsilon,1}) = -1+\eta, & \\ u_\varepsilon(t_{\varepsilon,2}) = 1-\eta, & \\ u_\varepsilon(r) > 1-\eta, & \text{for } r \in (t_{\varepsilon,2}, r_2+\theta]. \end{cases}$$

(b) *The function* $u_\varepsilon(r)$ *is increasing in* $(t_{\varepsilon,1}, t_{\varepsilon,2})$.

(c) *The inequality* $0 < R_1 \le \frac{t_{\varepsilon,2}-t_{\varepsilon,1}}{\varepsilon} \le R_2$ *holds, where* R_1 *and* R_2 *are two constants independent of* $\varepsilon > 0$.

(d) *If* $t_{\varepsilon_j,1},\ t_{\varepsilon_j,2} \to \bar{t}$ *for some positive sequence* $\{\varepsilon_j\}$ *converging to zero as* $j \to \infty$, *then* $\bar{t}$ *satisfies* $h(\bar{t})\bar{t}^{N-1} = \min_{s\in[r_1,r_2]} h(s)s^{N-1}$.

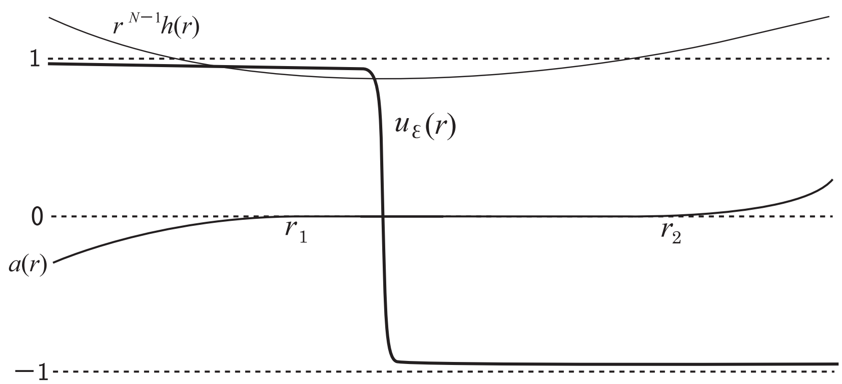

FIGURE 1. The profile of the global minimizer u_ε.

Remark 1. (i) We note that results from (a) to (c) both in cases (i) and (ii) are general results for unbalanced case and these results are not related to the presence of the function h. The effect of the function h appears in the result (d) in (i) and (ii).

(ii) If $\min_{s\in[r_1,r_2]} s^{N-1}h(s)$ is attained at a unique point $\bar{t} \in [r_1, r_2]$, we can show $t_{\varepsilon,1},\ t_{\varepsilon,2} \to \bar{t}$ as $\varepsilon \to 0$ without taking subsequences.

(iii) If the function $r^{N-1}h(r)$ is constant on $[r_1, r_2]$, it is a very difficult problem to know the location of the point $\bar{t} \in [r_1, r_2]$.

This paper is organized as follows. In Section 2, we prepare some preliminary results. We will prove Theorems 1.1 in Section 3.

2. **Preliminary Results.** In this section we prepare some preliminary results.

Let D is a bounded domain in $\mathbb{R}^N$. Let $\overline{f}(x,t)$ be a function defined on $\overline{D}\times\mathbb{R}$ which is bounded on $\overline{D}\times[-1,1]$. Suppose $\overline{f}$ is continuous on $t\in\mathbb{R}$ for each $x\in\overline{D}$ and is measurable in D for each $t \in \mathbb{R}$. We also assume

$$\overline{f}(x,t) > 0 \text{ for } x\in\overline{D},\ t<-1;\ \overline{f}(x,t)<0, \text{for } x\in\overline{D},\ t>1. \tag{2}$$

Consider the following minimization problem:

$$\inf\left\{\overline{J}_\varepsilon(u,D) := \int_D \left(\frac{\varepsilon^2}{2}|\nabla u|^2 - \overline{F}(x,u)\right)dx : u-\eta \in H_0^1(D)\right\}, \tag{3}$$

where $\eta \in H^1(D)$ with $-1 \le \eta \le 1$ on D and

$$\overline{F}(x,t) = \int_{-1}^{t} \overline{f}(x,s)ds.$$

We use next two lemmas about some properties of minimizers of the problem (3). Since these lemmas can be shown by methods similar to [4, Lemmas 2.2, 2.3], we omit proof. For the proofs, see [8, Lemmas 2.1, 2,2]. The readers can also refer to [6].

Lemma 2.1. *Suppose that $\overline{f}(x,t)$ satisfies (2). The minimizer u_ε of (3) satisfies $-1 \le u_\varepsilon \le 1$ on D.*

Lemma 2.2. *Suppose that $\overline{f}_1(x,t)$ and $\overline{f}_2(x,t)$ both satisfy (2) and the same regularity assumption on $\overline{f}$. Assume that $\eta_i \in H^1(D)$ satisfy $-1 \le \eta_i \le 1$ on D for $i = 1,2$. Let $u_{\varepsilon,i}$ be a corresponding minimizer of (3), where $\overline{f} = \overline{f}_i$ and $\eta = \eta_i$, $i = 1,2$. Suppose that $\overline{f}_1(x,t) \ge \overline{f}_2(x,t)$ for all $(x,t) \in \overline{D} \times [-1,1]$ and $1 \ge \eta_1 \ge \eta_2 \ge -1$. Then $u_{\varepsilon,1} \ge u_{\varepsilon,2}$.*

3. Proof of Main Theorem.

In this section we prove Theorem 1.1. The following proposition is the first part of Theorem 1.1.

Proposition 3.1. *Let u_ε be a global minimizer of the problem (1). Then u_ε satisfies*

$$u_\varepsilon \to \begin{cases} 1 & \textit{uniformly on any compact subset of } A_- \\ -1 & \textit{uniformly on any compact subset of } A_+ \end{cases}$$

as $\varepsilon \to 0$.

Proof. Let $x_0 \in A_-$. Choose $\delta > 0$ small so that $B_\delta(x_0) \subset\subset A_-$ holds. Take $b \in (\max_{z\in\overline{B_\delta(x_0)}} a(z), 0)$ and define $f_{x_0,\delta,b}(t) = (\min_{z\in B_\delta(x_0)} h(z)^2)(t-b)(1-t^2)$. Then for $x \in \overline{B_\delta(x_0)}$, $t \in [-1,1]$, we have $f(|x|,t) \ge f_{x_0,\delta,b}(t)$. Let $u_{\varepsilon,x_0,\delta,b}$ be the minimizer of

$$\inf\left\{\int_{B_\delta(x_0)} \left(\frac{\varepsilon^2}{2}|\nabla u|^2 - F_{x_0,\delta,b}(u)\right)dx : u+1 \in H_0^1(B_\delta(x_0))\right\},$$

where $F_{x_0,\delta,b}(t) = \int_{-1}^{t} f_{x_0,\delta,b}(s)ds$. It follows from Lemmas 2.1 and 2.2 that we have

$$u_{\varepsilon,x_0,\delta,b}(x) \le u_\varepsilon(x) \le 1, \text{ for } x \in B_\delta(x_0).$$

Since $\int_{-1}^{1} f_{x_0,\delta,b}(s)ds > 0$, it follows from [2, 3] that $u_{\varepsilon,x_0,\delta,b}(x) \to 1$ as $\varepsilon \to 0$ uniformly in $B_{\delta/2}(x_0)$, thus $u_\varepsilon(x) \to 1$ as $\varepsilon \to 0$ uniformly in $B_{\delta/2}(x_0)$. □

To prove the rest of Theorem 1.1, we have to see the minimizer u_ε is radially symmetric. We can say by following proposition.

Proposition 3.2. *Let u be a local minimizer of the following problem:*

$$\inf\left\{\int_{B_1(0)} \left(\frac{1}{2}|\nabla u|^2 - G(|x|,u)\right)dx : u \in H^1(B_1(0))\right\}.$$

Here $G(r,t) = \int_{-1}^{t} g(r,s)ds$, $g(r,t)$ is C^1 in $t \in \mathbb{R}$ for each $r \ge 0$, $g(r,t)$ and $g_t(r,t)$ are measurable on $[0,+\infty)$ for each $t \in \mathbb{R}$, $g(r,t) < 0$ if $t < -1$ or $t > 1$ and $|g(r,t)| + |g_t(r,t)|$ is bounded on $[0,k] \times [-2,2]$ for any $k > 0$. Then u is radial, i.e., $u(x) = u(|x|)$.

Proof. See [4, Proposition 2.6]. □

Before we prove the rest of Theorem 1.1, we prepare following lemma.

Lemma 3.3. *Let $\eta \in (0,1)$ be any fixed constant and w satisfies*

$$\begin{cases} -w_{zz} = w(1-w^2) & \text{on } \mathbb{R}, \\ w(0) = -1+\eta \ (\textit{resp. } w(0) = 1-\eta), & \\ w(z) \le -1+\eta \ (\textit{resp. } w(z) \ge 1-\eta) & \text{for } z \le 0, \\ w \textit{ is bounded on } \mathbb{R}. & \end{cases}$$

Then the function w is a unique solution of

$$\begin{cases} -w_{zz} = w(1-w^2) & \text{on } \mathbb{R}, \\ w(0) = -1+\eta \ (\textit{resp. } w(0) = 1-\eta), & \\ w'(z) > 0 \ (\textit{resp. } w'(z) < 0) & z \in \mathbb{R}, \\ w(z) \to \pm 1 \ (\textit{resp. } w(z) \to \mp 1) & \text{as } z \to \pm\infty. \end{cases}$$

Proof. We can show this lemma by using the phase plane analysis, see for example [7]. □

Now we prove the rest of Theorem 1.1.

Proof of Theorem 1.1. For the sake of simplicity, we prove for the case where $a(r) < 0$ on $[0, r_1)$, $a(r) = 0$ on $[r_1, r_2]$ and $a(r) > 0$ on $(r_2, 1]$ for some $0 < r_1 < r_2 < 1$ (see Figure 1 in Section 1). Note that by Proposition 3.2, u_ε is radially symmetric.

Part 1. First we show that u_ε converges uniformly near the boundary of $B_1(0)$, that is, $u_\varepsilon \to -1$ uniformly on $\overline{B_1(0)} \setminus B_{r_2+\tau}(0)$ for any small $\tau > 0$. We note that we have $u_\varepsilon \to -1$ uniformly on $\overline{B_{1-\tau}(0)} \setminus B_{r_2+\tau}(0)$ as $\varepsilon \to 0$. Now we claim that $u_\varepsilon(r) \le u_\varepsilon(1-\tau) =: T_\varepsilon$ for $r \in [1-\tau, 1]$. We define the function $\tilde{u}_\varepsilon$ as follows:

$$\tilde{u}_\varepsilon(r) = \begin{cases} u_\varepsilon(r) & \text{if } r \in [0, 1-\tau] \\ u_\varepsilon(r) & \text{if } u_\varepsilon(r) < T_\varepsilon \text{ and } r \in [1-\tau, 1], \\ T_\varepsilon & \text{if } u_\varepsilon(r) \ge T_\varepsilon \text{ and } r \in [1-\tau, 1]. \end{cases}$$

We note that $\tilde{u}_\varepsilon \in H^1(B_1(0))$ and $-F(r, T_\varepsilon) \le -F(r,t)$ for $\varepsilon > 0$ and $|r-1|$ small and $t \ge T_\varepsilon$. Hence we obtain $J_\varepsilon(\tilde{u}_\varepsilon) < J_\varepsilon(u_\varepsilon)$ and we have a contradiction if we assume that the measure of the set $\{r \in [0,1] | u_\varepsilon(r) > T_\varepsilon \text{ and } r \in [1-\tau, 1]\}$ is positive. Hence $-1 < u_\varepsilon(r) \le T_\varepsilon$ and $u_\varepsilon \to -1$ uniformly on $\overline{B_1(0)} \backslash B_{r_2+\tau}(0)$.

Part 2. In this part we show that u_ε has exactly one layer near the interval $[r_1, r_2]$. We first note that for any $t_2 > t_1$, u_ε is a minimizer of the following problem

$$\inf\{J_\varepsilon(u, B_{t_2}(0) \setminus \overline{B_{t_1}(0)}) : u - u_\varepsilon \in H_0^1(B_{t_2}(0) \setminus \overline{B_{t_1}(0)})\},$$

where

$$J_\varepsilon(u, M) = \int_M \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 - F(|x|, u) \right\} dx$$

for any open set M. Let $m_{\varepsilon, t_1, t_2}$ be the minimum value of this minimization problem.

Step 2.1. First we estimate the energy of transition layer. Let $\eta > 0$ and $\theta > 0$ be small numbers. Since $u_\varepsilon \to 1$ uniformly on $[0, r_1 - \theta]$, we can find $\overline{r}_\varepsilon \in (r_1 - \theta, r_2 + \theta)$ such that $u_\varepsilon(r) \ge 1 - \eta$ if $r \in [0, \overline{r}_\varepsilon]$, $u_\varepsilon(r) < 1 - \eta$ for $r - \overline{r}_\varepsilon > 0$ small. Since $u_\varepsilon \to -1$ uniformly on $[r_2 + \theta, 1 - \theta]$, we can find $\tilde{r}_\varepsilon > \overline{r}_\varepsilon$ be such that $u_\varepsilon(r) \le -1 + \eta$ if $r \in [\tilde{r}_\varepsilon, 1 - \theta]$, $u_\varepsilon(r) > -1 + \eta$ for $\tilde{r}_\varepsilon - r > 0$ small. We may assume that $\overline{r}_\varepsilon \to \overline{r} \in [r_1, r_2]$ and $\tilde{r}_\varepsilon \to \tilde{r} \in [r_1, r_2]$.

We employ the so-called blow-up argument. Let $v_\varepsilon(t) = u_\varepsilon(\varepsilon t + \overline{r}_\varepsilon)$. Then

$$-v_\varepsilon'' - \varepsilon \frac{N-1}{\varepsilon t + \overline{r}_\varepsilon} v_\varepsilon' = f(\varepsilon t + \overline{r}_\varepsilon, v_\varepsilon),$$

$-1 \le v_\varepsilon \le 1$ and $v_\varepsilon(0) = 1-\eta$. Since $\overline{r}_\varepsilon \to \overline{r} \in [r_1, r_2]$, it is easy to see that $v_\varepsilon \to v$ in $C^1_{\rm loc}(\mathbb{R})$ and

$$-v'' = h(\overline{r})^2(v - v^3),\ t \in \mathbb{R}.$$

and $v(t) \ge 1-\eta$ for $t \le 0$. If we set $v(t) = V(h(\overline{r})t)$, the function $V(t)$ satisfies

$$\begin{cases} -V'' = V - V^3 & \text{on } \mathbb{R}, \\ V(0) = 1-\eta, \\ V'(t) \ge 1-\eta & t \le 0. \end{cases} \tag{4}$$

Hence by Lemma 3.3, the function V is a unique solution for

$$\begin{cases} -V'' = V - V^3 & \text{on } \mathbb{R}, \\ V(0) = 1-\eta, \\ V'(t) < 0 & t \le 0. \\ V(t) \to \pm 1 & \text{as } t \to \mp\infty. \end{cases} \tag{5}$$

Thus, we can find an $R > 0$ large, such that $v(R) = -1+\eta$. Since $v_\varepsilon \to v$ in $C^1_{\rm loc}(\mathbb{R})$, we can find an $R_\varepsilon \in (R-1, R+1)$, such that $v'_\varepsilon(r) < 0$ if $r \in [0, R_\varepsilon]$ and $v_\varepsilon(R_\varepsilon) = -1+\eta$. Hence $u'_\varepsilon(r) < 0$ if $r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + \varepsilon R_\varepsilon]$ and $u_\varepsilon(\overline{r}_\varepsilon + \varepsilon R_\varepsilon) = -1+\eta$. Then we have

$$\begin{aligned} & J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon + \varepsilon R_\varepsilon}(0) \backslash \overline{B_{\overline{r}_\varepsilon}(0)}) \\ = \ & \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) \int_{\overline{r}_\varepsilon}^{\overline{r}_\varepsilon + \varepsilon R_\varepsilon} \left(\frac{\varepsilon^2}{2}|u'_\varepsilon|^2 - F(t, u_\varepsilon) \right) dt \\ = \ & \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) \varepsilon \int_0^{R_\varepsilon} \left(\frac{1}{2}|v'_\varepsilon|^2 - F(\varepsilon t + \overline{r}_\varepsilon, v_\varepsilon) \right) dt \\ = \ & \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1))(\beta_{h(\overline{r})} + O(\eta) + o_\varepsilon(1))\varepsilon, \end{aligned} \tag{6}$$

where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$, $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, $\beta_{h(s)}$ is the positive value defined by

$$\begin{aligned} \beta_{h(s)} &= \int_{-\infty}^{+\infty} \left\{ \frac{1}{2}|w'_{h(s)}(t)|^2 + h(s)^2 \frac{(w_{h(s)}(t)^2 - 1)^2}{4} \right\} dt \\ &= h(s) \int_{-\infty}^{+\infty} \left\{ \frac{1}{2}|V'(t)|^2 + \frac{(V(t)^2-1)^2}{4} \right\} dt \\ &= h(s)\beta_1 \end{aligned}$$

and $w_{h(s)}(t) = V(h(s)t)$ for $s \in [0,1]$. We note that although the function V depends on η, the value

$$\beta_1 = \int_{-\infty}^{+\infty} \left\{ \frac{1}{2}|V'(t)|^2 + \frac{(V(t)^2-1)^2}{4} \right\} dt$$

is independent of η.

Step 2.2. We claim that $u_\varepsilon(r)$ has exactly one layer near the interval $[r_1, r_2]$. To show this, it is sufficient to prove the following claim:

Claim. $\tilde{r}_\varepsilon = \overline{r}_\varepsilon + \varepsilon R_\varepsilon$.

Suppose that this claim is not true. Then we can find a point $t_\varepsilon > \overline{r}_\varepsilon + R_\varepsilon \varepsilon$ such that $u_\varepsilon(r) < -1+\eta$ for $r \in (\overline{r}_\varepsilon + R_\varepsilon\varepsilon, t_\varepsilon)$ and $u_\varepsilon(t_\varepsilon) = -1+\eta$ hold. Thus we can use the blow-up argument again at t_ε to deduce that there exists a point $\tilde{t}_\varepsilon = t_\varepsilon + \varepsilon \tilde{R}_\varepsilon$ such that $u'_\varepsilon(r) > 0$ for $r \in (t_\varepsilon, \tilde{t}_\varepsilon)$ and $u_\varepsilon(\tilde{t}_\varepsilon) = 1-\eta$ hold. We may assume that $t_\varepsilon, \tilde{t}_\varepsilon \to \overline{t}$ as $\varepsilon \to 0$ for some $\overline{t} \in [r_2, r_3]$. Moreover we have

$$J_\varepsilon(u_\varepsilon, B_{\tilde{t}_\varepsilon}(0) \backslash \overline{B_{t_\varepsilon}(0)}) = \omega_{N-1}(t_\varepsilon^{N-1} + o_\varepsilon(1))(\beta_{h(\overline{t})} + O(\eta))\varepsilon + o_\varepsilon(1) \tag{7}$$

Now we claim $\tilde{t}_\varepsilon \geq r_1$. Suppose $\tilde{t}_\varepsilon < r_1$.

Let $F_a(t) = \int_{-1}^{t}(v-a)(1-v^2)dv$. Then for any $t > 0$ small and $s \in [-1+t, 1-t]$,

$$\begin{aligned} & F_a(1-t) - F_a(s) \\ = \;& F_0(1-t) - F_0(s) + F_a(1-t) - F_0(1-t) - F_a(s) + F_0(s) \\ = \;& \left[\frac{(v^2-1)^2}{4}\right]_s^{1-t} - a\int_s^{1-t}(1-v^2)dv \end{aligned} \tag{8}$$

Thus it follows from (8) that if $a < 0$ then

$$F_a(1-t) - F_a(s) > 0 \tag{9}$$

for $s \in [-1+t, 1-t]$. Define

$$\overline{u}_\varepsilon(r) := \begin{cases} 1-\eta & r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + R_\varepsilon \varepsilon] \cup [t_\varepsilon, \tilde{t}_\varepsilon], \\ -u_\varepsilon(r) & r \in [\overline{r}_\varepsilon + R_\varepsilon\varepsilon, t_\varepsilon]. \end{cases}$$

By the assumption that $\tilde{t}_\varepsilon < r_1$ and using (9), we see $F(r, u_\varepsilon) < F(r, \overline{u}_\varepsilon)$ if $r \in [\overline{r}_\varepsilon, \tilde{t}_\varepsilon]$. Hence, we obtain

$$J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) < J_\varepsilon(u_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}).$$

Thus we obtain a contradiction. Therefore we have that $\tilde{t}_\varepsilon \geq r_1$.

Since $a(r) \geq 0$ for $r \in [r_1, 1]$, we see $F(r,t) \leq F(r,-1) = 0$ if $r \in [r_1, 1]$. Since $u_\varepsilon(r) \in (-1, -1+\eta)$ for $r \in [\overline{r}_\varepsilon + R_\varepsilon\varepsilon, t_\varepsilon]$, we have

$$\begin{aligned} m_{\varepsilon,\overline{r}_\varepsilon,\tilde{r}_\varepsilon} \;=\;& J_\varepsilon(\overline{u}_\varepsilon, B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{t}_\varepsilon}(0)\backslash\overline{B_{t_\varepsilon}(0)}) \\ & + J_\varepsilon(\overline{u}_\varepsilon, B_{t_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)}) + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon}(0)\backslash\overline{B_{\tilde{t}_\varepsilon}(0)}) \\ \geq\;& \omega_{N-1}(\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}\varepsilon + t_\varepsilon^{N-1}\beta_{h(\overline{t})}\varepsilon) + O(\eta\varepsilon) + o(\varepsilon) \\ & + \inf\left\{-\int_{B_{t_\varepsilon}(0)\backslash B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)} F(r,w)dx : -1 \leq w \leq 1+\eta\right\} \\ & + \inf\left\{-\int_{B_{\tilde{r}_\varepsilon}(0)\backslash B_{\tilde{t}_\varepsilon}(0)} F(r,w)dx : -1 \leq w \leq 1\right\} \\ \geq\;& \omega_{N-1}(\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}\varepsilon + t_\varepsilon^{N-1}\beta_{h(\overline{t})}\varepsilon) + O(\eta\varepsilon) + o(\varepsilon) \end{aligned} \tag{10}$$

Now we give an upper bound for $m_{\varepsilon,\overline{r}_\varepsilon,\tilde{r}_\varepsilon}$. Let $R > 0$ be such that $V(h(\overline{r})R) = -1+\eta$, where V is a unique solution to (5). Define $\overline{u}_\varepsilon$ as follows:

$$\overline{u}_\varepsilon(r) := \begin{cases} V(h(\overline{r})\frac{r-\overline{r}_\varepsilon}{\varepsilon}) & r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + \varepsilon R] \\ -1+\eta - \frac{\eta}{\varepsilon}(r - \overline{r}_\varepsilon - \varepsilon R) & r \in [\overline{r}_\varepsilon + \varepsilon R, \overline{r}_\varepsilon + \varepsilon R + \varepsilon] \\ -1 & r \in [\overline{r}_\varepsilon + \varepsilon R + \varepsilon, \tilde{r}_\varepsilon - \varepsilon] \\ -1 + \frac{\eta}{\varepsilon}(r - \tilde{r}_\varepsilon + \varepsilon) & r \in [\tilde{r}_\varepsilon - \varepsilon, \tilde{r}_\varepsilon] \end{cases} \tag{11}$$

Now we note that $|F(r,t)| = O(\eta)$ for $r \in [\overline{r}_\varepsilon, \tilde{r}_\varepsilon]$ and $-1 \leq t \leq -1+\eta$. Then we have

$$\begin{aligned} m_{\varepsilon,\overline{r}_\varepsilon,\tilde{r}_\varepsilon} \;\leq\;& J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) \\ \leq\;& J_\varepsilon(\overline{u}_\varepsilon, B_{\overline{r}_\varepsilon+R\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon}(0)}) + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon}(0)\backslash\overline{B_{\tilde{r}_\varepsilon-\varepsilon}(0)}) \\ & + J_\varepsilon(\overline{u}_\varepsilon, B_{\tilde{r}_\varepsilon-\varepsilon}(0)\backslash\overline{B_{\overline{r}_\varepsilon+\varepsilon R}(0)}) \\ \leq\;& \omega_{N-1}\overline{r}_\varepsilon^{N-1}(\beta_{h(\overline{r})} + O(\eta))\varepsilon + o(\varepsilon) + O(\varepsilon\eta) + o(\varepsilon) \\ =\;& \omega_{N-1}\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})} + O(\eta\varepsilon) + o(\varepsilon) \end{aligned} \tag{12}$$

By (10) and (12), we have

$$\omega_{N-1}(\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}+t_\varepsilon^{N-1}\beta_{h(\overline{t})})\varepsilon \le \omega_{N-1}\overline{r}_\varepsilon^{N-1}\beta_{h(\overline{r})}\varepsilon + O(\varepsilon\eta)+o(\varepsilon)$$

This is a contradiction. So we can conclude $\tilde{r}_\varepsilon = \overline{r}_\varepsilon + \varepsilon R_\varepsilon$.

Part 3. It remains to prove that if $\overline{r}_{\varepsilon_j} \to \overline{r}$ for some positive sequence $\{\varepsilon_j\}$ converging to zero as $j \to \infty$, then $\overline{r}$ satisfies

$$\overline{r}^{N-1}h(\overline{r}) = \min_{s\in[r_1,r_2]} s^{N-1}h(s). \tag{13}$$

Step 3.1. First we note that from Part 1, the function u_ε satisfies $-1 \le u_\varepsilon \le -1+\eta$ for $r \in [\overline{r}_\varepsilon + \varepsilon R_\varepsilon, 1]$ in this case.

Step 3.2. Set $H(s) = s^{N-1}h(s)$. Assume that the result (13) is not true. Then there exists a subsequence of $\{\overline{r}_\varepsilon\}$ (denoted by $\overline{r}_\varepsilon$) such that $\overline{r}_\varepsilon \to r' \in [r_1, r_2]$ and $H(r') > \min_{s\in[r_1,r_2]} H(s)$. Then we can find a point $\overline{t} \in (r_1, r_2)$ such that $H(r') > H(\overline{t})$.

Next we give a lower estimate for $J_\varepsilon(u_\varepsilon)$. We divide $J(u_\varepsilon)$ into three terms as follows:

$$\begin{aligned} J_\varepsilon(u_\varepsilon) \;&=\; J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon}(0)) + J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}(0)\setminus B_{\overline{r}_\varepsilon}(0)) \\ &\quad + J_\varepsilon(u_\varepsilon, B_1(0)\setminus \overline{B_{\overline{r}_\varepsilon+R_\varepsilon\varepsilon}(0)}). \end{aligned} \tag{14}$$

In order to estimate for $J(u_\varepsilon, B_{\overline{r}_\varepsilon}(0))$, we note that $1-\eta \le u_\varepsilon(r) \le 1$ for $r \le \overline{r}_\varepsilon$ and for sufficiently small $\eta > 0$, $-F$ satisfies $-F(r,u) \ge -F(r,1)$ for $u \in [1-\eta, 1]$. We also remark that since a satisfies $a(r) < 0$ for $r < r_1$, $a(r) = 0$ for $r_1 \le r \le r_2$ and $a(r) > 0$ for $r > r_2$, we have $-F(r,1) < 0$ for $r < r_1$ and $-F(r,1) = 0$ for $r_1 \le r \le r_2$ and $-F(r,1) > 0$ for $r > r_2$. Hence we have $-\int_{r_1}^{\overline{r}_\varepsilon} r^{N-1}F(r,1)dr \ge 0$ and we obtain the following estimate

$$\begin{aligned} J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon}(0)) \;&\ge\; -\int_0^{\overline{r}_\varepsilon} r^{N-1}F(r,u_\varepsilon)dr \\ &\ge\; -\int_0^{\overline{r}_\varepsilon} r^{N-1}F(r,1)dr \\ &=\; -\int_0^{r_1} r^{N-1}F(r,1)dr - \int_{r_1}^{\overline{r}_\varepsilon} r^{N-1}F(r,1)dr \\ &\ge\; -\int_0^{r_1} r^{N-1}F(r,1)dr =: A. \end{aligned}$$

We can obtain

$$J_\varepsilon(u_\varepsilon, B_{\overline{r}_\varepsilon+R_\varepsilon\varepsilon}(0)\setminus B_{\overline{r}_\varepsilon}(0)) \ge \omega_{N-1}H(r')\beta_1\varepsilon + O(\eta\varepsilon)+o(\varepsilon). \tag{15}$$

by methods similar to the proof of (6).

Similarly in order to estimate for $J(u_\varepsilon, B_1(0)\setminus\overline{B_{\overline{r}_\varepsilon+R_\varepsilon\varepsilon}(0)})$ we note that $-1 \le u_\varepsilon(r) \le -1+\eta$ for $r \ge \overline{r}_\varepsilon + \varepsilon R_\varepsilon$ and for sufficiently small $\eta > 0$, $-F$ satisfies $-F(r,u) \ge -F(r,-1) = 0$ for $u \in [-1, -1+\eta]$. Thus we obtain the following estimate:

$$\begin{aligned} J_\varepsilon(u_\varepsilon, B_1(0)\setminus B_{\overline{r}_\varepsilon+R_\varepsilon\varepsilon}(0)) \;&\ge\; -\int_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}^1 r^{N-1}F(r,u_\varepsilon)dr \\ &\ge\; -\int_{\overline{r}_\varepsilon+\varepsilon R_\varepsilon}^1 r^{N-1}F(r,-1)dr = 0. \end{aligned} \tag{16}$$

As the result we obtain following lower estimate for $J(u_\varepsilon)$:

$$J(u_\varepsilon) \geq A + \omega_{N-1} H(r')\beta_1 \varepsilon + O(\eta\varepsilon) + o(\varepsilon). \tag{17}$$

Next we give an upper bound for $J_\varepsilon(u_\varepsilon)$. Consider the following function $\overline{w}_\varepsilon$:

$$\overline{w}_\varepsilon(r) := \begin{cases} 1 & r \in [0, \bar{t} - \varepsilon] \\ 1 - \frac{\eta}{\varepsilon}(r - \bar{t} + \varepsilon) & r \in [\bar{t} - \varepsilon, \bar{t}] \\ V\left(h(\bar{t})\frac{r - \bar{t}}{\varepsilon}\right) & r \in [\bar{t}, \bar{t} + \varepsilon R'] \\ -1 - \frac{\eta}{\varepsilon}(r - \bar{t} - \varepsilon R' - \varepsilon) & r \in [\bar{t} + \varepsilon R', \bar{t} + \varepsilon R' + \varepsilon] \\ -1 & r \in [\bar{t} + \varepsilon R' + \varepsilon, 1], \end{cases}$$

where $R' > 0$ is the number satisfying $V(h(\bar{t})R') = -1 + \eta$. Then we can see

$$J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\overline{w}_\varepsilon) \leq A + \omega_{N-1} H(\bar{t})\beta_1 \varepsilon + O(\eta\varepsilon) + o(\varepsilon). \tag{18}$$

By (17) and (18) we have a contradiction. In more complicated case, we can show similarly(see [8]). The proof of Theorem 1.1 is completed. □

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REFERENCES

[1] S. B. Angenent, J. Mallet-Paret, and L. A. Peletier, *Stable transition layers in a semilinear boundary value problem*, J. Differential Equations, **67** (1987), 212-242.

[2] Ph. Clément and L. A. Peletier, *On a nonlinear eigenvalue problem occurring in population genetics*, Proc. Royal Soc. Edinburgh, **100A**(1985), 85-101.

[3] Ph. Clément and G. Sweers, *Existence of multiplicity results for a semilinear eigenvalue problem*, Ann. Scuola Norm. Sup. Pisa, **14**(1987), 97-121

[4] E. N. Dancer, S. Yan, *Construction of various type of solutions for an elliptic problem*, Calculus of Variations and Partial Differential Equations **20**(2004), 93-118.

[5] P.C. Fife, "Mathematical Aspects of Reacting and Diffusing Systems". Lecture Notes in Biomathematics, Vol 28, Springer-Verlag, Berlin 1979.

[6] K. Kurata and H. Matsuzawa, *Multiple stable patterns in a balanced bistable equation with heterogeneous environments*, in preparation.

[7] H. Matsuzawa, *Stable transition layers in a balanced bistable equation with degeneracy*, Nonlinear Analysis **58** (2004), 45-67.

[8] H. Matsuzawa, *Asymptotic profile of a radially symmetric solution with transition layers for an unbalanced bistable equation*, Electron. J. Differential Equations 2006, No. 5, 1-12

[9] A. S. do. Nascimento, *Stable transition layers in a semilinear diffusion equation with spatial inhomogeneities in N- dimensional domains*, J. Differential Equations, **190** (2003), 16-38.

[10] K. Nakashima, *Multi-layered stationary solutions for a spatially inhomogeneous Allen-Cahn equation*, J. Differential Equations, **191** (2003), 234-276.

[11] K. Nakashima, K. Tanaka, *Clustering layers and boundary layers in spatially inhomogeneous phase transition problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 107-143.

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