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WRONSKIAN SOLUTIONS TO INTEGRABLE EQUATIONS

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Abstract. Wronskian determinants are used to construct exact solution to integrable equations. The crucial steps are to apply Hirota's bilinear forms and explore linear conditions to guarantee the Plücker relations. Upon solving the linear conditions, the resulting Wronskian formulations bring solution formulas, which can yield solitons, negatons, positions and complexitons. The solution process is illustrated by the Korteweg-de Vries equation and applied to the Boussinesq equation.

1. **Introduction.** There are different solution techniques for integrable equations: the inverse scattering transform, the Hamiltonian-Jacobi method, Hirota's bilinear method, Bäcklund and Darboux transformations, symmetry and Lie group method, etc. The resulting solutions contain determinant type solutions: Wronskian, Casoratian, Grammian and Pfaffian solutions.

The so-called Wronskian technique is a tool for constructing Wronskian solutions to integrable equations [1]-[5]. With Wronskian formulations, soliton solutions and rational solutions are often expressed as some kind of rational functions of Wronskian determinants, and the involved determinants are made of eigenfunctions satisfying both uncoupled and coupled linear systems of differential equations.

One of the clear advantages of the Wronskian technique is that it can yield different kinds of exact solutions to integrable equations [4]. Let us first check the case of linear ordinary differential equations (ODEs) with constant-coefficients to see what kinds of solutions such ODEs can have. Take the following differential equation:

$$a_0\frac{d^3x}{dt^3}+a_1\frac{d^2x}{dt^2}+a_2\frac{dx}{dt}+a_3x=0,\ a_i=\text{const.},$$

as an example. Its characteristic equation $a_0\lambda^3+a_1\lambda^2+a_2\lambda+a_3=0$ has the following four cases of solutions:

- Three real roots: λ_1, λ_2 and λ_3.
- One real root and one double real root: $\lambda_1\neq\lambda_2=\lambda_3$.
- One triple real root: $\lambda_1=\lambda_2=\lambda_3$.
- One real root and two conjugate complex roots: λ_1 and $\lambda_{2,3}=\alpha\pm\beta i$ ($\beta\neq 0$).

It then follows that the considered differential equation has and only has the following elementary function solutions:

$$c_1\,e^{\lambda_1 t}+c_2\,e^{\lambda_2 t}+c_3\,e^{\lambda_3 t},\ c_1\,e^{\lambda_1 t}+c_2\,e^{\lambda_2 t}+c_3t\,e^{\lambda_2 t},$$
$$c_1\,e^{\lambda_1 t}+c_2t\,e^{\lambda_1 t}+c_3t^2\,e^{\lambda_1 t},\ c_1\,e^{\lambda_1 t}+e^{\alpha t}(c_2\cos\beta t+c_3\sin\beta t),$$

where c_1, c_2 and c_3 are arbitrary real constants. These solutions involve two kinds of transcendental functions: exponential functions and trigonometric functions. The

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first three classes of solutions correspond to negatons including solitons. The last class of solutions with $\alpha = 0$ and $c_1 = 0$ and the last class of solutions with $\alpha \neq 0$ correspond to positons and complexitons to integrable equations, respectively.

Now, it is natural to ask whether one can present those kinds of exact solutions to integrable equations. In this report, we would like to show that besides soliton solutions and rational solutions, the Wronskian technique can be used to construct positon solutions [3][6]-[8] – solutions involving one kind of transcendental waves: trigonometric waves, and complexitons solutions [9]-[11] – solutions involving two kinds of transcendental waves: exponential waves and trigonometric waves.

Complexitons are a novel kind of exact solutions to integrable equations, introduced recently and generated by analytical methods such as the Wronskian and Casoratian techniques [9, 10, 12] and the Darboux transformation technique [11, 13]. They correspond to complex eigenvalues of associated characteristic linear problems and yield solitons, negatons and positons as some limit cases of the complex eigenvalues [4, 14].

This report is organized as follows. In Section 2, the Wronskian technique is briefly illustrated through the Korteweg-de Vries equation. In Sections 3-5, a Wronskian formulation is established for the Boussinesq equation, the representative systems of the linear conditions are solved, and examples of Wronskian solutions are presented, along with an idea to generate Wronskian solutions of high order. Concluding remarks are given finally in Section 6.

2. **The Wronskian technique.** We adopt the compact notation introduced by Freeman and Nimmo [2, 15]:

$$\mathrm{W}(\phi_1, \phi_2, \cdots, \phi_N) = (\widehat{N-1}; \Phi) = (\widehat{N-1}) = \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \quad (1)$$

where

$$\Phi = (\phi_1, \cdots, \phi_N)^T,\ \phi_i^{(0)} = \phi_i,\ \phi_i^{(j)} = \frac{\partial^j}{\partial x^j}\phi_i,\ j \geq 1,\ 1 \leq i \leq N. \quad (2)$$

Taking advantage of the transformation $u = -2(\ln f)_{xx}$, the Korteweg-de Vries (KdV) equation $u_t - 6uu_x + u_{xxx} = 0$ becomes [16]

$$(D_x D_t + D_x^4) f \cdot f = 2(f_{xt} f - f_t f_x + f_{xxxx} f - 4 f_{xxx} f_x + 3 f_{xx}^2) = 0, \quad (3)$$

where D_x and D_t are Hirota derivatives [17]. Hirota's bilinear equations play an extremely important role in the field of integrable equations. Solutions determined by $u = -2(\ln f)_{xx}$ with $f = (\widehat{N-1})$ to the KdV equation are called Wronskian solutions.

Wronskian solutions require

$$-\phi_{i,xx} = \lambda_i \phi_i,\ \phi_{i,t} = -4\phi_{i,xxx},\ 1 \leq i \leq N. \quad (4)$$

Three types of eigenfunctions are given by

$$\begin{aligned} &\phi_i = c_{1i} x + c_{2i},\ \lambda_i = 0, \\ &\phi_i = c_{1i} \cosh(\eta_i x - 4\eta_i^3 t) + c_{2i} \sinh(\eta_i x - 4\eta_i^3 t),\ \eta_i = \sqrt{-\lambda_i},\ \lambda_i < 0, \\ &\phi_i = c_{1i} \sin(\eta_i x + 4\eta_i^3 t) + c_{2i} \cos(\eta_i x + 4\eta_i^3 t),\ \eta_i = \sqrt{\lambda_i},\ \lambda_i > 0. \end{aligned}$$

Soliton solutions $u = -2(\ln f)_{xx}$ are associated with

$$\begin{cases} \phi_i = \cosh(\eta_i x - 4\eta_i^3 t + \gamma_i), \ \gamma_i = \text{const.}, \ i \text{ odd}, \\ \phi_i = \sinh(\eta_i x - 4\eta_i^3 t + \gamma_i), \ \gamma_i = \text{const.}, \ i \text{ even}. \end{cases}$$

In particular, the 1-soliton and 2-soliton read

$$u = \frac{-2\eta_1^2}{\cosh^2(\eta_1 x - 4\eta_1^3 t + \gamma_1)},$$
$$u = \frac{4(\eta_1^2 - \eta_2^2)[(\eta_2^2 - \eta_1^2) + \eta_1^2 \cosh(2\theta_2) + \eta_2^2 \cosh(2\theta_1)]}{[(\eta_2 - \eta_1)\cosh(\theta_1 + \theta_2) + (\eta_2 + \eta_1)\cosh(\theta_1 - \theta_2)]^2]},$$

where $\theta_i = \eta_i x - 4\eta_i^3 t + \gamma_i, \ \gamma_i = \text{const.}, \ i = 1, 2.$

The generalized Wronskian technique [3, 6] uses derivatives of eigenfunctions with respect to parameters to generate Wronskian solutions:

$$W(\phi_0, \phi_1, \cdots, \phi_n) = W(\phi(\eta), \partial_\eta \phi(\eta), \cdots, \partial_\eta^n \phi(\eta)), \tag{5}$$

starting from

$$-(\phi(\eta))_{xx} = f(\eta)\phi_i(\eta), \ (\phi(\eta))_t = -4(\phi(\eta))_{xxx}, \tag{6}$$

where η is a constant and f is an arbitrary function. The resulting Wronskian solutions are a special class of Wronskian solutions associated with the following sufficient conditions:

$$-\phi_{i,xx} = \sum_{j=1}^{n} \lambda_{ij}\phi_j, \ \phi_{i,t} = -4\phi_{i,xxx}, \ 1 \le i \le N, \tag{7}$$

where the coefficient matrix $\Lambda = (\lambda_{ij})$ is an arbitrary real constant matrix [4, 6].

It is easy to see that two similar coefficient matrices produce the same Wronskian solution, and thus, we only need to consider two types of Jordan blocks of the coefficient matrix Λ:

$$\text{Type 1:} \quad \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i \times k_i},$$

$$\text{Type 2:} \quad \begin{bmatrix} A_i & & & 0 \\ I_2 & A_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & A_i \end{bmatrix}_{l_i \times l_i}, \ A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix},$$

where λ_i, α_i and $\beta_i > 0$ are real constants. The associated Wronskian solutions can be classified as [4, 18]:

- Rational solutions: Jordan block of Type 1 with $\lambda_i = 0$.
- Negaton solutions: Jordan block of Type 1 with $\lambda_i < 0$.
- Positon solutions: Jordan block of Type 1 with $\lambda_i > 0$.
- Complexiton solutions: Jordan block of Type 2, which corresponds to complex eigenvalues.

In addition to rational solutions [19, 20], negatons and positons [21] including harmonic breathers [22], the KdV equation has complexiton solutions [4], which are associated with

$$-\begin{bmatrix}\phi_{i1,xx}\\ \phi_{i2,xx}\end{bmatrix}=A_i\begin{bmatrix}\phi_{i1}\\ \phi_{i2}\end{bmatrix},\ \begin{bmatrix}\phi_{i1,t}\\ \phi_{i2,t}\end{bmatrix}=-4\begin{bmatrix}\phi_{i1,xxx}\\ \phi_{i2,xxx}\end{bmatrix},$$

where α_i and β_i are real constants and A_i are defined as before. The n-complexiton of order $(l_1,\cdots,l_n)$ is given by

$$u=-2\partial_x^2\ln W(\phi_{11},\phi_{12},\cdots,\partial_{\alpha_1}^{l_1}\phi_{11},\partial_{\alpha_1}^{l_1}\phi_{12};\cdots;\phi_{n1},\phi_{n2},\cdots,\partial_{\alpha_n}^{l_n}\phi_{n1},\partial_{\alpha_n}^{l_n}\phi_{n2})$$

associated with Jordan blocks of type 2. The n-complexiton is the special solution

$$u=-2\partial_x^2\ln W(\phi_{11},\phi_{12},\cdots,\phi_{n1},\phi_{n2}).$$

The 1-complexiton reads [9]:

$$\begin{aligned}u&=-2\partial_x^2\ln W(\phi_{11},\phi_{12})\\&=\frac{-4\beta_1^2\big[1+\cos(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)\cosh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)\big]}{\big[\Delta_1\sin(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)+\delta_1\sinh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)\big]^2}\\&\quad+\frac{4\alpha_1\beta_1\sin(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)\sinh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)}{\big[\Delta_1\sin(2\delta_1(x-\bar\beta_1 t)+2\kappa_1)+\delta_1\sinh(2\Delta_1(x+\bar\alpha_1 t)+2\gamma_1)\big]^2},\end{aligned}\tag{8}$$

where α_1, $\beta_1>0$, κ_1 and γ_1 are arbitrary real constants, and Δ_1, δ_1, $\bar\alpha_1$, and $\bar\beta_1$ are

$$\Delta_1,\delta_1=\sqrt{\frac{\sqrt{\alpha_1^2+\beta_1^2}\mp\alpha_1}{2}},\ \bar\alpha_1,\bar\beta_1=4\sqrt{\alpha_1^2+\beta_1^2}\pm 8\alpha_1.$$

To sum up, the whole solution process by the Wronskian technique consists of (a) transforming integrable equations into Hirota's bilinear equations, (b) determining linear conditions to guarantee Wronskian determinants to solve given integrable equations, and (c) solving the resulting linear conditions to present and classify Wronskian solutions.

3. **The Boussinesq equation.**

Let us consider the Boussinesq equation [23]-[25]:

$$u_{tt}+(u^2)_{xx}+u_{xxxx}=0,\tag{9}$$

and call it the Boussinesq equation I. Obviously, a general Boussinesq equation

$$v_{tt}+a_1v_{xx}+a_2(v^2)_{xx}+a_3v_{xxxx}=0,\tag{10}$$

where a_i, $1\le i\le 3$, are real numbers and $a_2a_3\ne 0$, is equivalent to the Boussinesq equation I in (9), under the transformation $v(x,t)=-\frac{a_1}{2a_2}+\frac{a_3}{a_2}u(x,\sqrt{a_3}\,t)$ in the case of $a_3>0$. Similarly, in the case of $a_3<0$, a general Boussinesq equation (10) is equivalent to

$$u_{tt}+(u^2)_{xx}-u_{xxxx}=0,\tag{11}$$

under the transformation $v(x,t)=-\frac{a_1}{2a_2}-\frac{a_3}{a_2}u(x,\sqrt{-a_3}\,t)$. We call the equation (11) the Boussinesq equation II.

If we take the transformation $u=6(\ln f)_{xx}$, then the Boussinesq equation I in (9) becomes a bilinear differential equation

$$(D_t^2+D_x^4)f\cdot f=2(ff_{tt}-f_t^2+ff_{xxxx}-4f_xf_{xxx}+3f_{xx}^2)=0,\tag{12}$$

where D_x and D_t are Hirota derivatives [17]. Similarly, under the transformation $u=-6(\ln f)_{xx}$, the Boussinesq equation II in (11) becomes [15, 26]

$$(D_t^2-D_x^4)f\cdot f=2(ff_{tt}-f_t^2-ff_{xxxx}+4f_xf_{xxx}-3f_{xx}^2)=0.\tag{13}$$

As in the KdV case [4], a similar analysis can yield the following consequence on Wronskian solutions described by $u=6(\ln f)_{xx}$ with $f=(\widehat{N-1})$.

Theorem 3.1. *Let $\varepsilon = \pm 1$. If a group of functions $\phi_i = \phi_i(x,t)$, $1 \le i \le N$, satisfies the following linear conditions*

$$\phi_{i,xxx} = \sum_{j=1}^{N} \lambda_{ij}(t)\phi_j,\ \ \phi_{i,t} = \varepsilon\sqrt{3}\,\phi_{i,xx},\ \ 1 \le i \le N, \tag{14}$$

where the λ_{ij}'s are arbitrary real functions of t, then $f = (\widehat{N-1})$ defined by (1) solves the bilinear Boussinesq equation I (12).

Theorem 3.1 tells us that if a group of functions $\phi_i(x,t)$, $1 \le i \le N$, satisfies the linear conditions in (14), then we can get a solution $f = (\widehat{N-1})$ to the bilinear Boussinesq equation I (12). This Wronskian formulation is different from the Wronskian formulation in [12], and one difference is that it can lead to rational solutions of (9) but the formulation in [12] can not. Before we proceed to solve (14), let us observe how the above Wronskian formulation generates solutions more carefully.

Observation I. From the compatibility conditions $\phi_{i,xxxt} = \phi_{i,txxx}$, $1 \le i \le N$, of the conditions (14), we have the equalities

$$\sum_{j=1}^{N} \lambda_{ij,t}\phi_j = 0,\ \ 1 \le i \le N, \tag{15}$$

and thus we see that the Wronskian determinant $\mathrm{W}(\phi_1, \phi_2, \cdots, \phi_N)$ becomes zero, if the coefficient matrix $\Lambda = (\lambda_{ij})$ is dependent on t, i.e., $\Lambda_t \ne 0$.

Observation II. If the coefficient matrix Λ is similar to another matrix M under an invertible constant matrix P, i.e., we have $\Lambda = P^{-1}MP$, then $\tilde{\Phi} = P\Phi$ solves

$$\tilde{\Phi}_{xxx} = M\tilde{\Phi},\ \ \tilde{\Phi}_t = \varepsilon\sqrt{3}\,\tilde{\Phi}_{xx},\ \ \varepsilon = \pm 1,$$

and the resulting Wronskian solutions to the Boussinesq equation (9) are the same:

$$\begin{aligned} u(\Lambda) &= 6\partial_x^2 \ln |\Phi^{(0)}, \Phi^{(1)}, \cdots, \Phi^{(N-1)}| \\ &= 6\partial_x^2 \ln |P\Phi^{(0)}, P\Phi^{(1)}, \cdots, P\Phi^{(N-1)}| = u(M). \end{aligned}$$

Based on Observation I, we only need to consider the reduced case of (14) under $d\Lambda/dt = 0$, i.e., the following conditions:

$$\phi_{i,xxx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j,\ \ \phi_{i,t} = \varepsilon\sqrt{3}\,\phi_{i,xx},\ \ 1 \le i \le N, \tag{16}$$

where $\varepsilon = \pm 1$ and the λ_{ij}'s are arbitrary real constants. Moreover, Observation II tells us that an invertible constant linear transformation on Φ in the Wronskian determinant does not change the corresponding Wronskian solution, and thus, we only have to solve (14) under the Jordan form of Λ.

4. **Solving the representative systems.** Note that the Jordan form of a real matrix Λ has two types of blocks. Therefore, in order to construct Wronskian solutions, we need to solve the following two representative systems:

$$\phi_{xxx} = \lambda\phi + h,\ \ \phi_t = \pm\sqrt{3}\,\phi_{xx}, \tag{17}$$

and

$$\phi_{1,xxx} = \alpha\phi_1 - \beta\phi_2 + h_1,\ \ \phi_{2,xxx} = \beta\phi_1 + \alpha\phi_2 + h_2, \tag{18}$$

$$\phi_{1,t} = \pm\sqrt{3}\,\phi_{1,xx},\ \ \phi_{2,t} = \pm\sqrt{3}\,\phi_{2,xx}, \tag{19}$$

where λ, α and $\beta > 0$ are real constants, and $h = h(x,t)$, $h_1 = h_1(x,t)$ and $h_2 = h_2(x,t)$ are three given functions satisfying the compatibility condition $g_t =$

$\pm\sqrt{3}\,g_{xx}$. The first system corresponds to real eigenvalues of the coefficient matrix Λ, while the second system corresponds to complex eigenvalues of the coefficient matrix Λ. In what follows, we will consider both homogenous and non-homogenous equations in all cases of real and complex eigenvalues.

4.1. **The case of real eigenvalues.** First, let us consider the first representative system (17). In terms of the eigenvalue λ, we will establish solution formulae for two situations of the representative system (17).

4.1.1. *The sub-case of* $\lambda = 0$*:* Obviously, the first differential equation of (17) has a general solution

$$\phi = c_1(t)x^2 + c_2(t)x + c_3(t) + \int_0^x \int_0^\xi \int_0^\eta h(\zeta, t)\, d\zeta\, d\eta\, d\xi. \tag{20}$$

This makes it possible to break down the second differential equation of (17) into

$$c_{1t} = \pm\frac{\sqrt{3}}{2}\, h_x(0,t),\ \ c_{2t} = \pm\sqrt{3}\, h(0,t),\ \ c_{3t} = \pm 2\sqrt{3}\, c_1.$$

Integrating these equations with respect to t yields the solution formula of (17):

$$\begin{aligned}\phi &= c_{1,0}x^2 + c_{2,0}x + c_{3,0} \pm 2\sqrt{3}\, c_{1,0}t \pm \frac{\sqrt{3}}{2}\, x^2 \int_0^t h_x(0,t')\, dt' \\ &\pm \sqrt{3}\, x \int_0^t h(0,t')\, dt' + 3\int_0^t \int_0^{t'} h_x(0,t'')\, dt''\, dt' + \int_0^x \int_0^\xi \int_0^\eta h(\zeta,t)\, d\zeta\, d\eta\, d\xi,\end{aligned} \tag{21}$$

where $c_{1,0}, c_{2,0}$ and $c_{3,0}$ are arbitrary real constants.

4.1.2. *The sub-case of* $\lambda \neq 0$*:* Now, the characteristic equation $\mu^3 = \lambda$ of the first differential equation of (17) has one real root and two conjugate complex roots:

$$\mu_1 = -2a,\ \mu_2 = a + bI,\ \mu_3 = a - bI,\ a = -\frac{1}{2}\sqrt[3]{\lambda},\ b = \frac{\sqrt{3}}{2}\sqrt[3]{\lambda},\ I = \sqrt{-1}. \tag{22}$$

Therefore, the homogeneous equation $\phi_{xxx} = \lambda\phi$ has three fundamental solutions e^{-2ax}, $\mathrm{e}^{ax}\sin(bx)$ and $\mathrm{e}^{ax}\cos(bx)$, and further, an application of variation of parameters leads to the general solution of the non-homogeneous equation $\phi_{xxx} = \lambda\phi + h$:

$$\begin{aligned}\phi &= c_1(t)e^{-2ax} + c_2(t)\mathrm{e}^{ax}\cos bx + c_3(t)\mathrm{e}^{ax}\sin bx \\ &+ \frac{1}{4b^2}\int_0^x \{\mathrm{e}^{-2a(x-\xi)} - \mathrm{e}^{a(x-\xi)}[\sqrt{3}\sin b(x-\xi) + \cos b(x-\xi)]\}\, h(\xi,t)\, d\xi.\end{aligned} \tag{23}$$

Owing to $h_t = \pm\sqrt{3}\, h_{xx}$, the second differential equation $\phi_t = \pm\sqrt{3}\,\phi_{xx}$ of (17) equivalently requires

$$c_{1,t} \mp 4\sqrt{3}\, a^2 c_1 \pm \frac{\sqrt{3}}{4b^2}\,[-h_x(0,t) + 2ah(0,t)] = 0, \tag{24}$$

$$c_{2,t} \mp \sqrt{3}\,[(a^2-b^2)c_2 + 2abc_3] \pm \frac{\sqrt{3}}{4b^2}\,[h_x(0,t) + (a+\sqrt{3}\, b)h(0,t)] = 0, \tag{25}$$

$$c_{3,t} \mp \sqrt{3}\,[(a^2-b^2)c_3 - 2abc_2] \pm \frac{\sqrt{3}}{4b^2}\,[\sqrt{3}\, h_x(0,t) + (\sqrt{3}\, a - b)h(0,t)] = 0. \tag{26}$$

Thus, the solution formula for c_1 is given by

$$c_1(t) = \mathrm{e}^{\pm 4\sqrt{3}\, a^2 t}\Big\{\, c_{1,0} \mp \frac{\sqrt{3}}{4b^2}\int_0^t \mathrm{e}^{\mp 4\sqrt{3}\, a^2 t'}[-h_x(0,t') + 2ah(0,t')]\, dt'\,\Big\}, \tag{27}$$

where $c_{1,0}$ is an arbitrary real constant. The solution of (25) and (26) is given by

$$\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \mathrm{e}^{At} \begin{bmatrix} c_{2,0} \\ c_{3,0} \end{bmatrix} + \mathrm{e}^{At} \int_0^t \mathrm{e}^{-As} \begin{bmatrix} g_2(s) \\ g_3(s) \end{bmatrix} ds, \tag{28}$$

where $c_{2,0}$ and $c_{3,0}$ are arbitrary real constants, the matrix A is defined by

$$A = \mp\sqrt{3} \begin{bmatrix} b^2 - a^2 & -2ab \\ 2ab & b^2 - a^2 \end{bmatrix}, \tag{29}$$

and g_2 and g_3 are defined by

$$\begin{bmatrix} g_2(s) \\ g_3(s) \end{bmatrix} = \mp\frac{\sqrt{3}}{4b^2} \begin{bmatrix} h_x(0,t) + (a + \sqrt{3}\,b)h(0,t) \\ \sqrt{3}\,h_x(0,t) + (\sqrt{3}\,a - b)h(0,t) \end{bmatrix}. \tag{30}$$

Evidently, the coefficient matrix A has a pair of conjugate eigenvalues

$$\tilde{a} + \tilde{b}I,\ \tilde{a} - \tilde{b}I \ \text{ where } \ \tilde{a} = \mp\sqrt{3}\,(b^2 - a^2),\ \tilde{b} = \mp 2\sqrt{3}\,ab. \tag{31}$$

It now follows that

$$\mathrm{e}^{At} = \mathrm{e}^{\tilde{a}t} \begin{bmatrix} \cos\tilde{b}t & -\sin\tilde{b}t \\ \sin\tilde{b}t & \cos\tilde{b}t \end{bmatrix}, \tag{32}$$

by which the general solution to the system of (25) and (26) can be given explicitly.

4.2. **The case of complex eigenvalues.** Let us now consider the second representative system defined by (18) and (19). If we set

$$\phi = \phi_1 + \phi_2 I,\ h = h_1 + h_2 I,\ \lambda = \alpha + \beta I,$$

the system is transformed into a complex form of (17):

$$\phi_{xxx} = \lambda\phi + h,\ \phi_t = \pm\sqrt{3}\,\phi_{xx}. \tag{33}$$

The characteristic equation $\mu^3 = \lambda$ of the associated ordinary differential equation has three distinct complex roots

$$\mu_1 = \sqrt[3]{\lambda},\ \mu_2 = (-\frac{1}{2} + \frac{\sqrt{3}}{2}I)\sqrt[3]{\lambda},\ \mu_3 = (-\frac{1}{2} - \frac{\sqrt{3}}{2}I)\sqrt[3]{\lambda}. \tag{34}$$

So, the solution formula for the first differential equation of (33) is determined by

$$\phi = \nu_1(t)\mathrm{e}^{\mu_1 x} + \nu_2(t)\mathrm{e}^{\mu_2 x} + \nu_3(t)\mathrm{e}^{\mu_3 x} + \frac{1}{(\mu_3 - \mu_2)(\mu_3 - \mu_1)(\mu_2 - \mu_1)} \times$$
$$\int_0^x [(\mu_3 - \mu_2)\mathrm{e}^{\mu_1(x-x')} - (\mu_3 - \mu_1)\mathrm{e}^{\mu_2(x-x')} + (\mu_2 - \mu_1)\mathrm{e}^{\mu_3(x-x')}]h(x',t)dx'. \tag{35}$$

Now, because of $h_t = \pm\sqrt{3}\,h_{xx}$, the second differential equation of (33) engenders

$$\nu_1 = \mathrm{e}^{\pm\sqrt{3}\,\mu_1^2 t}\{\,\nu_{1,0} \pm \frac{\sqrt{3}}{(\mu_3 - \mu_1)(\mu_2 - \mu_1)} \int_0^t \mathrm{e}^{\mp\sqrt{3}\,\mu_1^2 t'} \chi_1(t')\,dt'\,\}, \tag{36}$$

$$\nu_2 = \mathrm{e}^{\pm\sqrt{3}\,\mu_2^2 t}\{\,\nu_{2,0} \mp \frac{\sqrt{3}}{(\mu_3 - \mu_2)(\mu_2 - \mu_1)} \int_0^t \mathrm{e}^{\mp\sqrt{3}\,\mu_2^2 t'} \chi_2(t')\,dt'\,\}, \tag{37}$$

$$\nu_3 = \mathrm{e}^{\pm\sqrt{3}\mu_3^2 t}\{\,\nu_{3,0} \pm \frac{\sqrt{3}}{(\mu_3 - \mu_1)(\mu_3 - \mu_2)} \int_0^t \mathrm{e}^{\mp\sqrt{3}\mu_3^2 t'} \chi_3(t')\,dt'\,\}, \tag{38}$$

where $\chi_i(t') = h_x(0,t') + \mu_i h(0,t'),\ 1 \le i \le 3$, and $\nu_{1,0}, \nu_{2,0}$ and $\nu_{3,0}$ are arbitrary complex constants.

Therefore, the general solution of the second representative system of (18) and (19) is given by (35) with (36)-(38). The real solution to the system of (18) and (19) is obtained by taking the real and imaginary parts of the above general complex solution.

5. **Constructing Wronskian solutions.** Sections 3 and 4 provide a general procedure for constructing Wronskian solutions associated with two types of Jordan blocks of the coefficient matrix Λ. In what follows, we will present two specific examples of Wronskian solutions and an idea to construct Wronskian solutions of higher order. We will only consider the case of $\varepsilon = -1$. The case of $\varepsilon = 1$ is just an action of replacing t with $-t$ in the obtained solutions.

Let us first consider the coefficient matrix:

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -8k^3 \end{bmatrix}, \ k - \text{real const.}, \tag{39}$$

which possesses only real eigenvalues: $0, 0$ and $-8k^3$, and choose the following eigenfunctions of the corresponding linear conditions:

$$\phi_1 = 1, \ \phi_2 = \frac{1}{6}x^3 - \sqrt{3}\,xt, \ \phi_3 = c_1\, e^{-2\xi} + c_2\, e^{\xi}\cos\eta + c_3\, e^{\xi}\sin\eta, \tag{40}$$

where $\xi = k(x + 2\sqrt{3}\,kt)$, $\eta = \sqrt{3}\,k(x - 2\sqrt{3}\,kt)$, and c_1, c_2 and c_3 are arbitrary real constants. Then, the associated Wronskian solution to (12) reads

$$\begin{aligned} f = W(\phi_1,\phi_2,\phi_3) &= 2\,kc_1(kx^2 - 2\sqrt{3}\,kt + x)e^{-2\xi} \\ &-k(c_2kx^2 - \sqrt{3}\,c_3kx^2 - 2\sqrt{3}\,c_2kt + 6c_3kt + c_2x + \sqrt{3}\,c_3x)e^{\xi}\cos\eta \\ &-k(\sqrt{3}\,c_2kx^2 + c_3kx^2 - 6c_2kt - 2\sqrt{3}\,c_3kt - \sqrt{3}\,c_2x + c_3x)e^{\xi}\sin\eta. \end{aligned} \tag{41}$$

Let us second consider the coefficient matrix:

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k^3 \\ 0 & -k^3 & 0 \end{bmatrix}, \ k - \text{real const.}, \tag{42}$$

which possesses both real and complex eigenvalues: 0 and $\pm k^3 I$, and choose the following eigenfunctions of the corresponding linear conditions:

$$\begin{cases} \phi_1 = x, \ \phi_2 = c_1\, e^{\sqrt{3}\,k^2t}\cos(kx) + c_2\, e^{-\frac{1}{2}\eta}\cos[\frac{1}{2}k(x+3kt)], \\ \phi_3 = c_1\, e^{\sqrt{3}\,k^2t}\sin(kx) - c_2\, e^{-\frac{1}{2}\eta}\sin[\frac{1}{2}k(x+3kt)], \end{cases} \tag{43}$$

where $\eta = \sqrt{3}\,k(x + kt)$, and c_1 and c_2 are arbitrary real constants. Then, the associated Wronskian solution to (12) reads

$$\begin{aligned} f = W(\phi_1,\phi_2,\phi_3) &= \frac{1}{2}\,k^2\big[2kxc_1{}^2\, e^{2\sqrt{3}\,k^2t} - (kxc_2{}^2 + \sqrt{3}\,c_2{}^2)\, e^{-\eta} \\ &-2kxc_1c_2\, e^{\xi}\cos(\frac{\sqrt{3}}{2}\,\eta) + 3c_1c_2\, e^{\xi}\sin(\frac{\sqrt{3}}{2}\,\eta) - \sqrt{3}\,c_1c_2\, e^{\xi}\cos(\frac{\sqrt{3}}{2}\,\eta)\big], \end{aligned} \tag{44}$$

where $\xi = \frac{\sqrt{3}}{2}\,k(kt - x)$.

To construct Wronskian solutions associated with Jordan blocks of higher-order, we use the basic idea developed for the KdV equation [6]. Let $\phi_i(\lambda_i)$ satisfy

$$(\phi_i(\lambda_i))_{xxx} = \lambda_i\phi_i(\lambda_i), \ (\phi_i(\lambda_i))_t = -\sqrt{3}\,\phi_i(\lambda_i), \ \lambda_i - \text{real const.}$$

Differentiating $\phi_i(\lambda_i)$ with respect to λ_i leads to the vector function

$$\Phi_i = \Phi_i(\lambda_i) = (\phi_i(\lambda_i), \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i), \cdots, \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i))^T, \tag{45}$$

which satisfies

$$\Phi_{i,xxx} = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i\times k_i} \Phi_i,\ \ \Phi_{i,t} = -\sqrt{3}\,\Phi_{i,xx},$$

where ∂_{λ_i} denotes the derivative with respect to λ_i and k_i is an arbitrary natural number. This set of eigenfunctions produces a generalized Wronskian solution to the Boussinesq equation I (9):

$$u = 6\partial_x^2 \ln W(\phi_i(\lambda_i), \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i), \cdots, \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i)). \tag{46}$$

The case of complex eigenvalues can be similarly dealt with.

6. **Concluding remarks.** We have shed light on the existence of Wronskian solutions to integrable equations. Rational solutions, negatons, positons and complexitons to the Boussinesq equation I are special Wronskian solutions and their interaction solutions can be obtained within the established Wronskian formulation (see [4] for the case of the KdV equation and [10] for the case of the Toda lattice equation). There also exist algebro-geometric solutions to the Boussinesq equation I (9) on the circle [27]. Blow-up phenomena of solutions of (9) were shown in the case of special initial conditions [28]. If the time variable t is replaced with $\sqrt{-1}\,t$, real Wronskian solutions to the bilinear Boussinesq equation I (12) becomes complex Wronskian solutions to the bilinear Boussinesq equation II (13). All these results show the richness of the infinite-dimensional solution space of the Boussinesq equation. Our theory on Wronskian solutions is expected to help understand complete integrability of nonlinear differential equations.

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