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EXISTENCE OF NODAL SOLUTIONS OF MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. We study the nonlinear boundary value problem consisting of the equation $y''+w(t)f(y)=0$ on $[a,b]$ and a multi-point boundary condition. By relating it to the eigenvalues of a linear Sturm-Liouville problem with a two-point separated boundary condition, we obtain results on the existence and nonexistence of nodal solutions of this problem. We also discuss the changes of the existence of different types of nodal solutions as the problem changes.

1. **Introduction.** We are concerned with the boundary value problem (BVP) consisting of the equation

$$y'' + w(t)f(y) = 0,\ t \in (a,b), \tag{1.1}$$

and the multi-point boundary condition (BC)

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0,\pi),\\ &y'(b) - \textstyle\sum_{i=1}^m k_i y'(\eta_i) = 0, \end{aligned} \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $a < b$. We assume throughout, and without further mention, that the following conditions hold:

(H1) $w \in C^1[a,b]$ such that $w(t) > 0$ on $[a,b]$;
(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$, and f is locally Lipschitz on $(-\infty,0)\cup(0,\infty)$;
(H3) there exist extended real numbers $f_0, f_\infty \in [0,\infty]$ such that

$$f_0 = \lim_{y\to 0} f(y)/y \quad \text{and} \quad f_\infty = \lim_{|y|\to\infty} f(y)/y;$$

(H4) $\eta_i \in (a,b)$ and $k_i \in \mathbb{R}$ for $i = 1,\dots,m$.

The existence of solutions, especially positive solutions, of BVPs with multi-point BCs have been studied extensively, see, for example, [1, 3, 5, 6, 7, 13, 14, 25] and the references therein. In this paper, we study the existence of nodal solutions, i.e., solutions with a specific zero-counting property in (a,b), of the multi-point BVP (1.1), (1.2). Great progress has been made to the study of such solutions for nonlinear BVPs consisting of Eq. (1.1) (and more general forms of equations) and

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two-point separated BCs, see [8, 9, 15, 17, 18, 21]. As regard to multi-point BVPs, results on the existence of nodal solutions have been obtained only for the special BVP consisting of the equation

$$y'' + f(y) = 0,\ t \in (0, 1), \tag{1.3}$$

and the BC

$$y(0) = 0, \quad y(1) - \sum_{i=1}^{m} k_i y(\eta_i) = 0, \tag{1.4}$$

see Ma [16], Ma and O'Regan [19], Rynne [22], Sun, Xu, and O'Regan [23], and Xu [24]. Among them, [19] and [22] used a standard global bifurcation method to establish the existence of nodal solutions of BVP (1.3), (1.4) by relating it to the eigenvalues of the corresponding linear Sturm-Liouville problem (SLP) with BC (1.4). However, the establishment of these results relies heavily on the direct computation of the eigenvalues and eigenfunctions of the associated multi-point SLP and hence can not be extended to a general BVP with a variable coefficient function w. Moreover, to the best of the authors' knowledge, there is nothing done so far on the existence of nodal solutions of BVPs with the multi-point BC (1.2).

In view of the fact that eigenvalues are easy to calculate for all two-point linear self-adjoint SLPs using standard software packages such as those in [2], in this paper, we obtain results on the existence and nonexistence of nodal solutions of BVP (1.1), (1.2) by relating it to the eigenvalues of an associated linear SLP with a two-point separated BC rather than a multi-point BC. The shooting method and an energy function play key roles in the proofs. We also discuss the changes of the existence of different types of nodal solutions when some parameters in the problem change, more precisely, when the interval $[a, b]$ shrinks and when the function w increases in certain direction. Note that our results are for the general BVP (1.1), (1.2) with a variable w, a separated BC at the left endpoint a prescribed by an arbitrary α, and the multi-point BC given in (1.2), where k_i, $i = 1, \dots, m$, may assume either positive or negative values.

Let $\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues of the SLP consisting of the equation

$$y'' + \lambda w(t) y = 0,\ t \in (a, b), \tag{1.5}$$

and the two-point BC

$$\begin{aligned} &\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, \quad \alpha \in [0, \pi), \\ &y(b) = 0. \end{aligned} \tag{1.6}$$

It is well known that any eigenfunction associated with λ_n has n simple zeros in (a, b), see [26, Theorem 4.3.2].

2. **Existence and nonexistence of nodal solutions.**

We study the nodal solutions of BVP (1.1), (1.2) in the following classes.

Definition 2.1. A solution y of BVP (1.1), (1.2) is said to belong to class $\mathcal{S}_n^\gamma$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\gamma \in \{+, -\}$ if

(i) y has exactly n zeros in (a, b),

(ii) $\gamma y(t) > 0$ in a right-neighborhood of a.

In the sequel we let

$$w'_-(t) := \max\{-w'(t), 0\} \quad \text{and} \quad k_0 = \int_a^b \frac{w'_-(t)}{w(t)}\, dt.$$

We now present our main results on the existence and nonexistence of nodal solutions of BVP (1.1), (1.2) with the proofs given latter in this section after several technical lemmas are derived. The first theorem is about the existence of certain types of nodal solutions.

Theorem 2.1. *Assume either (i) $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$, or (ii) $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$, for some $n \in \mathbb{N}_0$. Suppose*

$$1 - \sum_{i=1}^{m} |k_i| e^{k_0/2} > 0. \tag{2.1}$$

Then BVP (1.1), (1.2) has two solutions $y_{n,\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ for $\gamma \in \{+,-\}$.

As a consequence of Theorem 2.1 we have the following corollary on the existence of an infinite number of different types of nodal solutions.

Corollary 2.1. *Assume (2.1) holds, and*

$$\text{either } f_0 = 0 \text{ and } f_\infty = \infty, \quad \text{or } f_\infty = 0 \text{ and } f_0 = \infty.$$

Then there exists $\alpha^* \in (\pi/2, \pi)$ such that

(i) when $\alpha \in [0, \alpha^*)$, BVP (1.1), (1.2) has a solution $y_n^{\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ for each $n \geq 0$ and $\gamma \in \{+,-\}$;

(ii) when $\alpha \in [\alpha^*, \pi)$, then BVP (1.1), (1.2) has a solution $y_n^{\gamma} \in \mathcal{S}_{n+1}^{\gamma}$ for each $n \geq 1$ and $\gamma \in \{+,-\}$.

Remark 2.1. The number α^* in the above theorem can be explicitly computed using the fundamental solutions of (1.5), see [4, Theorem 2.2] for details.

The next theorem is about the nonexistence of certain types of nodal solutions.

Theorem 2.2. *(i) Assume $f(y)/y < \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1), (1.2) has no solution in $\mathcal{S}_i^{\gamma}$ for all $i \geq n+1$ and $\gamma \in \{+,-\}$.*

(ii) Assume $f(y)/y > \lambda_n$ for some $n \in \mathbb{N}_0$ and all $y \neq 0$. Then BVP (1.1), (1.2) has no solution in $\mathcal{S}_i^{\gamma}$ for all $i \leq n$ and $\gamma \in \{+,-\}$.

To prove Theorem 2.1, we need some preliminaries. The first lemma is a known result on the global existence and uniqueness of solutions of initial value problems associated with Eq. (1.1) given by [20, Proposition 2.1].

Lemma 2.1. *Any initial value problem associated with Eq. (1.1) has a unique solution which exists on the whole interval $[a,b]$. Consequently, the solution depends continuously on the initial condition.*

For $\gamma \in \{+,-\}$, let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying

$$y(a) = \gamma\rho \sin\alpha \quad \text{and} \quad y'(a) = \gamma\rho\cos\alpha, \tag{2.2}$$

where $\rho > 0$ is a parameter. Let $\theta(t,\rho)$ be the Prüfer angle of $y(t,\rho)$, i.e., $\theta(\cdot,\rho)$ is a continuous function on $[a,b]$ such that

$$\tan\theta(t,\rho) = y(t,\rho)/y'(t,\rho) \quad \text{and} \quad \theta(a,\rho) = \alpha.$$

By Lemma 2.1, $\theta(t,\rho)$ is continuous in ρ on $(0,\infty)$ for any $t \in [a,b]$. The following results are from [8, Lemmas 4.1, 4.2, 4.4, and 4.5].

Lemma 2.2. *(i) Assume $f_0 < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b,\rho) < (n+1)\pi$ for all $\rho \in (0,\rho_*)$.*

(ii) Assume $\lambda_n < f_\infty$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b,\rho) > (n+1)\pi$ for all $\rho \in (\rho^*,\infty)$.

Lemma 2.3. *(i) Assume $f_\infty < \lambda_n$ for some $n \in \mathbb{N}_0$. Then there exists $\rho^* > 0$ such that $\theta(b,\rho) < (n+1)\pi$ for all $\rho \in (\rho^*, \infty)$.*

(ii) Assume $\lambda_n < f_0$ for some $n \in \mathbb{N}_0$. Then there exists $\rho_* > 0$ such that $\theta(b,\rho) > (n+1)\pi$ for all $\rho \in (0, \rho_*)$.

Proof of Theorem 2.1. We first prove it for the case where $f_0 < \lambda_n$ and $\lambda_{n+1} < f_\infty$. Without loss of generality we assume $\gamma = +$. The case with $\gamma = -$ can be proved in the same way. Let $y(t,\rho)$ be the solution of Eq. (1.1) satisfying (2.2) with $\gamma = +$, and $\theta(t,\rho)$ its Prüfer angle. By Lemma 2.2, there exist $0 < \rho_* < \rho^* < \infty$ such that

$$\theta(b,\rho) < (n+1)\pi \quad \text{for all } \rho \in (0, \rho_*)$$

and

$$\theta(b,\rho) > (n+2)\pi \quad \text{for all } \rho \in (\rho^*, \infty).$$

By the continuity of $\theta(t,\rho)$ in ρ, there exist $\rho_* \le \rho_{n+1} < \rho_{n+2} \le \rho^*$ such that

$$\theta(b,\rho_{n+1}) = (n+1)\pi \quad \text{and} \quad \theta(b,\rho_{n+2}) = (n+2)\pi, \tag{2.3}$$

and

$$(n+1)\pi < \theta(b,\rho) < (n+2)\pi \quad \text{for } \rho_{n+1} < \rho < \rho_{n+2}. \tag{2.4}$$

Define an energy function for $y(t,\rho)$ by

$$E(t,\rho) = \frac{1}{2}[y'(t,\rho)]^2 + w(t)F(y(t,\rho)), \quad t \in [a,b] \text{ and } \rho > 0, \tag{2.5}$$

where $F(y) = \int_0^y f(s)ds$. Then

$$E'(t,\rho) = w'(t)F(y(t,\rho)) \ge -\frac{w'_-(t)}{w(t)}E(t,\rho).$$

It follows that

$$\ln\frac{E(b,\rho)}{E(\eta_i,\rho)} = \int_{\eta_i}^b \frac{E'(t,\rho)}{E(t,\rho)}dt \ge -\int_a^b \frac{w'_-(t)}{w(t)}dt = -k_0.$$

Thus

$$E(\eta_i,\rho) \le e^{k_0}E(b,\rho), \quad i = 1,\dots,m. \tag{2.6}$$

We observe that for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$

$$E(\eta_i,\rho) \ge \frac{1}{2}[y'(\eta_i,\rho)]^2, \quad i = 1,\dots,m,$$

and

$$E(b,\rho) = \frac{1}{2}[y'(b,\rho)]^2.$$

Thus for $\rho = \rho_{n+1}$ and $\rho = \rho_{n+2}$

$$|y'(\eta_i,\rho)| \le \sqrt{2E(\eta_i,\rho)} \quad \text{and} \quad |y'(b,\rho)| = \sqrt{2E(b,\rho)}. \tag{2.7}$$

Define

$$\Gamma(\rho) = y'(b,\rho) - \sum_{i=1}^m k_i y'(\eta_i,\rho). \tag{2.8}$$

Assume $n = 2k - 1$ with $k \in \mathbb{N}_0$. Since $y'(b, \rho_{2k}) > 0$ and $y'(b, \rho_{2k+1}) < 0$, by (2.6)-(2.8) and (2.1),

$$\begin{aligned}
\Gamma(\rho_{2k}) &= y'(b, \rho_{2k}) - \sum_{i=1}^{m} k_i y'(\eta_i, \rho_{2k}) \\
&\geq y'(b, \rho_{2k}) - \sum_{i=1}^{m} |k_i|\ |y'(\eta_i, \rho_{2k})| \\
&\geq \sqrt{2E(b, \rho_{2k})} - \sum_{i=1}^{m} |k_i| \sqrt{2E(\eta_i, \rho_{2k})} \\
&\geq \sqrt{2E(b, \rho_{2k})} - \sum_{i=1}^{m} |k_i| \sqrt{2e^{k_0} E(b, \rho_{2k})} \\
&= \sqrt{2E(b, \rho_{2k})} \left(1 - \sum_{i=1}^{m} |k_i| e^{k_0/2}\right) > 0, \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(\rho_{2k+1}) &= y'(b, \rho_{2k+1}) - \sum_{i=1}^{m} k_i y'(\eta_i, \rho_{2k+1}) \\
&\leq y'(b, \rho_{2k+1}) + \sum_{i=1}^{m} |k_i|\ |y'(\eta_i, \rho_{2k+1})| \\
&\leq -\sqrt{2E(b, \rho_{2k+1})} + \sum_{i=1}^{m} |k_i| \sqrt{2E(\eta_i, \rho_{2k+1})} \qquad (2.10) \\
&\leq \sqrt{2E(b, \rho_{2k+1})} \left(-1 + \sum_{i=1}^{m} |k_i| e^{k_0/2}\right) < 0. \qquad (2.11)
\end{aligned}$$

By the continuity of $\Gamma(\rho)$, there exists $\bar{\rho} \in (\rho_{2k}, \rho_{2k+1})$ such that $\Gamma(\bar{\rho}) = 0$. Similarly, for $n = 2k$ with $k \in \mathbb{N}_0$, there exists $\bar{\rho} \in (\rho_{2k+1}, \rho_{2k+2})$ such that $\Gamma(\bar{\rho}) = 0$. In both cases, from (2.4)

$$(n+1)\pi < \theta(b, \bar{\rho}) < (n+2)\pi.$$

Since for $t \in [a, b]$

$$\theta'(t, \rho) = \cos^2 \theta(t, \rho) + w(t) \frac{f(y(t, \rho)) y(t, \rho)}{r^2(t, \rho)}, \qquad (2.12)$$

where $r = (y^2 + {y'}^2)^{1/2}$, we have that $\theta(\cdot, \rho)$ is strictly increasing on $[a, b]$. We note that $y(t) = 0$ if and only if $\theta(t, \rho) = 0 \pmod{\pi}$. Thus, y has exactly $n+1$ zeros in (a, b). Initial condition (2.2) implies that $y(t, \bar{\rho}) > 0$ in a right-neighborhood of a. Therefore, $y(t, \bar{\rho}) \in \mathcal{S}_{n+1}^{+}$.

The proof for the case where $f_\infty < \lambda_n$ and $\lambda_{n+1} < f_0$ is essentially the same as above except that the discussion is based on Lemma 2.3 instead of Lemma 2.2. □

Proof of Corollary 2.1. Consider the SLP consisting of Eq. (1.5) and the BC

$$\begin{aligned}
\cos\alpha\ y(a) - \sin\alpha\ y'(a) = 0, &\qquad \alpha \in [0, \pi), \\
\cos\beta\ y(b) - \sin\beta\ y'(b) = 0, &\qquad \beta \in (0, \pi].
\end{aligned}$$

Denote by $\lambda_n(\alpha,\beta)$ the n-th eigenvalue of this problem for $n \in \mathbb{N}_0$. It is easy to see that $\lambda_0(\pi/2,\pi/2) = 0$. In fact, $y_0(t) \equiv 1$ is an associated eigenfunction. From [12, Theorem 4.2] and [11, Lemma 3.32] we see that $\lambda_0(\alpha,\beta)$ is a continuous function of (α,β) on $[0,\pi)\times(0,\pi]$, and is strictly decreasing in α and strictly increasing in β. Furthermore, for any $\beta \in (0,\pi]$

$$\lim_{\alpha\to\pi-} \lambda_0(\alpha,\beta) = -\infty \quad \text{and} \quad \lim_{\alpha\to\pi-} \lambda_{n+1}(\alpha,\beta) = \lambda_n(0,\beta) \text{ for } n \in \mathbb{N}_0.$$

This shows that $\lambda_0(\pi/2,\pi) > 0$, and hence there exists $\alpha^* \in (\pi/2,\pi)$ such that $\lambda_0(\alpha,\pi) > 0$ for $\alpha \in [0,\alpha^*)$, and $\lambda_0(\alpha,\pi) \le 0$ and $\lambda_1(\alpha,\pi) > 0$ for $\alpha \in [\alpha^*,\pi)$. Note that $\beta = \pi$ if and only if $y(b) = 0$. Then the conclusion follows from Theorem 2.1. □

Proof of Theorem 2.2. (i) Assume to the contrary that BVP (1.1), (1.2) has a solution $y \in \mathcal{S}_i^\gamma$ for some $i \ge n+1$ and $\gamma \in \{+,-\}$. Let $\tilde{w}(t) = w(t)f(y(t))/y(t)$. Then $\tilde{w}(t)$ is continuous on $[a,b]$ by the continuous extension since $f_0 < \infty$. Let $\theta(t)$ be the Prüfer angle of $y(t)$ with $\theta(a) = \alpha$. Then $\theta(t)$ satisfies Eq. (2.12) and hence is strictly increasing on $[a,b]$. Note from the assumption that $\tilde{w}(t) < \lambda_n w(t) \le \lambda_{i-1}w(t)$ on $[a,b]$, we have

$$\theta'(t) < \cos^2\theta(t) + \lambda_{i-1}w(t)\sin^2\theta(t).$$

Let $u(t)$ be an eigenfunction of SLP (1.5), (1.6) associated with the eigenvalue λ_{i-1}, and $\phi(t)$ its Prüfer angle with $\phi(a) = \alpha$. Then

$$\phi'(t) = \cos^2\phi(t) + \lambda_{i-1}w(t)\sin^2\phi(t)$$

and $\phi(b) = i\pi$. By the theory of differential inequalities we find that $\theta(b) < \phi(b)$. This contradicts the assumption that $y \in \mathcal{S}_i^\gamma$.

(ii) It is similar to (i) and hence is omitted. □

3. **Dependence of nodal solutions on the problem.** In this section we investigate the changes of the existence of different types of nodal solutions of BVP (1.1), (1.2) as the problem changes. The first result is about the changes as the interval $[a,b]$ shrinks, more precisely, as $b \to a^+$. We discuss both the cases when one of f_0 and f_∞ is infinite and when both of them are finite.

Theorem 3.1. *Let Eq.* (1.1) *and BC* (1.2) *be fixed and let* (2.1) *hold.*

(i) Assume either $f_0 < \infty$ and $f_\infty = \infty$, or $f_\infty < \infty$ and $f_0 = \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a,b_n)$ and for any $i \ge n$, BVP (1.1), (1.2) has a solution $y_i^\gamma \in \mathcal{S}_{i+1}^\gamma$ for $\gamma \in \{+,-\}$.

(ii) Assume $f_0 < \infty$ and $f_\infty < \infty$. Then for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a,b_n)$ and for any $i \ge n+1$, BVP (1.1), (1.2) has no solutions in $\mathcal{S}_i^\gamma$ for $\gamma \in \{+,-\}$.

To prove Theorem 3.1 we need the following result about the dependence of the n-th eigenvalue of BVP (1.5), (1.6) on the right endpoint b which can be excerpted from [10, Theorems 2.2 and 2.3].

Lemma 3.1. *Consider the n-th eigenvalue of BVP (1.5), (1.6) as a function of b for $b \in (a,\infty)$, denoted by $\lambda_n(b)$. Then for $n \in \mathbb{N}_0$, $\lambda_n(b) \to \infty$ as $b \to a+$.*

Proof of Theorem 3.1. (i) Without loss of generality assume $f_0 < \infty$ and $f_\infty = \infty$. Let $\lambda_n(b)$ be defined as in Lemma 3.1. By Lemma 3.1, for any $n \in \mathbb{N}_0$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ we have $f_0 < \lambda_n(b) < f_\infty$ and hence $f_0 < \lambda_i(b) < f_\infty$ for all $i \geq n$. Then the conclusion follows from Theorem 2.1.

(ii) By Lemma 3.1, for any $n \in \mathbb{N}$, there exists $b_n > a$ such that for any $b \in (a, b_n)$ we have that $\lambda_n(b) > f^* := \sup\{f(y)/y : y \in (0, \infty)\}$. Then the conclusion follows from Theorem 2.2, (i). □

We then present a result on the nonexistence of certain types of nodal solutions of BVP (1.1), (1.2) as the function w increases in a given direction. More precisely, let $s \geq 0$ and $h \in C^1[a, b]$ such that $h(t) > 0$ on $[a, b]$, and consider the equation

$$y'' + [w(t) + sh(t)]f(y) = 0. \tag{3.1}$$

Theorem 3.2. *Let the interval $[a, b]$ and BC (1.2) be fixed and let (2.1) hold. Assume $f(y)/y \geq f_* > 0$ for all $y \neq 0$. Then for any $n \in \mathbb{N}_0$, there exists $s_n \geq 0$ such that for any $s > s_n$ and for any $i \leq n$, BVP (3.1), (1.2) has no solution $y_i^\gamma \in \mathcal{S}_i^\gamma$ for $\gamma \in \{+, -\}$.*

To prove Theorem 3.2 we need the following result about the dependence of the n-th eigenvalue of BVP (1.5), (1.6) on the function w which can be excerpted from [12, Theorem 4.2].

Lemma 3.2. *Consider the n-th eigenvalue of BVP (1.5), (1.6) as a function of w for $w \in C^1[a, b]$, denoted by $\lambda_n(w)$. Then $\lambda_n(w)$ is decreasing as long as it is positive, i.e., for $w_1, w_2 \in C^1[a, b]$ such that $w_1(t) \leq w_2(t)$ for $t \in [a, b]$, then $\lambda_n(w_1) \geq \lambda_n(w_2)$ as long as $\min\{\lambda_n(w_1), \lambda_n(w_2)\} \geq 0$.*

Proof of Theorem 3.2. For $s \geq 0$ and $i \in \mathbb{N}_0$, we denote by $\lambda_i(s)$ the i-th eigenvalue of the SLP consisting of the equation

$$y'' + \lambda[w(t) + sh(t)]y = 0 \quad \text{on } [a, b]$$

and BC (1.6). Let $h_* = \min\{h(t)/w(t) : t \in [a, b]\}$, and denote by $\mu_i(s)$ the i-th eigenvalue of the SLP consisting of the equation

$$y'' + \mu(1 + sh_*)w(t)y = 0 \quad \text{on } [a, b]$$

and BC (1.6). Since for $s \geq 0$

$$w(t) + sh(t) \geq (1 + sh_*)w(t),$$

by Lemma 3.2

$$\lambda_i(s) \leq \mu_i(s) \text{ for all } s \geq 0 \text{ and } i \geq 0, \text{ whenever } \lambda_i(s) \geq 0. \tag{3.2}$$

Note that for $i \geq 0$, $\mu_i(s)(1 + sh_*) = \mu_i(0)$, we have

$$\mu_i(s) = \frac{\mu_i(0)}{1 + sh_*} \to 0 \text{ as } s \to \infty.$$

This together with (3.2) implies that $\lambda_i(s) < f_*$ as $s \to \infty$. Then, for any $n \in \mathbb{N}_0$ there exists $s_n \geq 0$ such that $\lambda_n(s) < f_*$ for $s > s_n$. Therefore, the conclusion follows from Theorem 2.2, (ii). □

Remark 3.1. Theorems 3.1 and 3.2 show that we can "create" or "eliminate" certain types of nodal solutions by changing the interval $[a, b]$ and the function w. Since the eigenvalues of SLP (1.5), (1.6) can be easily computed using computer software such as that in [2], we are able to construct specific BVPs (1.1), (1.2) which have or do not have nodal solutions in $\mathcal{S}_n^\gamma$ for a prescribed $n \in \mathbb{N}_0$.

Remark 3.2. It is not clear how the existence of different types of nodal solutions of BVP (1.1), (1.2) is affected by the BC parameter α; however, we observe that the results of Theorems 2.1 and 2.2 are not affected by the locations of the points η_i, nor by the values or even the signs of k_i, for $i = 1, \dots, m$, as long as (H4) and (2.1) remain valid.

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