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A DUAL-PETROV-GALERKIN METHOD FOR EXTENDED FIFTH-ORDER KORTEWEG-DE VRIES TYPE EQUATIONS

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Abstract. This paper extends the dual-Petrov-Galerkin method proposed by Shen [16] and further developed by Yuan, Shen and Wu [23] to several integrable and non-integrable fifth-order KdV type equations. These fifth-order equations arise in modeling different wave phenomena and involve various nonlinear terms. The method is implemented to compute the solitary wave solutions of these equations and the numerical results imply that this scheme is capable of capturing, with very high accuracy, the details of these solutions with modest computational costs. It is also shown that the scheme is stable under a very mild stability constraint, and is second-order accurate in time and spectrally accurate in space.

1. Introduction.

Fifth-order Korteweg-de Vries (KdV) type equations

$$u_t - u_{xxxxx} = F(x, t, u, u_x, u_{xx}, u_{xxx})$$

arise naturally in modeling many different wave phenomena such as gravity-capillary waves, the propagation of shallow water waves over a flat surface and magneto-sound propagation in plasmas (see e.g. [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 18, 20, 21, 22]). This work focuses on two families of fifth-order KdV equations: the non-integrable family

$$u_t - \frac{2}{15} u_{xxxxx} - (\nu u - b) u_{xxx} - (3u + 2\nu u_{xx}) u_x = 0 \tag{1}$$

and the completely integrable family

$$u_t - u_{xxxxx} - \alpha\, u\, u_{xxx} - (\gamma\, u^2 + \beta\, u_{xx}) u_x = 0, \tag{2}$$

where ν, b, α, γ and β are real parameters. Both families are special cases in the general class of Hamiltonian equations studied in [13]. (1) models water waves with surface tension and reduces to the Kawahara equation when $\nu = 0$. The

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second family contains several well-known equations such as the Kaup-Kupershmidt equation ([4, 8, 14, 22])

$$u_t - u_{xxxxx} - 10uu_{xxx} - 25u_xu_{xx} - 20\,u^2u_x = 0, \tag{3}$$

the Lax equation ([2, 10, 15, 22])

$$u_t - u_{xxxxx} - 10u\,u_{xxx} - 20u_{xx}u_x - 30u^2u_x = 0 \tag{4}$$

and the Sawada-Kotera equation ([2, 11, 19, 22])

$$u_t - u_{xxxxx} - 15u\,u_{xxx} - 15u_{xx}u_x - 45u^2u_x = 0. \tag{5}$$

We study the initial- and boundary-value problem (IBVP) of these equations in the space-time domain $[-1,1]\times[0,T]$ with the following initial and boundary values

$$\begin{aligned} &u(-1,t) = g(t), \quad u_x(-1,t) = h(t), \\ &\qquad\qquad u(1,t) = u_x(1,t) = u_{xx}(1,t) = 0, \quad t \in [0,T] \\ &u(x,0) = u_0(x), \quad x \in [-1,1] \end{aligned} \tag{6}$$

and the goal is to understand the solutions of these IBVPs mainly through numerical computations. Due to the fifth-order terms in these equations, it is very difficult to compute the solutions of these equations accurately and efficiently. Recently, J. Shen [16] proposed a new dual-Petrov-Galerkin method for the third and higher odd-order equations and has proven to be very effective for the KdV type equations in bounded domains [16] and in semi-infinite intervals [17]. In [23], a numerical scheme based on the dual-Petrov-Galerkin method was proposed and implemented for the Kawahara and modified Kawahara equations. This paper extends this scheme to suit more general fifth-order KdV type equations including the two families mentioned above. We intentionally selected two families of equations with different integrability to test the universality of our numerical scheme.

To demonstrate the effectiveness of the numerical scheme, we compute for each one of the equations (1), (3), (4) and (5) some computationally challenging solitary waves. By tracking the L^2-differences between the exact solitary solutions and the numerical solutions at various times, we found that the numerical scheme is extremely accurate (sometimes as small as the order of 10^{-9}). In addition, we also quantitatively studied how these L^2-errors vary according to the time steps and the tables we constructed clearly demonstrate that the Crank-Nicholson-leap-frog discretization (in time) in our scheme is of the second order in time. We leave more details to Section 3.

These quantitative results also propelled us to rigorously prove that the numerical solutions of all these equations converge to the exact solutions with a second-order accuracy in time and spectrally in space. The exact convergence rate is given in Section 2.

2. **Numerical methods.**

The numerical scheme consists of a dual-Petrov-Galerkin method in space and the second-order Crank-Nicholson-leap-frog discretization in time.

As pointed out in [23], (6) can be converted into a homogenous boundary condition. In fact, if we consider $w(x,t) = u(x,t) - v(x,t)$ with $v(x,t) = \frac{(1-x)^3}{8}[(h(t) +$

$\frac{3}{2}g(t))(x+1)+g(t)]$, then w solves an IBVP with homogeneous boundary conditions. Therefore, we shall assume, without loss of generality, $g(t)=h(t)\equiv 0$.

The dual-Petrov-Galerkin method generates a sequence of approximate solutions that satisfy a weak form of the original differential equations as tested against polynomials in a dual space. To describe this method and the full discretization more precisely, we recall a few notation from [16, 23].

Let $I=(-1,1)$ and let $H^m_\omega(I)$ $(m=0,\pm1,\cdots)$ denote the weighted Sobolev spaces whose norms are denoted by $\|\cdot\|_{m,\omega}$. In particular, the norm and inner product of $L^2_\omega(I)=H^0_\omega(I)$ are denoted by $\|\cdot\|_\omega$ and $(\cdot,\cdot)_\omega$ respectively. Let $\omega(x)$ be a positive function (not necessarily in $L^1(I)$), we define

$$L^2_\omega(I)=\{u:(u,u)_\omega:=\int_I u^2(x)\omega(x)dx<+\infty\} \tag{7}$$

with the norm $\|\cdot\|_\omega=(u,u)^{\frac{1}{2}}_\omega$. We denote by C a generic constant that is independent of any parameters and functions.

For any constants α and β, let $\omega^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$ be the Jacobi weight function with index (α,β). We define a set of non-uniformly weighted Sobolev spaces as follows:

$$H^m_{\omega^{\alpha,\beta}}(I)=\{u\in L^2_{\omega^{\alpha,\beta}}(I):\partial^l_x u\in L^2_{\omega^{l-\alpha,l-\beta}}(I),\ 1\le l\le m\}. \tag{8}$$

Let P_N denote the space of polynomials of degree $\le N$ and set

$$\begin{aligned} W_N&=\{u\in P_N:\ u(\pm1)=u_x(\pm1)=u_{xx}(1)=0\},\\ W^*_N&=\{u\in P_N:\ u(\pm1)=u_x(\pm1)=u_{xx}(-1)=0\}. \end{aligned} \tag{9}$$

Let Π_N be the orthogonal projection from $L^2_{\omega^{-3,-2}}$ onto W_N defined by

$$(u-\Pi_N u,v_N)_{\omega^{-3,-2}}=0 \quad \text{for any } v_N\in W_N.$$

We are now ready to provide the numerical scheme for the IBVP of (1), namely

$$\begin{aligned} &u_t-\frac{2}{15}u_{xxxxx}-(\nu u-b)u_{xxx}-(3u+2\nu u_{xx})u_x=0, \quad x\in I, t\in(0,T],\\ &u(\pm1,t)=u_x(\pm1,t)=u_{xx}(1,t)=0, \quad t\in[0,T],\\ &u(x,0)=u_0(x), \quad x\in I. \end{aligned} \tag{10}$$

Assume (10) admits a unique solution u satisfying

$$u\in C^3([0,T];L^2_{\omega^{2,2}}(I))\cap C^1([0,T];H^m_{\omega^{-3,-2}}(I)) \quad \text{with } m\ge 3.$$

For a given Δt, we set $t_k=k\Delta t$ and let $u^0_N=\Pi u_0$ and u^1_N be a suitable approximation of $u(\cdot,t_1)$. Then, the second-order Crank-Nicolson-leap-frog scheme in time with a dual-Petrov-Galerkin approximation in space reads: for $k=1,2,\cdots,[T/\Delta t]-1$, find $u^{k+1}_N\in W_N$ such that

$$\begin{aligned} \frac{1}{2\Delta t}(u^{k+1}_N-u^{k-1}_N,\eta_N)&+\frac{1}{15}(\partial^2_x(u^{k+1}_N+u^{k-1}_N),\partial^3_x\eta_N)\\ &+\frac{b}{2}(\partial_x(u^{k+1}_N+u^{k-1}_N),\partial^2_x\eta_N)\\ =-\frac{3}{2}((u^k_N)^2,\partial_x\eta_N)&-\nu(u^k_N\partial^2_x u^k_N,\partial_x\eta_N)-\frac{\nu}{2}((\partial_x u^k_N)^2,\partial_x\eta_N) \end{aligned} \tag{11}$$

for any $\eta_N\in W^*_N$.

Notice that for any $v_N \in W_N$, we have $\omega^{-1,1} v_N \in W_N^*$. Thus, the above dual-Petrov-Galerkin formulation is equivalent to the following weighted spectral-Galerkin approximation: Find $u_N \in W_N$ such that

$$\begin{aligned}\frac{1}{2\Delta t}(u_N^{k+1} - u_N^{k-1}, v_N)_{\omega^{-1,1}} &+ \frac{1}{15}(\partial_x^2(u_N^{k+1} + u_N^{k-1}), \partial_x^3(v_N \omega^{-1,1})) \\ &+ \frac{b}{2}(\partial_x(u_N^{k+1} + u_N^{k-1}), \partial_x^2(v_N \omega^{-1,1})) = -\frac{3}{2}((u_N^k)^2, \partial_x(v_N\, \omega^{-1,1})) \\ &-\nu(u_N^k \partial_x^2 u_N^k, \partial_x(v_N \omega^{-1,1})) - \frac{\nu}{2}((\partial_x u_N^k)^2, \partial_x(v_N \omega^{-1,1})) \qquad (12)\end{aligned}$$

for any $v_n \in W_N$. The dual-Petrov-Galerkin formulation (11) is most suitable for implementation while the weighted Galerkin formulation (12) is more convenient for error analysis.

To study the convergence of (12), we denote

$$e_N^k = u(\cdot, t_k) - u_N^k, \quad \hat{e}_N^k = \Pi u(\cdot, t_k) - u_N^k \quad \text{and} \quad \tilde{e}_N^k = u(\cdot, t_k) - \Pi u(\cdot, t_k).$$

Clearly, $e_N^k = \hat{e}_N^k + \tilde{e}_N^k$. Following the argument in [23], we can establish the following convergence result.

Theorem 2.1. *Assume (10) admits a unique solution u satisfying*

$$u \in C^3([0,T]; L^2_{\omega^{2,2}}(I)) \cap C^1([0,T]; H^m_{\omega^{-3,-2}}(I)) \quad \text{with } m \geq 3.$$

Let $b \geq -\frac{3}{80}$ and ν be real parameters. Then, there exists a $c_0 > 0$ such that if $\Delta t N < c_0$, then (12) is unconditionally stable and the following error estimate holds for $1 \leq n \leq [T/\Delta t] - 1$,

$$\|e_N^{n+1}\|_{\omega^{-1,1}} \leq C\left((\Delta t)^2 + N^{1-m}\right),$$

$$\left(\Delta t \sum_{k=1}^{n} \|\partial_x^2(e_N^{k+1} + e_N^{k-1})\|^2_{\omega^{-1,0}}\right)^{\frac{1}{2}} \leq C\left((\Delta t)^2 + N^{1-m}\right),$$

where C is a constant that may depend on the parameters b and ν.

The condition $b > -\frac{3}{80}$ is imposed to guarantee the coercivity of the linear part of the equation (see [12]). The schemes for IBVP's of the other three equations (3), (4) and (5) and the convergence results are similar. We shall therefore omit further details.

3. Numerical results.

This section presents the numerical results for the IBVP's for all four equations (1), (3), (4) and (5). The dual-Petrov-Galerkin scheme detailed in the previous section is applied to simulate the solitary wave solutions of these equations. The L^2-errors at various times and corresponding to different time steps are recorded to test the accuracy and the convergence rate of the scheme. Solitary wave solutions are fundamental objects in the study of wave phenomena ([1, 3, 13]) and the computations of these waves serve as a test ground for judging the quality of numerical schemes.

3.1. **Solitary waves of the non-integrable equation.** We start with the numerical approximation of solitary wave solutions of (1). Without loss of generally, we set $\nu = 1$. It is known (see [21]) that (1) has a family of solitary wave solutions

$$u_s(x,t) = 3(b+\tfrac{1}{2})\,\text{sech}^2\left(\sqrt{\tfrac{3(2b+1)}{4}}(x-at)\right), \quad a = \tfrac{3}{5}(2b+1)(b-2), \quad b \geq -\tfrac{1}{2}. \tag{13}$$

Letting $a = 0$ and thus $b = 2$ yields the steady state solution

$$u_{\text{ex1}}(x) = 7.5\,\text{sech}^2\left(\frac{\sqrt{15}}{2}x\right)$$

and letting $a = -1.2$ and $b = 0$ generates the traveling wave solution

$$u_{\text{ex2}}(x,t) = 1.5\,\text{sech}^2\left(\frac{\sqrt{3}}{2}(x+1.2t+x_0)\right).$$

We simulate the solutions of (1) with these two initial conditions. In order to apply the dual-Petrov-Galerkin scheme detailed in the previous section, we let $x_0 = 0$ and restrict the problem to the finite interval $[-L, L]$ with L sufficiently large such that the solutions are essentially zero at $\pm L$ for $t \in [0,T]$. We apply the scaling $\tilde{x} = L^{-1}x$, $\tilde{t} = L^{-1}t$. For notational convenience, we still write (x,t) for $(\tilde{x}, \tilde{t})$. Then we are led to consider the following IBVP for the scaled equation

$$\begin{array}{l} u_t - \frac{2}{15L^4}u_{xxxxx} - \frac{1}{L^2}(u-2)u_{xxx} - (3u + \frac{2}{L^2}u_{xx})u_x = 0, \quad x \in (-1,1) \\ u(\pm 1) = u_x(\pm 1) = u_{xx}(1) = 0, \\ u(x,0) = u_{\text{ex1}}(Lx) = 7.5\,\text{sech}^2\left(\frac{\sqrt{15}}{2}Lx\right) \end{array} \tag{14}$$

and

$$\begin{array}{l} u_t - \frac{2}{15L^4}u_{xxxxx} - \frac{1}{L^2}u\,u_{xxx} - (3u + \frac{2}{L^2}u_{xx})u_x = 0, \quad x \in (-1,1) \\ u(\pm 1) = u_x(\pm 1) = u_{xx}(1) = 0, \\ u(x,0) = u_{\text{ex2}}(Lx,0) = 1.5\,\text{sech}^2\left(\frac{\sqrt{3}}{2}Lx\right). \end{array} \tag{15}$$

Below, we present some numerical results with $L = 200$ using $N = 1000$ in the dual-Petrov-Galerkin scheme. In Table 1, we provide the L^2- errors at $t = 1$ and $t = 2$, which is of the order 10^{-9}. The numerical scheme is extremely accurate for the steady-state solution. In Table 2, we list the L^2-errors at different times with two different time steps. Note that in these numerical tests, the spatial error (with $N = 1000$) is negligible and the error is dominated by the time discretization error. Table 2 clearly indicates that the Crank-Nicholson-leap-frog scheme is of second-order in time.

Time	$L^2(\Delta$ t=1.0E-4)
1	1.93 E-9
2	7.895 E-9

TABLE 1. L^2-errors for steady-state solution

Time	L^2-error(Δ t=1.0E-5)	L^2-error(Δ t=2.0E-5)	Rate
0.025	5.053E-7	1.997E-6	3.95
0.05	8.237E-7	3.243E-6	3.94
0.1	1.218E-6	4.778E-6	3.92

TABLE 2. L^2-errors for traveling wave solution of (15) with $L = 60$

3.2. **Solitary wave solutions of the integrable equations.** This subsection simulates the solitary wave solutions of the Kaup-Kupershmidt (KK) equation(3), the Lax equation (4) and the Sawada-Kortera (SK) equation (5).

(3), (4) and (5) are known to have the following solitary wave solutions

$$\begin{aligned} u_{\text{KK}}(x,t) &= \frac{24\exp(x+t)[4\exp(x+t)+\exp(2(x+t))+16]}{[16\exp(x+t)+\exp(2(x+t))+16]^2}, \\ u_{\text{Lax}}(x,t) &= 2\,\text{sech}^2(x+16t), \\ u_{\text{SK}}(x,t) &= 2\,\text{sech}^2(x+16t), \end{aligned}$$

respectively (see e.g. [2, 4, 8, 10, 11, 14, 15, 19, 22]). We compute these solitary wave solutions and track the L^2-errors between the numerical solutions and the corresponding exact solutions.

As mentioned in the previous subsection, in order to apply the dual-Petrov-Galerkin scheme, we rescale the equations with $(\tilde{x},\tilde{t}) = (-L^{-1}x, L^{-1}t)$ and still use (x,t) to denote $(\tilde{x},\tilde{t})$. Then we are led to consider the following IBVP's for the scaled equations:

$$\begin{aligned} & u_t - \frac{1}{L^4}u_{xxxxx} - \frac{10}{L^2}uu_{xxx} - \frac{25}{L^2}u_{xx}u_x - 20u^2u_x = 0, \quad x \in (-1,1), \\ & u(\pm 1,t) = u_x(\pm 1,t) = u_{xx}(1,t) = 0, \\ & u(x,0) = \frac{24\exp(Lx)[4\exp(Lx)+\exp(2Lx)+16]}{[16\exp(Lx)+\exp(2Lx)+16]^2}, \end{aligned} \tag{16}$$

$$\begin{aligned} & u_t - \frac{1}{L^4}u_{xxxxx} - \frac{10}{L^2}uu_{xxx} - \frac{20}{L^2}u_{xx}u_x - 30u^2u_x = 0, \quad x \in (-1,1), \\ & u(\pm 1,t) = u_x(\pm 1,t) = u_{xx}(1,t) = 0, \\ & u(x,0) = 2\,\text{sech}^2(Lx), \end{aligned} \tag{17}$$

and

$$\begin{aligned} & u_t - \frac{1}{L^4}u_{xxxxx} - \frac{15}{L^2}uu_{xxx} - \frac{15}{L^2}u_{xx}u_x - 45u^2u_x = 0, \quad x \in (-1,1), \\ & u(\pm 1) = u_x(\pm 1) = u_{xx}(1) = 0, \\ & u(x,0) = 2\,\text{sech}^2(Lx). \end{aligned} \tag{18}$$

Below, we present some numerical results with $L = 40$ using $N = 1000$ in the dual-Petrov-Galerkin scheme. In Tables 3-5, we list the L^2-errors at different times with two different time steps. Note that in these numerical tests, the spatial error (with $N = 1000$) is negligible and the error is dominated by the time discretization error. Tables 3-5 clearly indicates that the Crank-Nicholson-leap-frog scheme is of second-order in time.

Time	L^2-error(Δt=1.0E-4)	L^2-error(Δt=2.0E-4)	rate
0.125	2.641E-6	1.057 E-5	4.0
0.25	2.881E-6	1.152E-5	4.0
0.375	3.007E-6	1.203E-5	4.0

TABLE 3. Errors for the solitary wave solution of KK equation

Time	L^2-error(Δt=1.0E-6)	L^2-error(Δt=2.0E-6)	rate
0.01	2.131E-6	8.524E-6	4.0
0.02	2.706E-6	1.082E-5	4.0
0.03	3.219E-6	1.288E-5	4.0

TABLE 4. Errors for the solitary wave solution of Lax equation

Time	L^2-error(Δt=1.0E-6)	L^2-error(Δt=2.0E-6)	rate
0.01	6.83E-6	2.733E-5	4.0
0.02	8.658E-6	3.465E-5	4.0
0.03	1.147E-5	4.594E-5	4.01

TABLE 5. Errors for solitary wave solution of SK equation

In Figures (1) and (2), we plot the computed and exact solutions for the IBVPs (15), (16), (17) and (18). The computed solutions and the exact solutions are virtually indistinguishable.

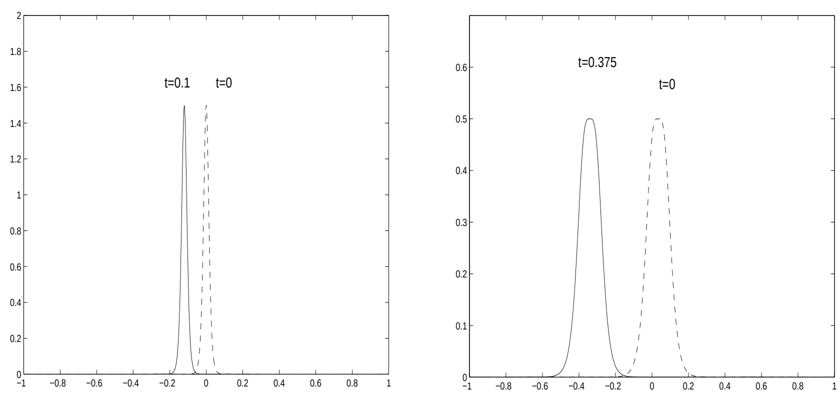

FIGURE 1. Solitary wave solutions of (15) (left) with Δt=1.0E-5 and of (16) (right) with Δt=1.0E-4

4. **Concluding remarks.** We presented a numerical scheme consists of dual-Petrov-Galerkin method in space and Crank-Nicholson-leap-frog in time for the extended fifth-order KdV equation which has been proposed to model many physical phenomena such as gravity-capillary waves and magneto-sound propagation in plasmas. At each time step, the scheme is reduced to a linear fifth-order equation with constant coefficients that can be very efficiently solved by the dual-Petrov-Galerkin method. It is shown that the scheme is stable under a very mild stability constraint, and

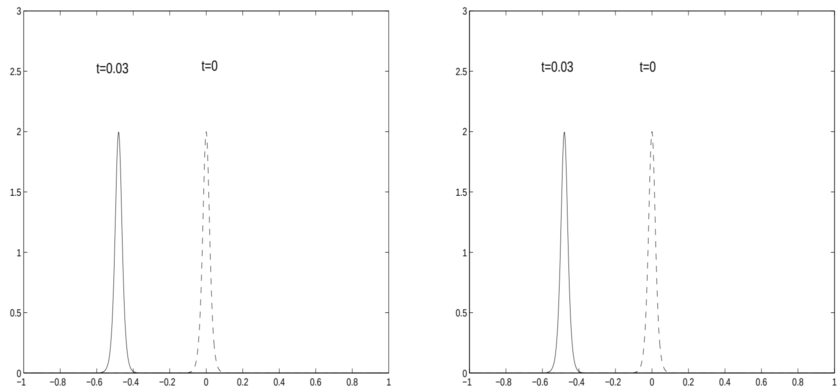

FIGURE 2. Solitary wave solutions of (17) (left) and of (18) (right) with Δt=1.0E-6

is second-order accurate in time and spectrally accurate in space. We used this scheme to compute solitary wave solutions of one non-integrable equation and three integrable ones including the Kaup-Kupershmidt equation, the Lax equation and the Sawada-Kotera equation, and our numerical results indicate that the scheme is capable of capturing, with very high accuracy, solitary wave solutions with modest computational costs.

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