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A ROBUST FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS

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Abstract. In this paper, we consider a convection-diffusion boundary value problem with singular perturbation. A finite element method (FEM) is proposed based on discontinuous Galerkin (DG) discretization of least-squares variational formulation. Numerical tests on representative problems reveal that the method is robust and efficient.

1. **Introduction.**

We consider the following convection-diffusion problem

$$\begin{cases} -\epsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x) & \text{in} \ \ \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $0 < \epsilon \ll 1$ is a parameter, $b \in W^{1,\infty}(\Omega)$ with $b \geq \beta > 0$ on $\overline{\Omega}$, $c \in L^\infty(\Omega)$ with $c > 0$ on $\overline{\Omega}$, $f \in L^2(\Omega)$, and

$$c - b'/2 \geq \gamma > 0 \quad \text{on} \quad \overline{\Omega}. \tag{2}$$

Here the constants β and γ are independent of ϵ. Note that hypothesis (2) can be ensured by a change of variable of the form $v = e^{-\sigma x}u$ with suitably chosen σ; see [27] for details. Under the assumptions of initial data, the problem (1) admits a unique solution in $H_0^1(\Omega) \cap H^2(\Omega)$; cf., e.g., [22, 26].

In the convection dominated case (i.e. $\epsilon \ll \beta$), the solution to (1) has a boundary layer at $x = 1$, which causes nonphysical oscillations in the numerical solutions by standard Galerkin FEMs. Over the years many stabilization techniques have been suggested, including upwind, Petrov-Galerkin, streamline diffusion FEMs, and anisotropic mesh adaptation. For an overview of these methods, we refer to, e.g., the books [14, 21, 22, 26] and the references therein. Nonetheless, singularly perturbed problems remain difficult to solve numerically.

The purpose of this paper is to develop a robust approximation of the singularly perturbed problem (1) based on DG discretization of least-squares variational formulation. The least-squares finite element method (LSFEM) has become increasingly popular lately for numerical solutions of boundary value problems. Least-squares principles give rise to unconstrained minimization problems through a variational framework of residual minimization. For linear differential equations, the LSFEM leads to symmetric positive-definite algebraic systems which can be efficiently solved by iterative methods. The LSFEM possesses a series of significant and valuable properties, such as freedom in choosing finite element spaces, optimal error estimates, easy application to a wide range of problems, etc. The LSFEM has been applied to solve many convection-reaction-diffusion problems, see, e.g.,

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[4, 6, 9, 10, 12, 18, 24, 25]. Extensive reviews and studies of the method can be found in [5] and [17].

In spite of its attractive features, the use of the LSFEM is restricted by several disadvantages [7, 9, 10]. In particular, comparing with standard Galerkin FEMs, the least-squares variational formulation requires higher regularity of solution spaces. Moreover, as illustrated in [19] and Section 4 of this paper, the classic LSFEM is inefficient for singularly perturbed problems, especially in and near boundary layers. In order to remedy the defects, discontinuous approximation spaces have been used to discretize least-squares formulations for solving a variety of problems. For example, Cao and Gunzburger [11] used least-squares methods with discontinuous elements to treat interface problems. Gerritsma and Proot [15] derived a discontinuous least-squares spectral element method for a sample first order ordinary differential equation. Bensow and Larson applied discontinuous LSFEMs to elliptic problems [2] and div-curl problems [3] with boundary singularities. Meanwhile, least-squares technique has also been used as a stabilizer of DG methods. For instance, Houston, Jensen, and Süli [16] investigated a general family of hp-discontinuous Galerkin FEMs with least-squares stabilization for symmetric systems of first-order partial differential equations.

Recently, the author proposed a discontinuously discretized LSFEM for singularly perturbed reaction-diffusion problems [19, 20]. The boundary value problem is decomposed into a first-order system to which a suitable weighted least-squares formulation is proposed. DG discretization is performed for the weak formulation. The numerical approach is stable and efficient. We hereby extend the method and develop a robust numerical approximation for singularly perturbed convection-diffusion problems.

The remainder of this paper is organized as follows. Section 2 includes notations utilized in this paper. In Section 3, we present the least-squares variational formulation and the LSFEM for the problem. Coercivity estimate of the bilinear form is proved in an associated norm. In Section 4, numerical examples are given. Conclusions are drawn in Section 5.

2. **Notation.**

In this paper we shall use C to denote a generic positive constant which is independent of the singular perturbation parameter ϵ and of the mesh used. Vectors and scalers will be denoted by bold and plain letters, respectively.

We shall use standard notations for Sobolev spaces throughout this paper. In particular, the inner product in $L^2(\Omega)$ and $[L^2(\Omega)]^2$ are denoted by

$$(v, w) = \int_0^1 vw \, dx \quad \text{and} \quad (\mathbf{v}, \mathbf{w}) = \int_0^1 \mathbf{v} \cdot \mathbf{w} \, dx,$$

respectively, where $\mathbf{v} \cdot \mathbf{w}$ is the vector inner product of $\mathbf{v}$ and $\mathbf{w}$. The associated L^2-norm is denoted as $|| \cdot ||_0$. For $s > 0$, the Sobolev space $H^s(\Omega)$ has norms $|| \cdot ||_t$ and seminorms $| \cdot |_t$, $1 \leq t \leq s$. Similar norms and seminorms can be defined for $[H^s(\Omega)]^2$, which will be denoted also as $|| \cdot ||_t$ and $| \cdot |_t$, respectively, $1 \leq t \leq s$. We recall the space $H_0^1(\Omega)$ consisting of all functions in $H^1(\Omega)$ that vanish at boundary points 0 and 1. By the Poincaré-Friedrichs inequality, $| \cdot |_1$ is a norm on $H_0^1(\Omega)$ equivalent to $|| \cdot ||_1$.

Let $\mathcal{T}_h = \{\Omega_k\}_{k=1}^M = \{[x_{k-1}, x_k]\}_{k=1}^M$ be an equidistance partition of Ω, where $x_k = kh$ for $k = 0, \ldots, M$ with mesh size parameter $h = 1/M$. Note that an anisotropic mesh (e.g. Shishkin mesh) will certainly improve numerical results, which is however not necessary for our method.

We shall use the following broken Sobolev space

$$H^1(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_{\Omega_k} \in H^1(\Omega_k), k = 1, \ldots, M \right\},$$

where $H^1(\Omega_k)$ is the Sobolev space of order 1 on Ω_k. The inner products and norms defined on Ω can be taken over the elements Ω_k, which are denoted by $(\cdot,\cdot)_{\Omega_k}$ and $||\cdot||_{1,\Omega_k}$, respectively. For $v \in H^1(\mathcal{T}_h)$, we define its norm and seminorm as

$$||v||_1^2 = \sum_{k=1}^{M} ||v||_{1,\Omega_k}^2 \quad \text{and} \quad |v|_1^2 = \sum_{k=1}^{M} |v|_{1,\Omega_k}^2, \tag{3}$$

respectively. In (3), we use the same norm notations as in the continuous Sobolev spaces, which will cause no ambiguity. We define $H_0^1(\mathcal{T}_h)$ the subspace of $H^1(\mathcal{T}_h)$ consisting of all functions that vanish at 0 and 1.

Finally, we define the finite element space associated with $\mathcal{T}_h$ as

$$\mathbf{V}^h = V^h \times V_0^h \subset H^1(\mathcal{T}_h) \times H_0^1(\mathcal{T}_h),$$

where $V^h \subset H^1(\mathcal{T}_h)$ is the space of piecewise linear polynomials allowing discontinuity at interelement nodes, and V_0^h is the subspace of V^h which consists of functions vanishing at 0 and 1. A basis of V^h can be $\left\{N_{k-1}^+, N_k^-\right\}_{k=1}^M$, where

$$N_{k-1}^+(x) = \begin{cases} \dfrac{x_k - x}{h} & x \in \overline{\Omega}_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_k^-(x) = \begin{cases} \dfrac{x - x_{k-1}}{h} & x \in \overline{\Omega}_k, \\ 0 & \text{otherwise.} \end{cases}$$

3. Weak formulation and discrete problem.

3. Weak formulation and discrete problem. Problem (1) is equivalent to

$$\begin{cases} p - u' = 0 & \text{in} \quad \Omega, \\ -\epsilon p' + bu' + cu = f & \text{in} \quad \Omega, \\ u(0) = u(1) = 0. \end{cases} \tag{4}$$

Let $\mathbf{u} = \begin{pmatrix} p \\ u \end{pmatrix} \in H^1(\Omega) \times H_0^1(\Omega)$. We define

$$A\mathbf{u} = \begin{pmatrix} \sqrt{\epsilon}(p - u') \\ -\epsilon p' + bu' + cu \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Notice that weight $\sqrt{\epsilon}$ is employed in the first component of $A\mathbf{u}$. Equation (4) can thus be written as

$$A\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega. \tag{5}$$

The homogenous boundary condition in (4) is satisfied since $u \in H_0^1(\Omega)$. Note that problem (1) has a unique solution in $H_0^1(\Omega) \cap H^2(\Omega)$. Therefore problem (5) has a unique solution in $H^1(\Omega) \times H_0^1(\Omega)$.

We need some *a priori* estimates for the solution u. The following stability result can be found in [26].

Lemma 3.1. *Let u be the solution of (1). Then the following a priori estimates hold*

$$|u^{(i)}(x)| \leq C \left(1 + \epsilon^{-i} \exp\left(-\beta \frac{1-x}{\epsilon}\right)\right)$$

for $x \in \Omega$ and $0 \leq i \leq 2$.

As immediate consequences of Lemma 3.1, we have $|u|_2 = O(\epsilon^{-3/2})$ and $|u|_1 = O(\epsilon^{-1/2})$. Therefore the standard Sobolev norm $||u||_2$ or $||u||_1$ does not provide an informative gauge when ϵ is small. It is natural to introduce the following ϵ-dependent norm in $H^1(\Omega)$ (cf. [26])

$$||v||_{1,\epsilon}^2 = \epsilon|v|_1^2 + ||v||_0^2.$$

In addition, we present ϵ-dependent norms in $H^1(\Omega) \times H_0^1(\Omega)$ as

$$\begin{aligned}||\mathbf{v}||_{0,\epsilon}^2 &= \epsilon||q||_0^2 + ||v||_0^2,\\ ||\mathbf{v}||_{1,\epsilon}^2 &= \epsilon^3|q|_1^2 + \epsilon|v|_1^2 + ||\mathbf{v}||_{0,\epsilon}^2,\end{aligned}$$

where $\mathbf{v} = \begin{pmatrix} q \\ v \end{pmatrix}$. Similarly, ϵ-dependent norms can be defined in $H^1(\mathcal{T}_h)$ as

$$||v||_{1,\epsilon}^2 = \sum_{k=1}^{M} ||v||_{1,\epsilon,\Omega_k}^2,$$

where

$$||v||_{1,\epsilon,\Omega_k}^2 = \epsilon|v|_{1,\Omega_k}^2 + ||v||_{0,\Omega_k}^2;$$

and in $H^1(\mathcal{T}_h) \times H_0^1(\mathcal{T}_h)$ as

$$\begin{aligned}||\mathbf{v}||_{0,\epsilon}^2 &= \sum_{k=1}^{M} ||\mathbf{v}||_{0,\epsilon,\Omega_k}^2,\\ ||\mathbf{v}||_{1,\epsilon}^2 &= \sum_{k=1}^{M} ||\mathbf{v}||_{1,\epsilon,\Omega_k}^2,\end{aligned}$$

where

$$\begin{aligned}||\mathbf{v}||_{0,\epsilon,\Omega_k}^2 &= \epsilon||q||_{0,\Omega_k}^2 + ||v||_{0,\Omega_k}^2,\\ ||\mathbf{v}||_{1,\epsilon,\Omega_k}^2 &= \epsilon^3|q|_{1,\Omega_k}^2 + \epsilon|v|_{1,\Omega_k}^2 + ||\mathbf{v}||_{0,\epsilon,\Omega_k}^2.\end{aligned}$$

3.1. **The least-squares formulation.** Define the least-squares functional $\mathcal{J}$: $H^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ as

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2}\|A\mathbf{v} - \mathbf{f}\|_0^2 = \frac{1}{2}\left(A\mathbf{v} - \mathbf{f}, A\mathbf{v} - \mathbf{f}\right). \tag{6}$$

A necessary condition that $\mathbf{u} \in H^1(\Omega) \times H_0^1(\Omega)$ be a minimizer of the functional $\mathcal{J}$ is that its first variation vanishes at $\mathbf{u}$, i.e.

$$\lim_{t\to 0} \frac{d}{dt}\mathcal{J}(\mathbf{u} + t\mathbf{v}) = (A\mathbf{u} - \mathbf{f}, A\mathbf{v}) = 0 \qquad \forall \mathbf{v} \in H^1(\Omega) \times H_0^1(\Omega).$$

The variational formulation corresponding to the least-squares functional (6) thus follows: Find $\mathbf{u} \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \qquad \forall \mathbf{v} \in H^1(\Omega) \times H_0^1(\Omega), \tag{7}$$

where the bilinear form B and the linear functional L are defined as

$$\begin{aligned}B(\mathbf{u}, \mathbf{v}) &= (A\mathbf{u}, A\mathbf{v}),\\ L(\mathbf{v}) &= (\mathbf{f}, A\mathbf{v}).\end{aligned} \tag{8}$$

It is clear that the bilinear form is symmetric. In addition, we have the following coercivity result.

Theorem 3.2. *Assume that* $\sqrt{\epsilon}||c||_{L^\infty(\Omega)} \le ||b||_{L^\infty(\Omega)}$. *There exists a constant* $\alpha > 0$ *independent of* ϵ *such that*

$$B(\mathbf{v},\mathbf{v}) \ge \alpha||\mathbf{v}||_{1,\epsilon}^2 \quad \forall \mathbf{v} \in H^1(\Omega) \times H_0^1(\Omega). \tag{9}$$

Proof. From definition (8), one gets

$$B(\mathbf{v},\mathbf{v}) = \epsilon||q - v'||_0^2 + ||-\epsilon q' + bv' + cv||_0^2.$$

Let $\sigma = \min\{\gamma, 1\}$. Then, we have

$$\begin{aligned} B(\mathbf{v},\mathbf{v}) &\ge \epsilon||q - v' + \sigma v'||_0^2 - 2\sigma\epsilon(q - v', v') - \sigma^2\epsilon||v'||_0^2 \\ &\ge -2\sigma\epsilon(q, v') + \sigma(2-\sigma)\epsilon|v|_1^2 \end{aligned}$$

and

$$\begin{aligned} B(\mathbf{v},\mathbf{v}) &\ge ||-\epsilon q' + bv' + cv - \sigma v||_0^2 + 2\sigma(-\epsilon q' + bv' + cv, v) - \sigma^2||v||_0^2 \\ &\ge -2\sigma\epsilon(q', v) + \sigma(b, (v^2)') + 2\sigma(cv, v) - \sigma^2||v||_0^2 \\ &= 2\sigma\epsilon(q, v') - \sigma(b'v, v) + 2\sigma(cv, v) - \sigma^2||v||_0^2 \\ &\ge 2\sigma\epsilon(q, v') + \sigma(2\gamma - \sigma)||v||_0^2, \end{aligned}$$

where integration by parts, homogeneous boundary conditions of v, and assumption (2) are used. It follows that

$$B(\mathbf{v},\mathbf{v}) \ge \frac{1}{2}\left(\sigma(2-\sigma)\epsilon|v|_1^2 + \sigma(2\gamma-\sigma)||v||_0^2\right) \ge \frac{\sigma^2}{2}||v||_{1,\epsilon}^2. \tag{10}$$

On the other hand, using hypothesis of the theorem, (10), and boundedness of coefficients, we obtain

$$\epsilon||q||_0^2 \le 2\epsilon||q - v'||_0^2 + 2\epsilon||v'||_0^2 \le 2B(\mathbf{v},\mathbf{v}) + \frac{4}{\sigma^2}B(\mathbf{v},\mathbf{v}) \tag{11}$$

and

$$\begin{aligned} \epsilon^3|q|_1^2 &\le 2\epsilon||-\epsilon q' + bv' + cv||_0^2 + 2\epsilon||bv' + cv||_0^2 \\ &\le 2B(\mathbf{v},\mathbf{v}) + 2||b||_{L^\infty(\Omega)}^2\epsilon|v|_1^2 + 2\epsilon||c||_{L^\infty(\Omega)}^2||v||_0^2 \\ &\le 2B(\mathbf{v},\mathbf{v}) + \frac{4}{\sigma^2}||b||_{L^\infty(\Omega)}^2 B(\mathbf{v},\mathbf{v}). \end{aligned} \tag{12}$$

Finally, (9) follows from (10), (11), and (12) by letting $\alpha = \frac{\sigma^2}{8(1+\sigma^2+||b||_{L^\infty(\Omega)}^2)}$. This completes the proof. ☐

The following result is an immediate consequence of Theorem 3.2.

Theorem 3.3. *Let the hypothesis of Theorem 3.2 hold true. Problem (7) has a unique solution in* $H^1(\Omega) \times H_0^1(\Omega)$.

Proof. The result follows from (9), boundedness of the bilinear form $B(\cdot,\cdot)$ and the linear functional $L(\cdot)$, and the Lax-Milgram lemma (cf. [8]). ☐

Remark 1. Numerical experiments show that, when $\epsilon \ll h$, a classic finite element discretization of the least-squares variational problem (7) is not efficient. See Section 4 for details. Therefore, we consider discontinuous finite element spaces.

3.2. **DG discretization.** We next discretize the least-squares formulation (7) with the DG method [1, 13]. Using integration by parts in each element, we get the discrete problem: Find $\mathbf{u}_h = \begin{pmatrix} p_h \\ u_h \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{M}(p_{i-1}^{+}N_{i-1}^{+} + p_i^{-}N_i^{-}) \\ \sum_{j=1}^{M}(u_{j-1}^{+}N_{j-1}^{+} + u_j^{-}N_j^{-}) \end{pmatrix} \in \mathbf{V}^h$ such that

$$B_h(\mathbf{u}_h, \mathbf{v}) = L_h(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}^h, \tag{13}$$

where p_{i-1}^{+}, p_i^{-}, u_{j-1}^{+}, and u_j^{-} are coefficients to be determined, $1 \le i, j \le M$, the bilinear form B_h and the linear functional L_h are defined by

$$\begin{aligned} B_h(\mathbf{u}_h, \mathbf{v}) &= \sum_{k=1}^{M} (A\mathbf{u}_h, A\mathbf{v})_{\Omega_k} \\ &+ \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} (\epsilon c p_h' v + \epsilon p_h (cv)' - bc u_h' v - (bcv)' u_h + \epsilon u_h q' + \epsilon u_h' q)\, dx \\ &+ \sum_{k=1}^{M} [-\epsilon c \widehat{p}_h v + bc\widehat{u}_h v - \epsilon \widehat{u}_h q]_{x_{k-1}}^{x_k} \end{aligned} \tag{14}$$

and

$$L_h(\mathbf{v}) = \sum_{k=1}^{M} (\mathbf{f}, A\mathbf{v})_{\Omega_K} = \sum_{k=1}^{M} (f, -\epsilon q' + bv' + cv)_{\Omega_k}.$$

Here, $\widehat{u}_h$ and $\widehat{p}_h$ are numerical fluxes defined by

$$\widehat{p}_h(x_k) = \lambda p_h(x_k^-) + (1-\lambda) p_h(x_k^+), \tag{15}$$

$$\widehat{u}_h(x_k) = \mu u_h(x_k^-) + (1-\mu) u_h(x_k^+), \tag{16}$$

where $1/2 \le \lambda, \mu \le 1$ are parameters, and $p_h(x_k^{\pm})$ and $u_h(x_k^{\pm})$ are the right-hand limit and left-hand limit of u and p at x_k, respectively, $k = 0, \ldots, M$. Note that $\mathbf{V}^h$ is in general not a subspace of $H^1(\Omega) \times H_0^1(\Omega)$. Our method is therefore nonconforming in this sense.

Remark 2. For continuous functions, the numerical fluxes defined in (15)-(16) are the restrictions of the corresponding functions at associated interelement nodes. A straightforward calculation shows that $B_h(\cdot,\cdot)$ coincides with $B(\cdot,\cdot)$ in space $H^1(\Omega) \times H_0^1(\Omega)$.

Remark 3. A difference between the *discontinuous LSFEMs* in [2, 3, 11, 15] and the method developed in this paper is symmetrization of the discrete problem. We note that, for discontinuous LSFEMs in the addressed papers, standard finite element discretization is conducted for some special least-squares functionals in discontinuous spaces, which leads to symmetric systems of the corresponding functionals. The method in this paper, on the other hand, discretizes a standard least-squares functional with the DG method, which is in general nonsymmetric.

The analogs of Theorems 3.2 and 3.3 can be obtained by similar proofs.

Theorem 3.4. *Let the hypothesis of Theorem 3.2 hold true. Let $\lambda = \mu = 1$. There exists a constant $\widehat{\alpha} > 0$ independent of ϵ such that*

$$B_h(\mathbf{v}, \mathbf{v}) \ge \widehat{\alpha} ||\mathbf{v}||_{1,\epsilon}^2 \quad \forall \mathbf{v} \in \mathbf{V}^h. \tag{17}$$

Theorem 3.5. *Let the hypothesis of Theorem 3.4 hold true. Problem (13) has a unique solution in $\mathbf{V}^h$.*

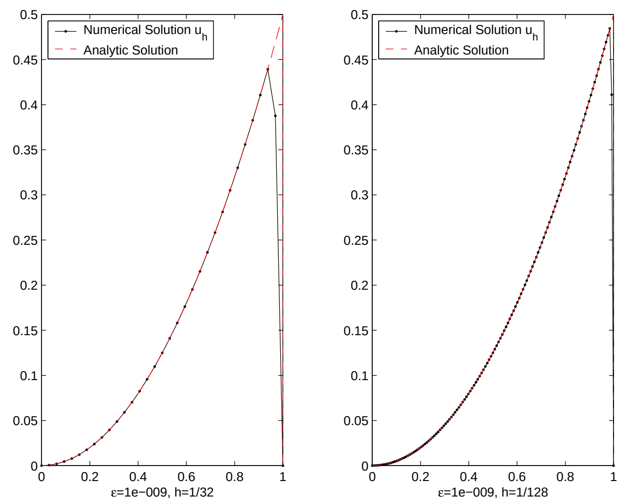

FIGURE 1. Numerical solutions by DG discretized LSFEM

Comparing to the classic LSFEM and DG FEM, the DG discretized LSFEM developed in this paper requires more degrees of freedom (i.e. roughly doubled). The shortcoming can be easily rectified by a natural discontinous/continuous least-squares finite element scheme. In particular, we divide the solution domain Ω into two regions: the *regular solution region* and the *layer region*. In the regular solution region, the exact solution is smooth and the derivatives of the exact solution can be bounded by a constant that is independent of ϵ, where we may use continuous finite elements. In the layer region, the exact solution has large derivatives (cf. Lemma 3.1), where discontinuous elements are employed. The combined method maintains desirable advantages of the discontinuous discretization, which is competitive with the classic LSFEM and DG method in the sense of degrees of freedom. See also [2, 3, 19] for details.

4. **Numerical experiments.** We have computed several test problems to assess the convergence property and efficiency of the method developed in Section 3. In this section we present numerical results of a test problem. The stiffness matrices and load vectors are calculated by symbolic algebra software (Maple$^{\text{TM}}$). High order Gaussian quadrature rules are used to calculate the norms of numerical errors over the computational regions (including the layers), which hereby causes no competitive extra errors in numerical integration. Linear finite elements are used in a set of equidistance meshes of decreasing size for all numerical tests.

Example 4.1. Consider the convection-diffusion equation

$$\begin{cases} -\epsilon u''(x) + e^{-x}u'(x) + u(x) = f \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{18}$$

where ϵ is a parameter to be specified in different tests, and f is selected such that the solution to (18) is

$$u(x) = x\left(\frac{x}{2} + \epsilon\right) - \frac{\left(\frac{1}{2} + \epsilon\right)\left(e^{(x-1)/\epsilon} - e^{-1/\epsilon}\right)}{1 - e^{-1/\epsilon}}. \tag{19}$$

The solution u features a boundary layer at $x = 1$.

TABLE 1. Numerical errors $||\mathbf{u}-\mathbf{u}_h||_{1,\epsilon}$ by DG discretized LSFEM for different ϵ values

$\epsilon\backslash h$	1/16	1/32	1/64	1/128	r.o.c.
10^{-3}	7.9515893e-2	5.6767294e-2	4.0393762e-2	2.8390349e-2	0.495281
10^{-6}	7.9599192e-2	5.6274836e-2	3.9782178e-2	2.8125784e-2	0.500287
10^{-9}	7.9598746e-2	5.6274329e-2	3.9781548e-2	2.8124954e-2	0.500298

We first inspect the accuracy and performance of the proposed DG discretized LSFEM. In Figure 1, we present the computational results of the method with $\epsilon = 10^{-9}$ and mesh parameters $h = 1/32$ and $1/128$, respectively. Here we pick $\lambda = \mu = 1$. The numerical solutions have two traces u_h^+ and u_h^- (one-sided limits) at each interelement node. The u_h plotted are averages of u_h^+ and u_h^-. Figure 1 shows that, even for a quite large mesh size ($h = 1/32$ comparing to $\epsilon = 10^{-9}$), the numerical solutions of the proposed method match the analytical solutions very well. This method hence is very robust and efficient.

In Table 1 we present numerical errors $||\mathbf{u}-\mathbf{u}_h||_{1,\epsilon}$ for $\epsilon = 10^{-3}$, 10^{-6}, and 10^{-9}, and the rate of convergence (r.o.c.) in each case. Here $\lambda = \mu = 1$ are used for all cases. It is observed that the DG discretized LSFEM is numerically independent of ϵ. In addition, when a singular perturbation occurs, we observe error convergence order $\mathcal{O}(h^{1/2})$. This implies that the convergence rate of error $||u - u_h||_{1,\epsilon}$ is also uniformly $\mathcal{O}(h^{1/2})$. The convergence rate can be compared with those obtained in [23] and [27], where uniform error estimates of $\mathcal{O}(h^{1/2})$ are obtained in a similar ϵ-dependent norm for singularly perturbed elliptic problems by using exponentially fitted spline elements. This convergence rate is optimal; cf. also [26].

Problem (18) is also solved using classic LSFEM and upwind method for comparison purpose. In Figure 2, we present the LSFEM numerical results in equidistance meshes with $\epsilon = 10^{-3}$. The continuous least-squares method has a performance depending on the mesh parameter h, which smears out the boundary layer even when h is very small (i.e. less than ϵ). Therefore, the standard LSFEM is inefficient for solving singularly perturbed problems. DG discretization contributes essentially to improving accuracy of the numerical solutions. In Figure 3, we present the results by plain version upwind finite difference/element schemes (cf. [26]) in equidistance meshes with $\epsilon = 10^{-9}$, which visually match the boundary layer. It is well known that, without special treatment, the upwind method does not guarantee convergence in/near the layer (cf. [26]), which has been observed in our numerical experiments. Nevertheless, the convergence rates of our method in Table 1 hold in the entire region including the boundary layer.

5. **Conclusions.** A singularly perturbed convection-diffusion problem with homogeneous Dirichlet boundary conditions is considered in this paper. We have developed a robust numerical approach based on DG discretization of least-squares formulation, which needs no special treatment or mesh. The coercivity of the bilinear form has been proven. Numerical examples illustrate efficiency of the method. This paper provides an alternative to numerical approaches for solving singularly perturbed convection-diffusion problems.

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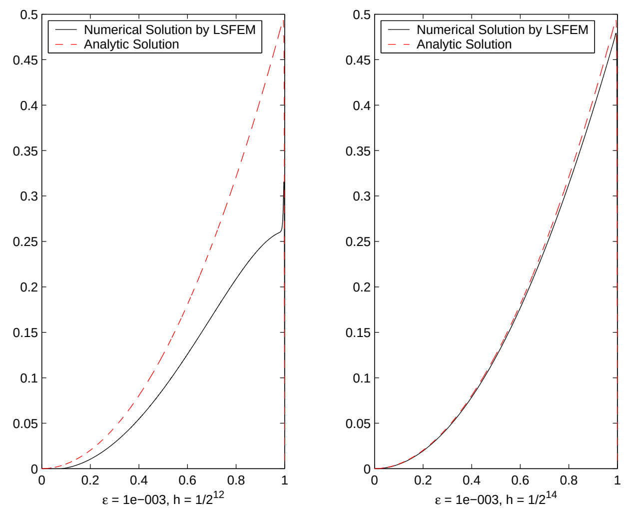

Figure 2. Numerical solutions by classic LSFEM

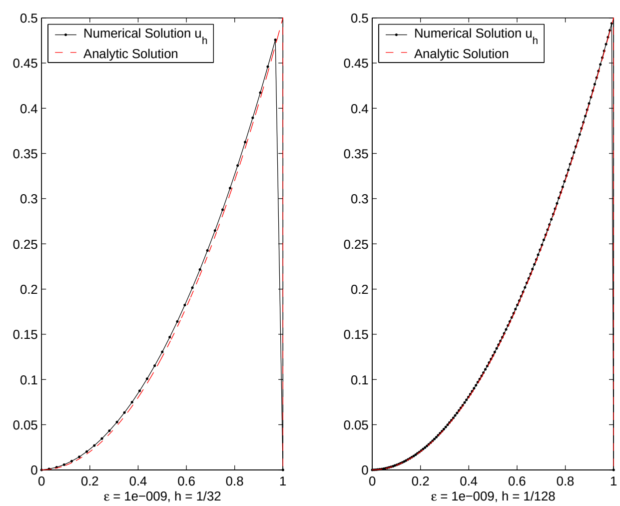

Figure 3. Numerical solutions by upwind method

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