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APPROXIMATING PROBLEMS OF VECTORIAL SINGULAR DIFFUSION EQUATIONS WITH INHOMOGENEOUS TERMS AND NUMERICAL SIMULATIONS

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Abstract. We consider a vectorial nonlinear diffusion equation with inhomogeneous terms in one-dimensional space. In this paper we study approximating problems of singular diffusion equations with a piecewise constant initial data. Also we consider the relationship between the singular diffusion problem and its approximating ones. Moreover we give some numerical experiments for the approximating equation with inhomogeneous terms and a piecewise constant initial data.

1. **Introduction.** In this paper we consider the following vectorial nonlinear diffusion equation with inhomogeneous terms in one-dimensional space, denoted by $(\mathrm{P})^\varepsilon$:

Problem $(\mathrm{P})^\varepsilon$. Find a function $\mathbf{u}^\varepsilon : (0,T)\times(0,L) \to \mathbb{R}^N$ such that

$$\begin{cases} \mathbf{u}^\varepsilon_t - \dfrac{1}{b(x)} \left(a(x) \dfrac{\mathbf{u}^\varepsilon_x}{\sqrt{|\mathbf{u}^\varepsilon_x|^2 + \varepsilon^2}} \right)_x = 0 & \text{a.e. in } (0,T)\times(0,L), \\ \mathbf{u}^\varepsilon(t,0) = \mathbf{g}_0, \quad \mathbf{u}^\varepsilon(t,L) = \mathbf{g}_L & \text{a.e. } t \in (0,T), \\ \mathbf{u}^\varepsilon(0) = \mathbf{u}_0 & \text{a.e. in } (0,L), \end{cases}$$

where $\varepsilon \in (0,1]$ is a given fixed constant, T, L and N are given positive constants with $N \in \mathbb{N}$, the inhomogeneous term $a(x)$ is a given positive continuous function on $[0,L]$, and $b(x)$ is a given positive piecewise continuous function on $[0,L]$. Also $\mathbf{g}_0,\ \mathbf{g}_L \in \mathbb{R}^N$ are given boundary conditions, and $\mathbf{u}_0 \in \mathbb{R}^N$ is a given initial data.

We easily see that $(\mathrm{P})^\varepsilon$ is the approximating problem of the following singular diffusion equation, denoted by (P):

Problem (P). Find a function $\mathbf{u} : (0,T)\times(0,L) \to \mathbb{R}^N$ such that

$$\begin{cases} \mathbf{u}_t - \dfrac{1}{b(x)} \left(a(x) \dfrac{\mathbf{u}_x}{|\mathbf{u}_x|} \right)_x = 0 & \text{a.e. in } (0,T)\times(0,L), \\ \mathbf{u}(t,0) = \mathbf{g}_0, \quad \mathbf{u}(t,L) = \mathbf{g}_L & \text{a.e. } t \in (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{a.e. in } (0,L). \end{cases}$$

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Recently, many mathematicians studied the singular diffusion problem (P) from various view-points (cf. [2, 3, 5, 8, 12, 14, 16, 17]). In the case when $a(x) = b(x) \equiv 1$, the singular diffusion equation appears in the image processing, phase transition phenomena and so on. In the case when $a(x) \neq b(x)$ and $a(x) \neq 1$, the problem (P) appears in material science, such as grain boundary problems (cf. [13]).

Mathematically, Giga et al. [8, 12] showed some properties of solutions to (P) in the case when $N = 1$, and gave some numerical experiments for approximating problems of (P) by approximating the energy functional for (P) (cf. Remark 3.1 below). In the case when $N \geq 1$, Kuroda [14] studied (P) and proved the structure of stationary solutions to (P).

Since (P) has the singular diffusion term $-\left(a(x)\dfrac{\mathbf{u}_x}{|\mathbf{u}_x|}\right)_x$, it is very difficult to study (P) from the view-point of numerical analysis. So, the main object of this paper is to study the approximating problem $(\mathrm{P})^\varepsilon$ by using the idea of [8, 9, 14]. More precisely, for each piecewise constant initial data, we find the piecewise constant solution to $(\mathrm{P})^\varepsilon$, and we give some numerical experiments of $(\mathrm{P})^\varepsilon$.

The plan of this paper is as follows. In Section 2 we mention the main theoretical results of this paper, which are concerned with the solvability of $(\mathrm{P})^\varepsilon$ and the relationship between $(\mathrm{P})^\varepsilon$ and (P). In Section 3 we give some numerical results of $(\mathrm{P})^\varepsilon$ for the fixed (sufficient) small parameter $\varepsilon > 0$.

Assumptions. Throughout this paper, we assume the following conditions:

(A1) $a(x)$ is positive and continuous on $[0, L]$;

(A2) $a(x)$ has local minimums at x_i, $i = 1, \cdots, m-1$, $(m \geq 3)$, where $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = L$;

(A3) $a(x_1) \leq a(x)$ for all $x \in [x_0, x_1]$, $a(x_{m-1}) \leq a(x)$ for all $x \in [x_{m-1}, x_m]$ and $a(x)$ is concave on each interval $[x_i, x_{i+1}]$, $i = 1, \cdots, m-2$, $(m \geq 3)$. Also the function $b(x)$ is positive constant on each interval $(x_i, x_{i+1}]$, $i = 0, \cdots, m-1$, $(m \geq 3)$;

(A4) $\mathbf{u}_0$ is piecewise constant on each interval (x_i, x_{i+1}), $i = 0, \cdots, m-1$, $(m \geq 3)$;

(A5) $\mathbf{g}_0$, $\mathbf{g}_L \in \mathbb{R}^N$.

Remark 1.1. In [8, 14] the following condition (A3)' for $a(x)$ and $b(x)$ is assumed:

(A3)' The function b is positive and continuous on $[0, L]$, $a(x_1) \leq a(x)$ for all $x \in [x_0, x_1]$, $a(x_{m-1}) \leq a(x)$ for all $x \in [x_{m-1}, x_m]$, and

$$\{a(x_{i+1}) - a(x_i)\} / \int_{x_i}^{x_{i+1}} b(\tau)\, d\tau \leq \{a(x) - a(x_i)\} / \int_{x_i}^{x} b(\tau)\, d\tau \quad \text{for all } x \in (x_i, x_{i+1}], \tag{1.1}$$

for all $i = 1, \ldots, m-2$, $(m \geq 3)$.

We easily see that (1.1) holds if the given functions $a(x)$ and $b(x)$ satisfy the condition (A3). Hence we can study Problem (P) by the similar arguments as in [8, 14]. Thus, for simplicity, we assume (A3) in this paper.

2. Mathematical Results.

Throughout this paper, we denote by $L^2(0, L; \mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued square integrable functions. Also we denote by $L^2(0, L; \mathbb{R}^N)_b$ the real Hilbert space $L^2(0, L; \mathbb{R}^N)$ with the inner product

$$\langle \mathbf{z}, \mathbf{v} \rangle_b := \int_0^L b(x)(\mathbf{z}(x), \mathbf{v}(x))dx, \qquad \forall \mathbf{z}, \mathbf{v} \in L^2(0, L; \mathbb{R}^N)_b,$$

where $(\cdot,\cdot)$ is the standard inner product in $\mathbb{R}^N$. Note that $L^2(0,L;\mathbb{R}^N)_b$ is equivalent to $L^2(0,L;\mathbb{R}^N)$ because of the assumption (A3).

Let H_Δ be the set of all $\mathbb{R}^N$-valued piecewise constant functions on $\bigcup_{i=0}^{m-1}(x_i,x_{i+1})$, i.e.,

$$H_\Delta = \left\{ \mathbf{z} = \sum_{i=0}^{m-1} \mathbf{h}_i \chi_{(x_i,x_{i+1})} \ ; \ \begin{array}{c} \mathbf{h}_i \in \mathbb{R}^N \ \text{ for } i = 0,\cdots,m-1 \\ \mathbf{h}_0 = \mathbf{g}_0,\ \mathbf{h}_{m-1} = \mathbf{g}_L \end{array} \right\},$$

where $\chi_{(x_i,x_{i+1})}$ is the characteristic function of (x_i,x_{i+1}) for $i = 0,\cdots,m-1$. Then we easily see that H_Δ is the subset of $L^2(0,L;\mathbb{R}^N)$, and the total variation of $\mathbf{u} \in H_\Delta$ with $a(\cdot)$ is given by this form:

$$\int_0^L a(x)|\mathbf{z}_x| = \sum_{i=1}^{m-1} a(x_i)|\mathbf{h}_i - \mathbf{h}_{i-1}| \quad \text{if } \mathbf{z} \in H_\Delta.$$

For the precise definition and basic properties of total variation, see monographs by Ambrosio et al. [1], Evans-Gariepy [7] or Giusti [10], for instance.

Now we reformulate the problem $(\mathrm{P})^\varepsilon$ as in some evolution equation. To do so, for each $\varepsilon \in (0,1]$, we put

$$\varphi^\varepsilon_\Delta(\mathbf{z}) = \begin{cases} \displaystyle\sum_{i=1}^{m-1} a(x_i)\sqrt{|\mathbf{h}_i - \mathbf{h}_{i-1}|^2 + \varepsilon^2} & \text{if } \mathbf{z} \in H_\Delta, \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

Then it follows from [7, 10] that $\varphi^\varepsilon_\Delta$ is the proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous) and convex function on $L^2(0,L;\mathbb{R}^N)_b$.

By using these notations as above, we easily see that the problem $(\mathrm{P})^\varepsilon$ under (A1)–(A5) can be reformulated as in the following form (cf. [8, 9, 14]):

$$\begin{cases} \dfrac{d}{dt}\mathbf{u}^\varepsilon(t) + \partial\varphi^\varepsilon_\Delta(\mathbf{u}^\varepsilon(t)) = 0 \quad \text{in } L^2(0,L;\mathbb{R}^N)_b \quad \text{for a.e. } t \in (0,T), \\ \mathbf{u}^\varepsilon(0) = \mathbf{u}_0 \in H_\Delta, \end{cases} \tag{2.2}$$

where $\partial\varphi^\varepsilon_\Delta$ is the subdifferential of $\varphi^\varepsilon_\Delta$ in the topology of $L^2(0,L;\mathbb{R}^N)_b$. For the precise definition and basic properties of subdifferential, we refer to the monograph by Brézis [6].

Now let us give the definition of a solution to $(\mathrm{P})^\varepsilon$ under (A1)–(A5).

Definition 2.1. Let $0 < T < \infty$ and $\varepsilon \in (0,1]$. For given data $\mathbf{g}_0,\ \mathbf{g}_L \in \mathbb{R}^N$ and $\mathbf{u}_0 \in H_\Delta$, a function $\mathbf{u}^\varepsilon : [0,T] \times (0,L) \to \mathbb{R}^N$ is called a solution of $(\mathrm{P})^\varepsilon$, if $\mathbf{u}^\varepsilon \in W^{1,2}(0,T;L^2(0,L;\mathbb{R}^N)_b)$ and (2.2) holds.

Now we state the first main result in this paper, which is concerned with the existence-uniqueness of solutions to $(\mathrm{P})^\varepsilon$ for each $\varepsilon \in (0,1]$.

Theorem 2.2 (cf. [8, 12, 14])**.** *Assume* (A1)–(A5)*. Then, for each* $\varepsilon \in (0,1]$ *and* $\mathbf{u}_0 \in H_\Delta$*, there is a unique solution* $\mathbf{u}^\varepsilon$ *of* $(\mathrm{P})^\varepsilon$ *in the sense of Definition 2.1 such that* $\mathbf{u}^\varepsilon(t)$ *is also piecewise constant on each interval* (x_i,x_{i+1}) *for any* $t > 0$ $(i = 0,\cdots,m-1)$.

Proof. We easily see that the problem $(\mathrm{P})^\varepsilon$ can be reformulated as in the evolution equation (2.2). Therefore, by applying the abstract theory established by Brézis [6], we can get the unique solution $\mathbf{u}^\varepsilon$ of $(\mathrm{P})^\varepsilon$ in the sense of Definition 2.1.

Furthermore, by taking account of (2.1), (2.2) and $\mathbf{u_0} \in H_\Delta$, we observe that the problem $(\mathrm{P})^\varepsilon$ can be reduced to the following ODE system (cf. [8, 12, 14]):

(ODE). Find a unique function $\mathbf{u}^\varepsilon(t) = \sum_{i=0}^{m-1} \mathbf{h}_i(t)\chi_{(x_i,x_{i+1})}$ on $[0,\infty)$ such that

$$\begin{cases} \dfrac{d}{dt}\mathbf{h}_i(t) = \dfrac{1}{\int_{x_i}^{x_{i+1}} b(\tau)\,d\tau}\left(a(x_{i+1})\dfrac{\mathbf{h}_{i+1}-\mathbf{h}_i}{\sqrt{|\mathbf{h}_{i+1}-\mathbf{h}_i|^2+\varepsilon^2}} - a(x_i)\dfrac{\mathbf{h}_i-\mathbf{h}_{i-1}}{\sqrt{|\mathbf{h}_i-\mathbf{h}_{i-1}|^2+\varepsilon^2}}\right), \\ \hfill (i=1,2,\cdots,m-2), \\ \dfrac{d}{dt}\mathbf{h}_0(t) = \dfrac{d}{dt}\mathbf{h}_{m-1}(t) = 0, \\ \mathbf{u}^\varepsilon(0) = \mathbf{u}_0 \in H_\Delta. \end{cases}$$

By applying the classical theory of ordinary differential equations (e.g. Cauchy-Lipschitz Theorem), we can get the unique global solution $\mathbf{u}^\varepsilon$ on $[0,\infty)$ to (ODE). Hence we observe that the solution $\mathbf{u}^\varepsilon(t)$ of $(\mathrm{P})^\varepsilon$ is also piecewise constant on each interval (x_i, x_{i+1}) for any $t > 0$ $(i = 0,\cdots,m-1)$. Thus the proof of Theorem 2.2 has been completed. □

By a similar argument, we see that the problem (P) under (A1)–(A5) has a unique solution. In fact, we define

$$\varphi_\Delta(\mathbf{z}) = \begin{cases} \displaystyle\sum_{i=1}^{m-1} a(x_i)|\mathbf{h}_i - \mathbf{h}_{i-1}| & \text{if } \mathbf{z} \in H_\Delta, \\ \infty & \text{otherwise.} \end{cases} \tag{2.3}$$

Then we observe from [7, 10] that φ_Δ is the proper, l.s.c. and convex function on $L^2(0,L;\mathbb{R}^N)_b$. Also we easily see that the problem (P) under (A1)–(A5) can be reformulated as in the following form (cf. [8, 12, 14]):

$$\begin{cases} \dfrac{d}{dt}\mathbf{u}(t) + \partial\varphi_\Delta(\mathbf{u}(t)) \ni 0 \quad \text{in } L^2(0,L;\mathbb{R}^N)_b \quad \text{for a.e. } t \in (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0 \in H_\Delta, \end{cases} \tag{2.4}$$

where $\partial\varphi_\Delta$ is the subdifferential of φ_Δ in the topology of $L^2(0,L;\mathbb{R}^N)_b$.

Now we state the second main result in this paper, which is concerned with the relationship between $(\mathrm{P})^\varepsilon$ and (P).

Theorem 2.3. *Assume* (A1)–(A5). *For each* $\varepsilon \in (0,1]$, *let* $\mathbf{u}^\varepsilon$ *be a unique solution of* $(\mathrm{P})^\varepsilon$ *in the sense of Definition 2.1. Then there is a unique function* $\mathbf{u} \in W^{1,2}(0,T;L^2(0,L;\mathbb{R}^N)_b)$ *such that for each* $T > 0$

$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \quad \text{in } C([0,T];L^2(0,L;\mathbb{R}^N)_b) \quad \text{as } \varepsilon \to 0 \tag{2.5}$$

and (2.4) *holds, namely,* $\mathbf{u}$ *is a unique solution to* (P) *under* (A1)–(A5).

Proof. At first, note from the definitions (2.1) and (2.3) that $\varphi_\Delta^\varepsilon$ converges to φ_Δ in the sense of Mosco as $\varepsilon \to 0$, namely, the following two conditions are satisfied:

(i) For any subsequence $\{\varphi_\Delta^{\varepsilon_{n_k}}\} \subset \{\varphi_\Delta^{\varepsilon_n}\}$, if $\mathbf{z}_k \to \mathbf{z}$ weakly in $L^2(0,L;\mathbb{R}^N)_b$ and $\varepsilon_{n_k} \to 0$ as $k \to \infty$, then, $\liminf_{k\to\infty} \varphi_\Delta^{\varepsilon_{n_k}}(\mathbf{z}_k) \geq \varphi_\Delta(\mathbf{z})$;

(ii) For any $\mathbf{z} \in D(\varphi_\Delta) = H_\Delta$, there are a sequence $\{\varepsilon_n\} \subset (0,1]$ and a sequence $\{\mathbf{z}_n\}$ in $L^2(0,L;\mathbb{R}^N)_b$ such that $\mathbf{z}_n \to \mathbf{z}$ in $L^2(0,L;\mathbb{R}^N)_b$, $\varepsilon_n \to 0$ and $\varphi_\Delta^{\varepsilon_n}(\mathbf{z}_n) \to \varphi_\Delta(\mathbf{z})$ as $n \to \infty$.

For the fundamental properties of Mosco convergence, we refer to [4, 11, 15].

Here, by applying the abstract theory in [4, 6], we see that there is a unique function $\mathbf{u} \in W^{1,2}(0,T;L^2(0,L;\mathbb{R}^N)_b)$ such that for each $T > 0$

$$\mathbf{u}^\varepsilon \longrightarrow \mathbf{u} \quad \text{in } C([0,T];L^2(0,L;\mathbb{R}^N)_b) \quad \text{as } \varepsilon \to 0$$

and (2.4) holds, namely, the solution $\mathbf{u}^\varepsilon$ of $(\mathrm{P})^\varepsilon$ converges to one $\mathbf{u}$ of (P) in the sense of (2.5) as $\varepsilon \to 0$. For the detailed proof, we refer to [4, 6]. Thus the proof of Theorem 2.3 has been completed. □

3. **Numerical results.**

By Theorem 2.3, we see that $(\mathrm{P})^\varepsilon$ is the approximating problem of (P). Since there is no singularity in $(\mathrm{P})^\varepsilon$, it is easy to solve $(\mathrm{P})^\varepsilon$ numerically.

In this section we show some numerical experiments of $(\mathrm{P})^\varepsilon$ under (A1)–(A5) (i.e. (2.2)) for the sufficient small parameter ε in the two cases: $N = 1$ and $N = 2$. Then we can observe mathematical results obtained in [8, 12, 14]. We note from Theorem 2.2 that the problem (2.2) can be reduced to the system (ODE) of ordinary differential equations. So we conduct some numerical simulations of (ODE) by using the standard time-discretization methods.

Here we give the fundamental data for numerical experiments. Assume that

$$\varepsilon = 0.001, \quad T = 10 \ \text{ and the mesh size in time is } \triangle t = 0.001.$$

Moreover, suppose that $L = 8.5$ and the interval $[0,L] = [0,8.5]$ consists of six intervals $[x_i, x_{i+1}]$, $i = 0, 1, \cdots, 5$ so that $[0,L] = \bigcup_{i=0}^{5}[x_i, x_{i+1}]$ with

$$x_0 = 0, \quad x_1 = 2, \quad x_2 = 3.5, \quad x_3 = 4, \quad x_4 = 5, \quad x_5 = 7, \quad x_6 = 8.5 (= L).$$

For simplicity, we assume that the given function

$$b(x) \equiv 1 \text{ for any } x \in [0,L] (= [0,8.5]).$$

3.1. **Scalar-valued setting when $N = 1$.**

We assume that the boundary conditions $g_0 = -1$ and $g_L = 1$. Also we suppose the initial data $u_0 = \sum_{i=0}^{5} h_i \chi_{(x_i,x_{i+1})}$ given by

$$h_0 = g_0 = -1, \quad h_1 = 2, \quad h_2 = h_3 = -1, \quad h_4 = h_5 = g_L = 1.$$

Now let us consider the following two cases, denoted by (a) and (b), respectively.

Case (a). We assume that the given function $a(x)$ has local minimums at x_i, $(i = 1, 2, \cdots, 5)$ such that

$$a(x_1) = 1.3, \quad a(x_2) = 1, \quad a(x_3) = 0.7, \quad a(x_4) = 1.5, \quad a(x_5) = 1.3. \tag{3.1}$$

The graphs of the initial data u_0 and the function $a(x)$ are illustrated in Figure 1. In this case, the function $a(x)$ has a unique global minimum at $x_3 = 4$. For such situations, Kuroda [14] has already proved that there is a unique stationary solution to (P) having a discontinuity only at $x_3 = 4$. For the detailed proof, refer to [14].

We observe from Figure 2 that mathematical results established in [14] holds, although Figure 2 is concerned with the numerical result of $(\mathrm{P})^\varepsilon$ with $\varepsilon = 0.001$ at

$T = 10$. For the detailed behavior of all segments, see Figure 3. We easily see from Figure 3 that the solution of $(\mathrm{P})^{\varepsilon}$ becomes a step function (cf. Figure 2) having a discontinuity only at $x_3 = 4$ after some finite time.

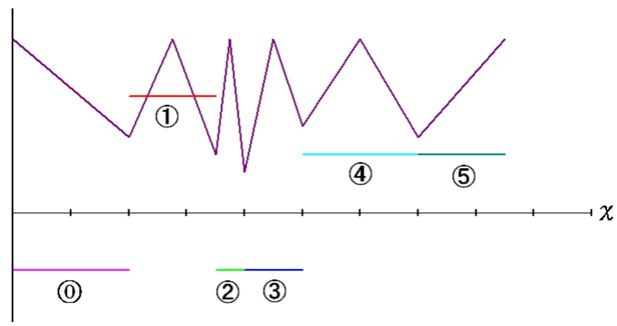

FIGURE 1. Graphs of initial data u_0 and the function $a(x)$ given by (3.1).

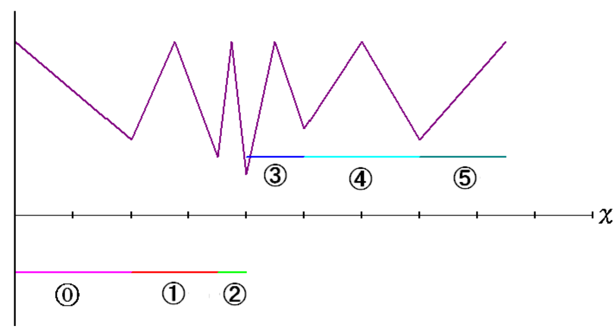

FIGURE 2. $T = 10$ in the case (a).

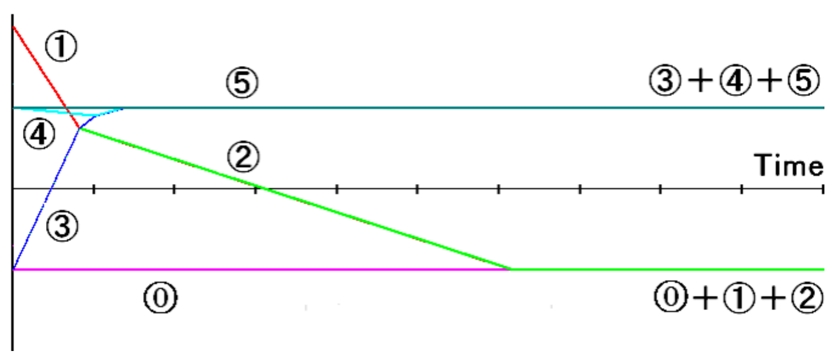

FIGURE 3. Behavior of all segments in the case (a).

Case (b). We assume that the given function $a(x)$ has local minimums at x_i, $(i = 1, 2, \cdots, 5)$ such that

$$a(x_1) = 1.3, \quad a(x_2) = 1, \quad a(x_3) = 0.7, \quad a(x_4) = 1.5, \quad a(x_5) = 0.7. \tag{3.2}$$

In this case, the function $a(x)$ has two global minimums at $x_3 = 4$ and $x_5 = 7$. Then Kuroda [14] has shown that there is at least one stationary solutions to (P) having discontinuities at $x_3 = 4$ and $x_5 = 7$. For the detailed proof, refer to [14].

We observe from Figure 4 that mathematical results established in [14] holds, although Figure 4 is concerned with the numerical result of $(\mathrm{P})^\varepsilon$ with $\varepsilon = 0.001$ at $T = 10$. For the detailed behavior of all segments, see Figure 5. We easily see from Figure 5 that the solution of $(\mathrm{P})^\varepsilon$ becomes a step function (cf. Figure 4) having discontinuities at $x_3 = 4$ and $x_5 = 7$ after some finite time.

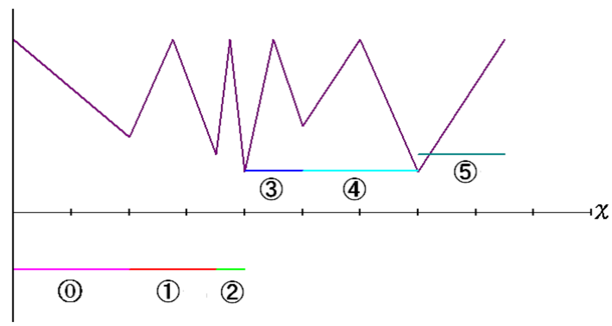

FIGURE 4. $T = 10$ in the case (b).

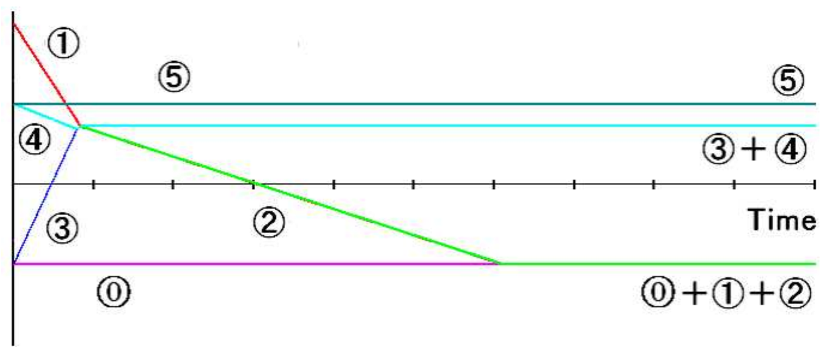

FIGURE 5. Behavior of all segments in the case (b).

Remark 3.1. Giga et al. [8] considered the following approximating problem of (P), denoted by $(\mathrm{P})^\gamma$:

$$(\mathrm{P})^\gamma \begin{cases} u_t^\gamma - \dfrac{1}{b(x)}\left(a(x)\chi_\gamma(u_x^\gamma)u_x^\gamma\right)_x = 0 & \text{a.e. in } (0,T)\times(0,L), \\ u^\gamma(t,0) = g_0, \quad u^\gamma(t,L) = g_L & \text{a.e. } t \in (0,T), \\ u^\gamma(0) = u_0 & \text{a.e. in } (0,L), \end{cases}$$

where $\gamma > 0$ is a given fixed sufficient large constant and $\chi_\gamma(p) = (\tanh \gamma p)/p$. Then Giga et al. [8] gave some numerical results for $(\mathrm{P})^\gamma$. Note that the problem $(\mathrm{P})^\gamma$ is different from $(\mathrm{P})^\varepsilon$, but, we note that the numerical results for $(\mathrm{P})^\varepsilon$ under sufficiently small ε are similar to those for $(\mathrm{P})^\gamma$ under sufficiently large γ. Namely, the large-time behavior of solutions u^γ for $(\mathrm{P})^\gamma$ is similar to one of u^ε for $(\mathrm{P})^\varepsilon$ (cf. Figures 3 and 5).

3.2. **Vector-valued setting when $N = 2$.** We assume that the boundary conditions $\mathbf{g}_0 = (1, -1)$ and $\mathbf{g}_L = (2, 1)$. Also we suppose the initial data $\mathbf{u}_0 = \sum_{i=0}^{5} \mathbf{h}_i \chi_{(x_i, x_{i+1})}$ given by

$$\mathbf{h}_0 = \mathbf{g}_0 = (1, -1), \quad \mathbf{h}_1 = (3, 2), \quad \mathbf{h}_2 = (1, -1), \quad \mathbf{h}_3 = (-0.5, 1),$$
$$\mathbf{h}_4 = (1, -2), \quad \mathbf{h}_5 = \mathbf{g}_L = (2, 1).$$

Now let us consider the following two cases, denoted by (a) and (b), respectively.

Case (a). We consider the same function $a(x)$ defined by (3.1). The graphs of the initial data $\mathbf{u}_0$ and the function $a(x)$ are illustrated in Figure 6. In this case, the function $a(x)$ has a unique global minimum at $x_3 = 4$. For such a case, Kuroda [14] proved that the result of vectorial case is similar to that of scalar case, namely, there is a unique stationary solution to (P) having a discontinuity only at $x_3 = 4$. For the detailed statements, refer to [14].

We observe from Figure 7 that the theoretical results obtained in [14] holds, although Figure 7 is concerned with the numerical result of $(\mathrm{P})^\varepsilon$ with $\varepsilon = 0.001$ at $T = 10$. In Figure 8, we can see the detailed behavior of angle θ between the x-axis and the vector, in which the value θ is taken so that $\theta \in (-\pi, \pi]$. We easily see from Figure 8 that the solution of $(\mathrm{P})^\varepsilon$ becomes a piecewise constant vector-valued function (cf. Figure 7) having a discontinuity only at $x_3 = 4$ after some finite time.

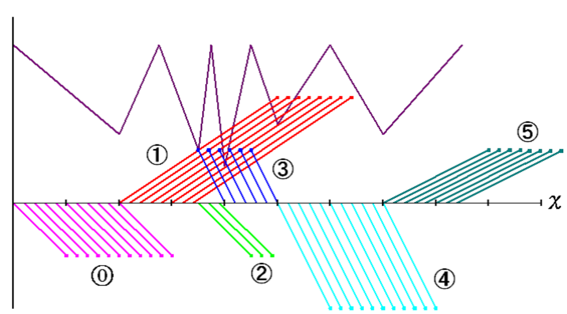

FIGURE 6. Graphs of initial data $\mathbf{u}_0$ and the function $a(x)$ given by (3.1).

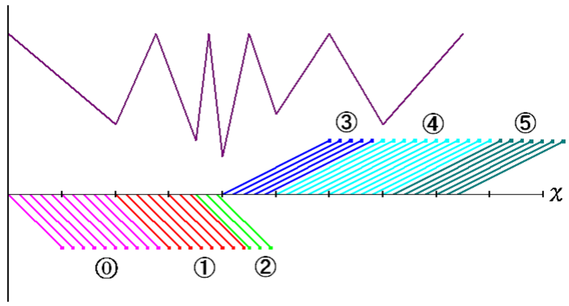

FIGURE 7. $T = 10$ in the case (a).

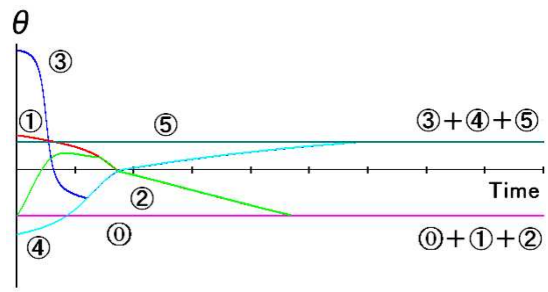

FIGURE 8. Behavior of angle of all vectors in the case (a).

Case (b). We consider the same function $a(x)$ defined by (3.2). In this case, the function $a(x)$ has two global minimums at $x_3 = 4$ and $x_5 = 7$. Then, Kuroda [14] has proved that there is at least one stationary solutions to (P) having discontinuities at $x_3 = 4$ and $x_5 = 7$. For the detailed proof, refer to [14].

We observe from Figure 9 that mathematical results established in [14] holds, although Figure 9 is concerned with the numerical result of $(\mathrm{P})^\varepsilon$ with $\varepsilon = 0.001$ at $T = 10$. For the detailed behavior of angle θ between the x-axis and the vector, see Figure 10. We easily see from Figure 10 that the solution of $(\mathrm{P})^\varepsilon$ becomes a piecewise constant vector-valued function (cf. Figure 9) having discontinuities at $x_3 = 4$ and $x_5 = 7$ after some finite time.

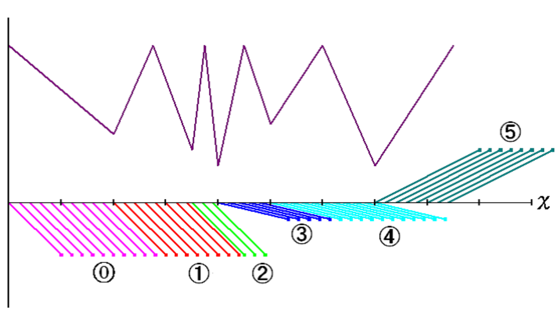

FIGURE 9. $T = 10$ in the case (b)

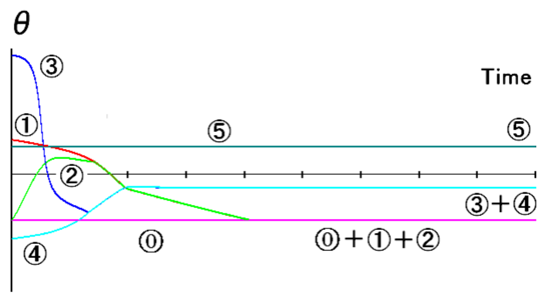

FIGURE 10. Behavior of angle of all vectors in the case (b).

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