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GENERALIZED SOLUTIONS OF A NON-ISOTHERMAL PHASE SEPARATION MODEL

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Abstract. We study a non-isothermal phase separation model of the Penrose-Fife type. We introduce the notion of a generalized solution and prove its unique existence.

1. **Introduction.** We study the following non-isothermal phase separation model of the Penrose-Fife type (cf. [9]): **Problem (P)**

$$e_t - \Delta\tilde{\alpha} = f,\ e = u + \lambda_0 w,\ \tilde{\alpha} \in \alpha(u) \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$w_t = \Delta\{-\kappa\Delta w + g(w) + \xi - \lambda_0\tilde{\alpha}\},\ \xi \in \beta(w) \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$\nabla\{-\kappa\Delta w + g(w) + \xi - \lambda_0\tilde{\alpha}\} \cdot n = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.3}$$

$$w \geq l_0,\ \nabla w \cdot n \geq 0,\ (w - l_0)\nabla w \cdot n = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.4}$$

$$\tilde{\alpha} = h \quad \text{on } \Gamma \times (0,T), \tag{1.5}$$

$$e(0) = e_0,\ w(0) = w_0 \quad \text{in } \Omega. \tag{1.6}$$

Here, Ω is a bounded domain of $\mathbf{R}^N (N = 1,2,3)$ and $\Gamma := \partial\Omega$ is a smooth boundary. Also, n is the unit normal on Γ and α and β are maximal monotone graphs in $\mathbf{R} \times \mathbf{R}$. Moreover, $\nu > 0, \kappa > 0$ and $\lambda_0 \in \mathbf{R}$ are positive constants and g is a sufficiently smooth function from $\mathbf{R}$ into itself.

The original model of our system was proposed by Penrose and Fife in [9] to describe the non-isothermal spinodal decomposition of a binary alloy composed of two components. Physically, e, u and w represent respectively, the internal energy,

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the temperature and the order parameter that describes the concentration of one of the components.

Much research has been done on the system $\{(1.1), (1, 2)\}$ with various boundary conditions. For early papers, we refer to the book by Brokate and Sprekels [2].

Usually, the Neumann boundary condition

$$\nabla w \cdot n = 0 \quad \text{on } \Gamma \times (0, T) \tag{1.7}$$

is imposed on the order parameter instead of (1.4).

Regarding temperature, Kubo, Ito and Kenmochi [6] and Ito, Kubo and Kenmochi [3] considered respectively, the third boundary condition

$$\nabla \tilde{\alpha} \cdot n + n_0 \tilde{\alpha} = h \quad \text{on } \Gamma \times (0, T) \tag{1.8}$$

and the Neumann boundary condition

$$\nabla \tilde{\alpha} \cdot n = 0 \quad \text{on } \Gamma \times (0, T). \tag{1.9}$$

Recently, Ito, Kenmochi and Niezgódka [4] studied the problem with the Signorini boundary condition (1.4) for the order parameter and the third boundary condition (1.8) for the temperature. On the other hand, Kumazaki, Ito and Kubo [8] studied the problem with condition (1.4) for the order parameter and the Dirichlet condition (1.5) for the temperature, by introducing the viscosity term $\nu \Delta w_t$ ($\nu > 0$) in (1.3):

$$w_t = \Delta\{\nu w_t - \kappa \Delta w + \cdots\}.$$

In [8], the term $\nu \Delta w_t$ plays an essential role in deriving uniform estimates of approximate solutions.

The present paper continues the study reported in [8] and considers the case $\nu = 0$.

In order to derive uniform estimates of approximate solutions without the viscosity term $\nu \Delta w_t$, we limit ourselves to a special case of the problem in [8]. In [8], we considered $e = u + \lambda(w)$ with a general function λ and the time-dependent boundary value $h(t)$ for the temperature. Now, however, we set $\lambda(w) = \lambda_0 w$ with the constant $\lambda_0 \in \mathbf{R}$ and assume that the boundary value h is independent of time.

Moreover, since we do not have an $L^2(\Omega)$-estimate of the temperature u, we have to employ a result of Kubo and Lu [7] to handle the relation $\tilde{\alpha} \in \alpha(u)$ in a generalized sense (Theorem 2.1). Then, we introduce the notion of a generalized solution (Definition 2.2) and show its unique existence (Main Theorem). The uniqueness together with Theorem 2.1 justify the introduction of the notion of a generalized solution (Remark 2.3).

The Main Theorem is stated in Section 2 together with a proof of uniqueness, and is proved in Section 3 by using a uniform estimate (Proposition 3.1) that is derived in Section 4.

1.1. **Notation and assumptions.** Throughout this paper, we use the notations given below.

In general, for a Hilbert space H, we denote by $(\cdot, \cdot)_H$ and $|| \cdot ||_H$ the inner product and norm, respectively. $H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces. The Hilbert space structure of $H_0^1(\Omega)$ is given by

$$(z_1, z_2)_{H_0^1(\Omega)} := \int_\Omega \nabla z_1 \cdot \nabla z_2 dx, \qquad \forall z_1, z_2 \in H_0^1(\Omega).$$

We denote by F and $\langle \cdot, \cdot \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ the duality mapping from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$ and the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, respectively. Then, $H^{-1}(\Omega)$

is a Hilbert space with the inner product defined as

$$(z_1, z_2)_{H^{-1}(\Omega)} = \langle z_1, F^{-1} z_2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \quad \forall z_1, z_2 \in H^{-1}(\Omega).$$

$L_0^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ defined as

$$L_0^2(\Omega) := \{z \in L^2(\Omega) | \int_\Omega z = 0\}.$$

Moreover, we denote by π_0 the projection operator from $L^2(\Omega)$ onto $L_0^2(\Omega)$;

$$\pi_0[z] := z - \frac{1}{|\Omega|} \int_\Omega z \quad \forall z \in L^2(\Omega).$$

$V_0 := H^1(\Omega) \cap L_0^2(\Omega)$ and V_0 is a Hilbert space with the inner product

$$(z_1, z_2)_{V_0} = \int_\Omega \nabla z_1 \cdot \nabla z_2 \quad \forall z_1, z_2 \in V_0.$$

Moreover, we denote by F_0 the duality mapping from V_0 onto its dual V_0^* and by $\langle \cdot, \cdot \rangle_{V_0^*, V_0}$ the duality pairing between V_0^* and V_0. Then, V_0^* is a Hilbert space with the inner product defined as

$$(z_1, z_2)_{V_0^*} = \langle z_1, F_0^{-1} z_2 \rangle_{V_0^*, V_0} \quad \forall z_1, z_2 \in V_0^*.$$

Next, we give the assumptions for the prescribed data. First, we note from (1.2) and (1.3) that

$$\frac{d}{dt} \int_\Omega w(t) = 0 \quad \text{a.e. } t \in (0, T).$$

Therefore,

$$\frac{1}{|\Omega|} \int_\Omega w(t) = \frac{1}{|\Omega|} \int_\Omega w(0) =: m_0 \quad \forall t \in [0, T].$$

(A1) α and β are maximal monotone graphs in $\mathbf{R} \times \mathbf{R}$. $\hat{\alpha}$ and $\hat{\beta}$ are proper, l.s.c, convex functions on $\mathbf{R}$ such that $\partial\hat{\alpha} = \alpha$ and $\partial\hat{\beta} = \beta$. Assume that there exists constants σ_*, σ^* such that

$$\overline{D(\hat{\beta})} = [\sigma_*, \sigma^*], \ -\infty < \sigma_* < \sigma^* < \infty.$$

(A2) $g \in C^1(\mathbf{R})$, $\hat{g}' = g$ and

$$\sup_{r \in \mathbf{R}} |\hat{g}(r)| + \sup_{r \in \mathbf{R}} |g(r)| + \sup_{r \in \mathbf{R}} |g'(r)| < +\infty.$$

(A3) $f \in L^2(0, T; L^2(\Omega))$.
(A4) $h \in H^1(\Omega)$ and there exists $\tilde{h} \in H^1(\Omega)$ such that $h \in \alpha(\tilde{h})$ a.e. in Ω.
(A5) $m_0 \in (\sigma_*, \sigma^*)$.
(A6) $e_0 \in L^2(\Omega)$ with $\hat{\alpha}(e_0 - \lambda(w_0)) \in L^1(\Omega)$.
(A7) $l_0 \in (\sigma_*, \sigma^*)$.
(A8) $w_0 \in H^1(\Omega)$ with $\hat{\beta}(w_0) \in L^1(\Omega)$ and $w_0 \geq l_0$ a.e. on Γ.

2. **Main Theorem.** Before stating the main theorem, we prepare the following theorem.

Theorem 2.1. *(* [7, Theorem 2.1]*) Under condition (A4), there exists a proper, l.s.c, convex function $\psi : H^{-1}(\Omega) \to \boldsymbol{R} \cup \{+\infty\}$ such that the following holds.*
For $z \in L^2(\Omega)$ and $z^* \in H^{-1}(\Omega), z^* \in \partial\psi(z)$ if and only if there exists $\tilde{z} \in L^2(\Omega)$ such that $\tilde{z} \in \alpha(z)$a.e. in $\Omega, \tilde{z} - h \in H_0^1(\Omega)$ and $z^* = F(\tilde{z} - h)$.

With the help of the convex function ψ, we introduce the notion of a generalized solution.

Definition 2.2. A function $(e, w) : [0, T] \to H^{-1}(\Omega) \times V_0^*$ is called *a generalized solution* of (P), if the following items are satisfied.

(S1) $e \in W^{1,2}(0, T; H^{-1}(\Omega))$,
$w \in W^{1,2}(0, T; V_0^*) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$.

(S2) For a.e. $t \in (0, T)$ the following equation holds:
$$e^{'}(t) + u^*(t) = f(t) + \Delta h, \quad u^*(t) \in \partial\psi(u(t)),$$
where $u(t) := e(t) - \lambda_0 w(t)$.

(S3) There exists $\xi \in L^2(0, T; L^2(\Omega))$ such that $\xi \in \beta(w)$ a.e. in $\Omega \times (0, T)$ and
$$F_0^{-1} w^{'}(t) + \pi_0[-\kappa\Delta w(t) + \xi(t)] + \pi_0[g(w(t)) - \lambda_0\tilde{\alpha}(t)] = 0 \quad \text{a.e. } t,$$
$$w \geq l_0,\ \nabla w \cdot n \geq 0,\ (w - l_0)\nabla w \cdot n = 0 \quad \text{a.e. on } \Gamma \times (0, T),$$
where $\tilde{\alpha}(t) := F^{-1}u^*(t) + h$.

(S4) $e(0) = e_0,\ w(0) = w_0$.

Now, we state the main theorem.

Main Theorem. *There exists a unique generalized solution of (P).*

Remark 2.3. By Theorem 2.1, if $u(t) \in L^2(\Omega)$, (S2) is equivalent to
$$\tilde{\alpha} \in \alpha(u) \quad \text{a.e. } \Omega \times (0, T)$$
and
$$\langle e^{'}(t), z\rangle + \int_\Omega \nabla\tilde{\alpha}(t) \cdot \nabla z = (f(t), z)_{L^2(\Omega)} \qquad \forall z \in H_0^1(\Omega).$$

Remark 2.4. The solution of the main theorem is obtained by a viscosity vanishing of the solution obtained in [8].

2.1. **Proof of uniqueness.** Let $(e_i, w_i)(i = 1, 2)$ be two generalized solutions of (P) and set $W := w_1 - w_2$ and $E := e_1 - e_2$. Then, for all $z \in H_0^1(\Omega)$, we have
$$\langle E(t)^{'}, z\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle u_1^*(t) - u_2^*(t), z\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = 0.$$

Therefore, we set $z = F^{-1}(E(t))$ and have
$$\langle E(t)^{'}, F^{-1}(E(t))\rangle + \langle u_1^*(t) - u_2^*(t), F^{-1}(E(t))\rangle = 0.$$

By using the monotonicity of $\partial\psi$, we obtain
$$(u_1^*(t) - u_2^*(t), u_1(t) - u_2(t))_{H^{-1}(\Omega)} \geq 0.$$

Therefore,
$$\frac{1}{2}\frac{d}{dt}||E(t)||^2_{H^{-1}(\Omega)} + \lambda_0(u_1^*(t) - u_2^*(t), W(t))_{H^{-1}(\Omega)} \leq 0. \tag{2.1}$$

Next, we take the inner product of $(S3)$ and $W(t)$ to derive
$$\frac{1}{2}\frac{d}{dt}||W(t)||^2_{V_0^*} + \kappa||\nabla W(t)||^2_{L^2(\Omega)} + (g(w_1(t)) - g(w_2(t)), W(t))_{L^2(\Omega)}$$
$$-\lambda_0(\tilde{\alpha}_1(t) - \tilde{\alpha}_2(t), W(t))_{L^2(\Omega)} \leq 0. \tag{2.2}$$

We calculate (2.1) + (2.2) to derive

$$\frac{1}{2}\frac{d}{dt}\left(||E(t)||^2_{H^{-1}(\Omega)}+||W(t)||^2_{V_0^*}\right)+\kappa||\nabla W(t)||^2_{L^2(\Omega)}+\lambda_0(u_1^*(t)-u_2^*(t),W(t))_{H^{-1}(\Omega)}$$
$$\leq |(g(w_1(t))-g(w_2(t)),W(t))_{L^2(\Omega)}|+\lambda_0(\tilde{\alpha}_1(t)-\tilde{\alpha}_2(t),W(t))_{L^2(\Omega)}.$$

Now, by noting that $\tilde{\alpha}_i(t) := F^{-1}u_i^*(t)+h$, we have

$$(\tilde{\alpha}_1(t)-\tilde{\alpha}_2(t),W(t))_{L^2(\Omega)} = (u_1^*(t)-u_2^*(t),W(t))_{H^{-1}(\Omega)}.$$

Note that

$$||z||^2_{L^2_0(\Omega)} \leq ||z||_{V_0^*}||z||_{V_0} \qquad \forall z\in V_0.$$

Hence, we obtain by the Lipschitz continuity of g (cf. (A2)),

$$\frac{1}{2}\frac{d}{dt}\left(||E(t)||^2_{H^{-1}(\Omega)}+||W(t)||^2_{V_0^*}\right)+\frac{\kappa}{2}||\nabla W(t)||^2_{L^2(\Omega)}$$
$$\leq C\left(||W(t)||^2_{V_0^*}+||E(t)||^2_{H^{-1}(\Omega)}\right).$$

Here C is a positive constant. By applying Gronwall's lemma, we have $w_1 = w_2$ in V_0^* and $e_1 = e_2$ in $H^{-1}(\Omega)$.

3. **Proof of Main Theorem.** First, for each $\nu \in (0,1)$, we consider the following approximate problem: $(P)_\nu$

$$e_t - \Delta\tilde{\alpha} = f,\ e = u+\lambda_0 w,\ \tilde{\alpha}\in\alpha(u) \quad \text{in } \Omega\times(0,T),$$
$$w_t = \Delta\{\nu w_t - \kappa\Delta w + g(w)+\xi-\lambda_0\tilde{\alpha}\},\ \xi\in\beta(w) \quad \text{in } \Omega\times(0,T),$$
$$\nabla\{\nu w_t-\kappa\Delta w+g(w)+\xi-\lambda_0\tilde{\alpha}\}\cdot n = 0 \quad \text{on } \Gamma\times(0,T),$$
$$w\geq l_0,\ \nabla w\cdot n\geq 0,\ (w-l_0)\nabla w\cdot n = 0 \quad \text{on } \Gamma\times(0,T),$$
$$\tilde{\alpha} = h \quad \text{on } \Gamma\times(0,T),$$
$$e(0)=e_0,\ w(0)=w_0 \quad \text{in } \Omega.$$

By using [8], this problem $(P)_\nu$ has a unique solution (e_ν, w_ν) on $[0,T]$ satisfying the following properties:

(APS1) $e_\nu \in W^{1,2}(0,T;H^{-1}(\Omega))\cap L^\infty(0,T;L^2(\Omega)),\ w_\nu\in W^{1,2}(0,T;L^2(\Omega))$ $\cap L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$.

(APS2) There exists $\tilde{\alpha}_\nu \in L^2(0,T;H^1(\Omega))$ such that $\tilde{\alpha}_\nu\in\alpha(e_\nu-\lambda_0 w_\nu)$ a.e. in $\Omega\times(0,T)$, $\tilde{\alpha}_\nu = h$ a.e. on $\Gamma\times(0,T)$ and for all $z\in H^1_0(\Omega)$ and a.e. $t\in(0,T)$ the following equality holds:

$$\langle e_\nu'(t),z\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \int_\Omega \nabla\tilde{\alpha}_\nu(t)\cdot\nabla z = (f(t),z)_{L^2(\Omega)}.$$

(APS3) There exists $\xi_\nu\in L^2(0,T;L^2(\Omega))$ such that $\xi_\nu\in\beta(w_\nu)$ a.e. in $\Omega\times(0,T)$ and

$$(F_0^{-1}+\nu I)w_\nu'(t)+\pi_0[-\kappa\Delta w_\nu(t)+\xi_\nu(t)]+\pi_0[g(w_\nu(t))-\lambda_0\tilde{\alpha}_\nu(t)] = 0 \quad \text{a.e. } t,$$
$$w_\nu\geq l_0,\ \nabla w_\nu\cdot n\geq 0,\ (w_\nu-l_0)\nabla w_\nu\cdot n = 0 \quad \text{a.e. on } \Gamma\times(0,T).$$

(APS4) $e_\nu(0)=e_0,\ w_\nu(0)=w_0$.

Now, we have the following uniform estimate.

Proposition 3.1 There exists a constant $K > 0$ such that

$$||w_\nu^{'}||_{L^2(0,T;V_0^*)} + ||e_\nu^{'}||_{L^2(0,T;H^{-1}(\Omega))} + \sqrt{\nu}||w_\nu^{'}||_{L^2(0,T;L^2(\Omega))} + \sup_{0\le t\le T}||\nabla w_\nu(t)||^2_{L^2(\Omega)}$$

$$+||\tilde{\alpha}_\nu||_{L^2(0,T;H^1(\Omega))} + \sup_{0\le t\le T}||\hat{\beta}(w_\nu(t))||_{L^1(\Omega)} + ||w_\nu||_{L^2(0,T;H^2(\Omega))} + ||\xi_\nu||_{L^2(0,T;L^2(\Omega))}$$

$$\le K \quad \forall \nu \in (0,1).$$

This proposition is proved in the next section.

By Proposition 3.1, we can take a sequence $\{\nu_n\} \subset (0,1)$, which converges to 0 as $n \to \infty$ such that the following convergences are fulfilled for some $(e, w, \xi, \tilde{\alpha})$:

$$e_{\nu_n} \to e \begin{cases} \text{weakly in } W^{1,2}(0,T;H^{-1}(\Omega)) \\ \text{weakly-}\star \text{ in } L^\infty(0,T;H^{-1}(\Omega)), \end{cases}$$

$$w_{\nu_n} \to w \begin{cases} \text{weakly-}\star \text{ in } L^\infty(0,T;H^1(\Omega)) \\ \text{weakly in } L^2(0,T;H^2(\Omega)), \end{cases}$$

$$w_{\nu_n}^{'} \to w^{'} \quad \text{weakly in } L^2(0,T;V_0^*),$$

$$\pi_0[w_{\nu_n}] \to \pi_0[w] \begin{cases} \text{weakly in } W^{1,2}(0,T;V_0^*) \\ \text{strongly in } C([0,T];V_0^*), \end{cases}$$

$$u_{\nu_n} \to u \quad \text{weakly-}\star \text{ in } L^\infty(0,T;H^{-1}(\Omega)),$$

$$\xi_{\nu_n} \to \xi \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$

$$\tilde{\alpha}_{\nu_n} \to \tilde{\alpha} \quad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

It is easily seen that $(e, w, \tilde{\alpha}, \xi)$ satisfies the following properties:

$$e^{'}(t) + F(\tilde{\alpha}(t) - h) = f(t) + \Delta h \quad \text{a.e. } t,$$

$$F_0^{-1}w^{'}(t) + \pi_0[-\kappa\Delta w(t) + \xi(t)] + \pi_0[g(w(t)) - \lambda_0\tilde{\alpha}(t)] = 0 \quad \text{a.e. } t,$$

$$\xi \in \beta(w) \quad \text{a.e. in } \Omega \times (0,T),$$

$$w \ge l_0,\ \nabla w \cdot n \ge 0,\ (w - l_0)\nabla w \cdot n = 0 \quad \text{a.e. on } \Gamma \times (0,T),$$

$$e(0) = e_0,\ w(0) = w_0.$$

To complete the proof, we have to show that $F(\tilde{\alpha}(t) - h) \in \partial\psi(u(t))$ for a.e. t. In $(P)_{\nu_n}$, we see that $u_{\nu_n}(t) \in L^2(\Omega)$. Hence, for a.e. $t \in (0,T)$, $(APS2)$ is equivalent to the following (cf. Theorem 2.1, Remark 2.3).

$$e_{\nu_n}^{'}(t) + u_{\nu_n}^*(t) = f(t) + \Delta h, \quad u_{\nu_n}^*(t) = F(\alpha_{\nu_n}(t) - h) \in \partial\psi(u_{\nu_n}(t)).$$

For all $v \in L^2(0,T;H^{-1}(\Omega))$, we have

$$\int_0^T (f(t) + \Delta h - e_{\nu_n}^{'}(t), v(t) - u_{\nu_n}(t))_{H^{-1}(\Omega)}dt \le \int_0^T \psi(v(t))dt - \int_0^T \psi(u_{\nu_n}(t))dt.$$

Note that

$$\begin{aligned}
&\int_0^T (e'_{\nu_n}(t), u_{\nu_n}(t))_{H^{-1}(\Omega)} dt \\
&= \int_0^T (e'_{\nu_n}(t), e_{\nu_n}(t))_{H^{-1}(\Omega)} dt - \lambda_0 \int_0^T (e'_{\nu_n}(t), w_{\nu_n}(t))_{H^{-1}(\Omega)} dt \\
&= \int_0^T \frac{1}{2}\frac{d}{dt} ||e_{\nu_n}(t)||^2_{H^{-1}(\Omega)} - \lambda_0 \int_0^T (e'_{\nu_n}(t), w_{\nu_n}(t))_{H^{-1}(\Omega)} dt \\
&= \frac{1}{2}\left(||e_{\nu_n}(T)||^2_{H^{-1}(\Omega)} - ||e_{\nu_n}(0)||^2_{H^{-1}(\Omega)} \right) - \lambda_0 \int_0^T (e'_{\nu_n}(t), w_{\nu_n}(t))_{H^{-1}(\Omega)} dt.
\end{aligned}$$

Now, $e_{\nu_n}(0) = e_0$ and for all t and $z \in H^1_0(\Omega)$, we have

$$\langle e_{\nu_n}(t), z \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_0^t \langle e'_{\nu_n}(s), z \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} ds + \langle e_{\nu_n}(0), z \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}.$$

By the convergence of e'_{ν_n}, we have for all $t \in [0, T]$

$$e_{\nu_n}(t) \to e(t) \quad \text{weakly in } H^{-1}(\Omega).$$

Moreover, w_{v_n} converges to w strongly in $L^2(0, T; H^1(\Omega))$. Therefore,

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T (e'_{\nu_n}(t), u_{\nu_n}(t))_{H^{-1}(\Omega)} dt \\
&\geq \frac{1}{2}\left(||e(T)||^2_{H^{-1}(\Omega)} - ||e(0)||^2_{H^{-1}(\Omega)} \right) - \lambda_0 \int_0^T (e'(t), w(t))_{H^{-1}(\Omega)} dt \\
&= \int_0^T (e'(t), u(t))_{H^{-1}(\Omega)} dt.
\end{aligned}$$

Hence, by the lower semicontinuity of ψ, we see that

$$\int_0^T (f(t) + \Delta h - e'(t), v(t) - u(t))_{H^{-1}(\Omega)} dt \leq \int_0^T \psi(v(t)) dt - \int_0^T \psi(u(t)) dt.$$

Hence, we have $u^*(t) := F(\tilde{\alpha}(t) - h) \in \partial\psi(u(t))$ for a.e. t (cf. [1, Prop. 2.16]).

4. **Proof of Proposition 3.1.** First, for each $\varepsilon \in (0, 1)$, we consider the following problem: $(P)_{\nu,\varepsilon}$

$$\begin{gathered}
e_t - \Delta\alpha_\varepsilon(u) = f, \ e = u + \lambda_0 w \quad \text{in } \Omega \times (0, T), \\
w_t = \Delta\{\nu w_t - \kappa\Delta w + g(w) + \xi - \lambda_0\alpha_\varepsilon(u)\}, \ \xi \in \beta(w) \quad \text{in } \Omega \times (0, T), \\
\nabla\{\nu w_t - \kappa\Delta w + g(w) + \xi - \lambda_0\alpha_\varepsilon(u)\} \cdot n = 0 \quad \text{on } \Gamma \times (0, T), \\
w \geq l_0, \ \nabla w \cdot n \geq 0, \ (w - l_0)\nabla w \cdot n = 0 \quad \text{on } \Gamma \times (0, T), \\
\alpha_\varepsilon(u) = h_\varepsilon \quad \text{on } \Gamma \times (0, T), \\
e(0) = e_{0,\varepsilon} (= u_{0,\varepsilon} + \lambda_0 w_0), \ w(0) = w_0 \quad \text{in } \Omega.
\end{gathered}$$

Here, $\alpha_\varepsilon, h_\varepsilon$ and $u_{0,\varepsilon}$ are defined below. We approximate $\hat{\alpha}$ by $\hat{\alpha}_\varepsilon$ defined as

$$\hat{\alpha}_\varepsilon(r) = \inf_{z \in \mathbf{R}} \left\{ \frac{1}{2\varepsilon}|z - r|^2 + \hat{\alpha}(z) \right\} + \frac{\varepsilon}{2}|r|^2 \quad \forall r \in \mathbf{R}$$

and we set $\alpha_\varepsilon = \dfrac{d\hat{\alpha}_\varepsilon}{dr} = \partial\hat{\alpha}_\varepsilon$. Also, we can approximate the data h, $\tilde{h}$ and u_0 by h_ε, $\tilde{h}_\varepsilon$ and $u_{0,\varepsilon}$, respectively, which satisfy the following properties (cf. [5]):

$$h_\varepsilon := h + \varepsilon(\tilde{h} + \varepsilon h) = \alpha_\varepsilon(\tilde{h}_\varepsilon) \to h, \ \tilde{h}_\varepsilon := \tilde{h} + \varepsilon h \to \tilde{h} \text{ in } H^1(\Omega),$$

$\{u_{0,\varepsilon}\} \subset H^1(\Omega),\ u_{0,\varepsilon} \to u_0$ in $L^2(\Omega), u_{0,\varepsilon} = \tilde{h}_\varepsilon(0)$ in $\partial\Omega, \left\{\int_\Omega \hat{\alpha}_\varepsilon(u_{0,\varepsilon})\right\}_\varepsilon$ is bounded.

Next, by using [8] again, $(P)_{\nu,\varepsilon}$ has a unique solution $(e_{\nu,\varepsilon}, w_{\nu,\varepsilon})$ on $[0,T]$ satisfying the following properties:

$(APS1)'$ $e_{\nu,\varepsilon} \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\ w_{\nu,\varepsilon} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$.

$(APS2)'$ For a.e. $t \in (0,T)$ the following evolution equation in $H^{-1}(\Omega)$ holds:

$$e'_{\nu,\varepsilon}(t) + F(\alpha_\varepsilon(u_{\nu,\varepsilon}(t)) - h_\varepsilon) = f(t) + \Delta h_\varepsilon.$$

$(APS3)'$ There exists $\xi_{\nu,\varepsilon} \in L^2(0,T;L^2(\Omega))$ such that $\xi_{\nu,\varepsilon} \in \beta(w_{\nu,\varepsilon})$ a.e. in $\Omega\times(0,T)$ and

$$(F_0^{-1}+\nu I)w'_{\nu,\varepsilon}(t)+\pi_0[-\kappa\Delta w_{\nu,\varepsilon}(t)+\xi_{\nu,\varepsilon}(t)]+\pi_0[g(w_{\nu,\varepsilon}(t))-\lambda_0\alpha_\varepsilon(u_{\nu,\varepsilon}(t))] = 0 \text{ a.e. } t,$$

$$w_{\nu,\varepsilon} \geq l_0,\ \nabla w_{\nu,\varepsilon}\cdot n \geq 0,\ (w_{\nu,\varepsilon} - l_0)\nabla w_{\nu,\varepsilon}\cdot n = 0 \quad \text{a.e. on } \Gamma\times(0,T).$$

$(APS4)'$ $e_{\nu,\varepsilon}(0) = e_{0,\varepsilon},\ w_{\nu,\varepsilon}(0) = w_0$.

Now, we take the duality pairing between $(APS2)'$ and $F^{-1}e'_{\nu,\varepsilon}(t)$ to derive

$$\begin{aligned} &||e'_{\nu,\varepsilon}(t)||^2_{H^{-1}(\Omega)} + \frac{d}{dt}\Phi_\varepsilon(u_{\nu,\varepsilon}(t)) + \lambda_0(\alpha_\varepsilon(u_{\nu,\varepsilon}(t)), w'_{\nu,\varepsilon}(t))_{L^2(\Omega)} \\ &\quad = \langle f(t) + \Delta h_\varepsilon, F^{-1}e'_{\nu,\varepsilon}(t)\rangle + \lambda_0(h_\varepsilon, w'_{\nu,\varepsilon}(t))_{L^2(\Omega)}. \end{aligned} \tag{4.1}$$

where $\Phi_\varepsilon : H^{-1}(\Omega) \to \mathbf{R}\cup\{+\infty\}$ is defined as

$$\Phi_\varepsilon(z) := \begin{cases} \int_\Omega \hat{\alpha_\varepsilon}(z)dx - (h_\varepsilon, z)_{L^2(\Omega)}, & \text{if } z \in L^2(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Next, we take the inner product of $(APS3)'$ and $w'_{\nu,\varepsilon}(t)$ to derive

$$\begin{aligned} &||w'_{\nu,\varepsilon}(t)||^2_{V_0^*} + \nu||w'_{\nu,\varepsilon}(t)||^2_{L^2(\Omega)} + \frac{d}{dt}\left\{\Psi(\pi_0[w_{\nu,\varepsilon}(t)]) + \int_\Omega \hat{g}(w_{\nu,\varepsilon}(t))\right\} \\ &\quad = \lambda_0(\alpha_\varepsilon(u_{\nu,\varepsilon}(t)), w'_{\nu,\varepsilon}(t))_{L^2(\Omega)}. \end{aligned} \tag{4.2}$$

where $\Psi : L^2_0(\Omega) \to \mathbf{R}\cup\{+\infty\}$ is defined as

$$\Psi(z_0) := \begin{cases} \frac{\kappa}{2}||z_0||^2_{V_0} + \int_\Omega \hat{\beta}(z_0 + m_0)dx, & \text{if } z_0 \in D_0 \\ +\infty, & \text{if } z_0 \in L^2_0(\Omega)\setminus D_0, \end{cases}$$

where D_0 is defined as

$$D_0 := \{z_0 \in V_0 \mid \hat{\beta}(z_0 + m_0) \in L^1(\Omega) \text{ and } z_0 + m_0 \geq l_0 \text{ a.e. on } \Gamma\}.$$

Then, we calculate (4.1) + (4.2) to derive

$$\begin{aligned} &||w'_{\nu,\varepsilon}(t)||^2_{V_0^*} + \nu||w'_{\nu,\varepsilon}(t)||^2_{L^2(\Omega)} + ||e'_{\nu,\varepsilon}(t)||^2_{H^{-1}(\Omega)} \\ &+\frac{d}{dt}\left\{\Phi_\varepsilon(u_{\nu,\varepsilon}(t)) + \Psi(\pi_0[w_{\nu,\varepsilon}(t)]) + \int_\Omega \hat{g}(w_{\nu,\varepsilon}(t))\right\} \\ = \;& \langle f(t) + \Delta h_\varepsilon, F^{-1}e'_{\nu,\varepsilon}(t)\rangle + \lambda_0(h_\varepsilon, w'_{\nu,\varepsilon}(t))_{L^2(\Omega)}. \end{aligned}$$

Note that

$$\begin{aligned}(h_\varepsilon, w'_{\nu,\varepsilon}(t))_{L^2(\Omega)} &= (\pi_0[h_\varepsilon], w'_{\nu,\varepsilon}(t))_{L^2(\Omega)} \\ &= \langle w'_{\nu,\varepsilon}(t), \pi_0[h_\varepsilon]\rangle_{V_0^*,V_0} \\ &\le ||w'_{\nu,\varepsilon}(t)||_{V_0^*}||\nabla h_\varepsilon||_{L^2(\Omega)}.\end{aligned}$$

Therefore, there exists a constant $C_1 > 0$ such that

$$\begin{aligned}&\frac{1}{2}||w'_{\nu,\varepsilon}(t)||^2_{V_0^*} + \nu||w'_{\nu,\varepsilon}(t)||^2_{L^2(\Omega)} + \frac{1}{2}||e'_{\nu,\varepsilon}(t)||^2_{H^{-1}(\Omega)} \\ &+\frac{d}{dt}\left\{\Phi_\varepsilon(u_{\nu,\varepsilon}(t)) + \Psi(\pi_0[w_{\nu,\varepsilon}(t)]) + \int_\Omega \hat{g}(w_{\nu,\varepsilon}(t))\right\} \\ \le\ & \frac{1}{2}||f(t) + \Delta h_\varepsilon||^2_{H^{-1}(\Omega)} + \frac{\lambda_0^2}{2}||\nabla h_\varepsilon||^2_{L^2(\Omega)} \\ \le\ & C_1(||f(t)||^2_{L^2(\Omega)} + ||\nabla h_\varepsilon||^2_{L^2(\Omega)}) \quad \text{a.e. } \ t \in (0,T).\end{aligned}$$

Then, by integrating over [0,t], we have

$$\begin{aligned}&\frac{1}{2}\int_0^t ||e'_{\nu,\varepsilon}(s)||^2_{H^{-1}(\Omega)}ds + \nu\int_0^t ||w'_{\nu,\varepsilon}(s)||^2_{L^2(\Omega)}ds + \frac{1}{2}\int_0^t ||w'_{\nu,\varepsilon}(s)||^2_{V_0^*}ds \\ &+\Phi_\varepsilon(u_{\nu,\varepsilon}(t)) + \Psi(\pi_0[w_{\nu,\varepsilon}(t)]) + \int_\Omega \hat{g}(w_{\nu,\varepsilon}(t)) \\ \le\ & \Phi_\varepsilon(u_{\nu,\varepsilon}(0)) + \Psi(\pi_0[w_{\nu,\varepsilon}(0)]) \\ &+\int_\Omega \hat{g}(w_{\nu,\varepsilon}(0)) + C_1||f||^2_{L^2(0,T;L^2(\Omega))} + C_1T||\nabla h_\varepsilon||^2_{L^2(\Omega)}.\end{aligned}$$

By noting that there exists a constant $M > 0$ such that for all $\varepsilon \in (0,1)$ and $r \in \mathbf{R}$,

$$\hat{\alpha}_\varepsilon(r) \ge -M(|r| + 1)$$

and that $h_\varepsilon = \alpha_\varepsilon(\tilde{h}_\varepsilon)$, we have

$$\begin{aligned}\Phi_\varepsilon(z) &= \int_\Omega \hat{\alpha}_\varepsilon(z)dx - (h_\varepsilon, z) \ge \int_\Omega \hat{\alpha}_\varepsilon(\tilde{h}_\varepsilon)dx - (h_\varepsilon, \tilde{h}_\varepsilon) \\ &\ge \int_\Omega -M(|\tilde{h}_\varepsilon| + 1)dx - (h_\varepsilon, \tilde{h}_\varepsilon) \ge -M'\end{aligned}$$

for a constant $M' > 0$ and for all $z \in L^2(\Omega)$. Therefore, we see that there exists a positive constant K such that

$$\begin{aligned}&||e'_{\nu,\varepsilon}||_{L^2(0,T;H^{-1}(\Omega))} + ||w'_{\nu,\varepsilon}||_{L^2(0,T;V_0^*)} + \sqrt{\nu}||w'_{\nu,\varepsilon}||_{L^2(0,T;L^2(\Omega))} + \\ &+||\alpha_\varepsilon(u_{\nu,\varepsilon})||_{L^2(0,T;H^1(\Omega))} + \sup_{0\le t\le T}||\nabla w_{\nu,\varepsilon}(t)||_{L^2(\Omega)} + \sup_{0\le t\le T}||\hat{\beta}(w_{\nu,\varepsilon}(t))||_{L^1(\Omega)} \le K.\end{aligned}$$

Here, the estimate of $||\alpha_\varepsilon(u_{\nu,\varepsilon})||_{L^2(0,T;H^1(\Omega))}$ follows from (cf. $(APS2)'$)

$$\alpha_\varepsilon(u_{\nu,\varepsilon}(t)) = F^{-1}(f(t) + \Delta h_\varepsilon - e'_{\nu,\varepsilon}) + h_\varepsilon.$$

Moreover, we note the following result.

Lemma 4.1. *(cf. [4]) Let $\bar{f} \in L^2(\Omega)$ and set $z := z_0 + m_0$. Then, $\pi_0[\bar{f}] \in \partial_{L^2_0(\Omega)}\Psi(z_0)$ if and only if $z_0 \in H^2(\Omega)$ and there exists a function $\xi := \xi_{\bar{f}} \in L^2(\Omega)$ such that*

$$\pi_0[-\kappa\Delta z + \xi] = \pi_0[\bar{f}] \quad \text{a.e. in } \ \Omega,$$

$$\xi \in \beta(z) \quad \text{a.e. in } \ \Omega,$$

$$z \geq l_0,\ \nabla z \cdot n \geq 0,\ (z - l_0)\nabla z \cdot n = 0 \quad a.e.\ on\ \ \Gamma.$$

Moreover, there exists a constant $K_1 > 0$ *such that*

$$||z||_{H^2(\Omega)} + ||\xi||_{L^2(\Omega)} \leq K_1(||\bar{f}||_{L^2(\Omega)} + 1).$$

From this lemma and $(APS3)'$, the estimates of $||w_{\nu,\varepsilon}||_{L^2(0,T;H^2(\Omega))}$ and $||\xi_{\nu,\varepsilon}||_{L^2(0,T;L^2(\Omega))}$ follows. Therefore, by letting $\varepsilon \to 0$, we obtain Proposition 3.1.

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