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SAMPLING - RECONSTRUCTION PROCEDURE WITH JITTER OF MARKOV CONTINUOUS PROCESSES FORMED BY STOCHASTIC DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

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Abstract. To describe sampling - reconstruction procedure (SRP) of Markov processes the conditional mean rule is used. There are two types of stochastic differential equations under consideration: 1) linear with varying in time coefficients; 2) non linear coefficients. In the first Gaussian case it is sufficiently to obtain the expression for conditional covariance function and then to calculate the reconstruction function and the error reconstruction function. In the case 2 it is necessary to obtain the solution of the corresponding Fokker - Plank - Kolmogorov equation for the conditional probability density functions (pdf). We obtain the required conditional pdf with two fixed samples and then determine the reconstruction function and the error reconstruction function. The jitter effect is described by random variable with the beta-distribution. Some examples are given.

1. **Introduction.** We consider the statistical description of Sampling - Reconstruction Procedures (SRP) with jitter of some continuous Markov Processes. First of all we give some definitions of terms which will be used in the present paper.

1) All stochastic processes are formed by systems described by the stochastic differential equation of the first order with the affecting force as white noise. In this case the output process will be one dimensional Markovian.process.

2) Usually samples of any process realization are located in the deterministic known points. However, a generator of sampled impulses can have some instabilities. Then the point location of samples will be random. This effect is called as the *jitter*. The jitter effect must be described by a model of a random variable.

3) In all cases (without jitter or with jitter), after sampling of one realization we have a set of samples. Knowing the sample set it is necessary to reconstruct the form of the initial realization. It is impossible to determine the exact form of realization so we need to estimate its value in any time (this is the reconstruction function) and to find the estimation error (this is the reconstruction error function).

4) In order to find the both principal statistical characteristics of SRP (the reconstruction function and the reconstruction error function) we have to use a statistical criterium. It is very useful to apply the conditional mean rule The estimation

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following to this rule provides a possibility to minimize the mean square error automatically. Then the conditional mean function is the reconstruction function and the conditional variance is the reconstruction error function.

5) The known arbitrary set of samples is the fixed variables in the expressions of the conditional mean and the conditional.variance. The reconstruction procedure *between* any pairs of samples is called the *interpolation* procedure. The reconstruction procedure in future (with respect of the last sample) is called the *extrapolation* procedure.

6) In the Gaussian case the reconstruction function is a sum of some samples multiplying by corresponding basic functions. These basic functions must be determined in the SRP investigation

Some first papers devoted to the statistical description of SRP with jitter of random processes are related to 60-th years of last century [1]-[3]. During the last decades this problem is permanently investigated by various methods (see for instance [4]-[6]) There are some principal drawbacks of majority mentioned publications: 1) the number of samples is equal to infinity, 2) the information about the probability density function (pdf) of the sampled process is not used, 3) all samples have the same jitter distribution. In order to overcome these drawbacks one can use the known conditional mean rule [7]. This approach has been applied in some SRP descriptions of different random processes [8]-[11]. In the present paper the SRP with jitter of Markov continuous processes is analyzed. Any continuous Markov process is described by its stochastic equation of the first order with white noise as an affecting force. Below we analyze two non trivial problems: SRP of *Gaussian* processes on the output of a linear dynamic system with *varying* in time parameters and SRP of *non Gaussian* processes. The mathematical model of jitter effect is chosen as the random variable with the beta distribution [11]. In the restriction of the Markov property the interpolation reconstruction algorithm with two samples is considered. The sampling interval can be arbitrary. There are two goals of the SRP investigation: to determine the reconstruction function and the error reconstruction function. It is necessary to find out the influence of all parameters of the sampled process, of the sampling intervals and of the type of jitter effect on the both mentioned principal SRP characteristics. The scientific novelty of the present paper is connected with the statistical description of SRP with jitter of continuous Markov processes.on the base of author's investigation method. We obtain the optimal reconstruction functions and the minimum error reconstruction functions for two non trivial above mentioned cases. The results of the investigation can be used: 1) in the calculation of the reconstruction quality when the sampling interval is given; 2) in the optimal choice of the sampling interval when the maximum error reconstruction function is given.

2. **SRP of Gaussian processes on the output of the dynamic systems with varying in time parameters without jitter.** The stochastic equation for the output process $x(t)$ has the view

$$\dot{x}(t) = -\alpha(t)x(t) + \alpha(t)n(t) \tag{1}$$

where $\alpha(t)$ is a time varying coefficient, $n(t)$ is white noise with the spectral density $N_0/2$.

The Gaussian process $x(t)$ is non stationary. Putting x_0, t_0 as an initial condition one can write the solution of the equation (1):

$$x(t) = \left\{ \int_{t_0}^{t} \frac{\alpha(u)n(u)du}{\exp\left[-\int_{t_0}^{t}\alpha(z)dz\right]} + x_0 \right\} \exp\left[-\int_{t_0}^{t}\alpha(z)dz\right]. \quad (2)$$

The mathematical expectation of the output process is:

$$\langle x(t)\rangle = x_0 \exp\left[-\int_{t_0}^{t}\alpha(z)dz\right] \quad (3)$$

Using (2) and (3) one can find the expression for the covariance function of the process $x(t)$:

$$K(t_1,t_2) = \langle \mathring{x}(t_1)\mathring{x}(t_2)\rangle = \exp\left[-\int_{t_1}^{t_2}\alpha(z)dz\right]\int_{t_0}^{t_1}\alpha^2(u) \times \quad (4)$$

$$\times \exp\left[-\int_{u}^{t_1}\alpha(z)dz\right] du, t_2 \geq t_1.$$

The expressions (3) and (4) describe the output non stationary Gaussian process completely.

Now we shell use the general formulas for the conditional mean $\tilde{m}(t)$ and the conditional variance $\tilde{\sigma}^2(t)$ of Gaussian process with the fixed set of samples $X, T = \{x(T_1), x(T_2), ..., x(T_N)\}$:

$$\tilde{m}(t) = m(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} K(t,T_i)a_{ij}\left[x(T_j) - m(T_j)\right] \quad (5)$$

$$\tilde{\sigma}^2(t) = \sigma^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K(t,T_i)a_{ij}K(T_j,t) \quad (6)$$

where $m(t)$ and $\sigma^2(t)$ are the mathematical expectation and the variance of the sampled process, a_{ij} is the element of the inverse covariance function with fixed time arguments

$$A = K^{-1}(T_i,T_j) = \begin{bmatrix} K(T_1,T_1) & K(T_1,T_2) & ... & K(T_1,T_{1N}) \\ K(T_2,T_1) & K(T_2,T_2) & ... & K(T_2,T_N) \\ ... & ... & ... & ... \\ K(T_N,T_1) & K(T_N,T_2) & ... & K(T_N,T_N) \end{bmatrix}^{-1}. \quad (7)$$

The expression (5) determines the reconstruction function and the expression (6) gives us the error reconstruction function in the general case. We pay our attention to the error reconstruction function $\tilde{\sigma}^2(t)$ does not depend on the samples. It depends on the sampling intervals only. It is clear that the reconstruction function $\tilde{m}(t)$ depends on the sample values $x(T_j)(j = 1, ...N)$. In order to characterize the reconstruction procedure without samples it is possible to introduce the basic function $B_j(t)$

$$B_j(t) = \sum_{i=1}^{N} K(t, T_i) a_{ij} \tag{8}$$

Then the reconstruction function $\tilde{m}(t)$ can be expressed by the formula

$$\tilde{m}(t) = \sum_{j=1}^{N} B_j(t) x(T_j) + m(t) \left[1 - \sum_{j=1}^{N} B_j(t) \right]. \tag{9}$$

There are two SRP variants for the Markov processes: $N = 1$ for the extrapolation reconstruction and $N = 2$ for the interpolation reconstruction. Other samples can not influence on the reconstruction procedure owing to Markov's property. Below we consider the second case only.

Example 1. SRP of Gaussian non stationary process without jitter

Let us concretize the function $\alpha(t)$ as a linear function:

$$\alpha(t) = \begin{cases} \alpha_0, 0 < t < t_\alpha \\ \alpha_0 + V_\alpha(t - t_\alpha), t > t_\alpha \end{cases}. \tag{10}$$

Then using (10) and (4) one can find the covariance function for the output process under consideration:

$$K(t_1, t_2) = \exp\left[-\alpha_0 \left(\tau_2 + \tau_1\right)\right] \exp\left[-0.5 V_\alpha \left(\tau_2^2 + \tau_1^1\right)\right] \left[\exp\left(V_\alpha \tau_1^2\right) \times \right. \\ \left. \times \left(1 - \exp\left(-2\alpha_0 t_\alpha\right)\right) + \frac{2}{\alpha_0} \int_0^{\tau_1} \left(\alpha_0 + V_\alpha \tau\right)^2 \exp\left(2\alpha\tau + V_\alpha \tau^2\right) d\tau \right] \tag{11}$$

where $\tau_1 = t_1 - t_\alpha, \tau_2 = t_2 - t_\alpha$.

Substituting (11) into (5) and (6) one can obtain two principal SRP characteristics. Fig. 1 illustrates the error reconstruction function $\tilde{\sigma}^2(t)$ for $N = 2$ and for various values of V_α with the sampling interval $\Delta T = 0.5$.

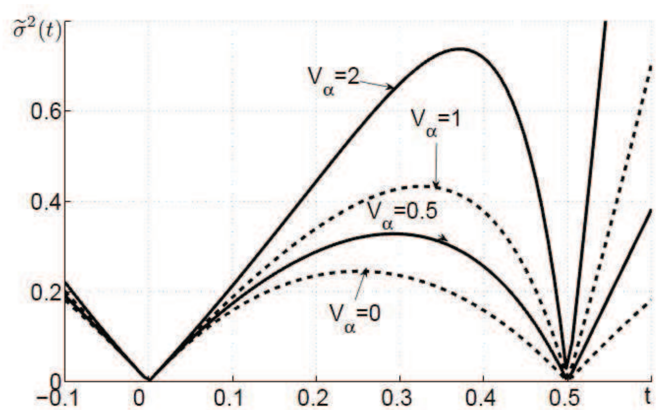

FIGURE 1. Error reconstruction function of a Gaussian Process without jitter

3. **SRP of Gaussian processes on the output of the dynamic systems with varying in time parameters with jitter.** Let us assume that sampling times have a jitter. It means that all sampling times (or some of them) can be random, i.e.

$$\breve{T}_j = T_j + \varepsilon_j \tag{12}$$

where ε_j is a random variable with a known pdf.

It is convenient to use the Beta distribution for the description of jitter effect [11]:

$$w(\varepsilon_j) = \frac{(\varepsilon_j - a_j)^{\beta-1} (\varepsilon_j - b_j)^{\gamma-1}}{B(\beta,\gamma)(b_j - a_j)^{\beta+\gamma-1}}, a_j \le \varepsilon_j \le b_j \tag{13}$$

where a_j and b_j are the interval bounds of jitter, $B(\beta,\gamma)$ is the Beta function.

In the jitter presence all fixed times T_i and T_j (or some of them) must be changed by the values $\breve{T}_i$ and $\breve{T}_j$ in all above mentioned formulas..(Below we use the same upper sign for other functions depended on $\breve{T}_i$ and $\breve{T}_j$) In order to determine the both principal SRP characteristics for the discussed case it is necessary to carry out the statistical average operation of the expressions (9) and (6) with respect of the pdf (13). It means, instead of (6) - (8) we have:

$$\langle B_j(t) \rangle = \left\langle \sum_{i=1}^{N} K(t, \breve{T}_i) \breve{a}_{ij} \right\rangle \tag{14}$$

$$\left\langle \breve{A} \right\rangle = \left\langle K^{-1}\left(\breve{T}_i, \breve{T}_j\right) \right\rangle \tag{15}$$

$$\left\langle \tilde{\sigma}^2(t) \right\rangle = \sigma^2(t) - \left\langle \sum_{i=1}^{N}\sum_{j=1}^{N} K(t, \breve{T}_i) \breve{a}_{ij} K(\breve{T}_j, t) \right\rangle \tag{16}$$

here the angular parenthesis mean the statistical operation.

The expressions (14) - (16) are final.

Example 2. SRP of Gaussian non stationary process with jitter

For the sake of the illustration convenience let us consider three samples here. We give the graphs of the error reconstruction function in Fig. 2 for the following conditions: the first and the third samples have not jitter and the second sample has jitter with the uniform distribution ($\beta = 1, \gamma = 1$), $\Delta T = 0.5, b_2 - a_2 = 0.1$ and with various values of the parameter V_α. As one can see the error reconstruction function is not equal to zero in the point of the second sample.

4. **SRP of Markov Non Gaussian processes.** Markov non Gaussian processes are formed by nonlinear dynamic systems driven by white noise. In the case the stochastic differential equation has the view:

$$\dot{x}(t) = f(x,t) + g(x,t)n(t) \tag{17}$$

where $f(x)$ and $g(x)$ are some deterministic nonlinear functions.

On the basis of (17) one can determine [13] kinetic coefficients $K_i(x,t)\,(i = 1; 2)$:

$$K_1(x,t) = f(x,t) + \frac{N_0}{4} g(x,t) \frac{\partial g(x,t)}{\partial x}; K_2(x,t) + N_0 g^2(x,t)/2 \tag{18}$$

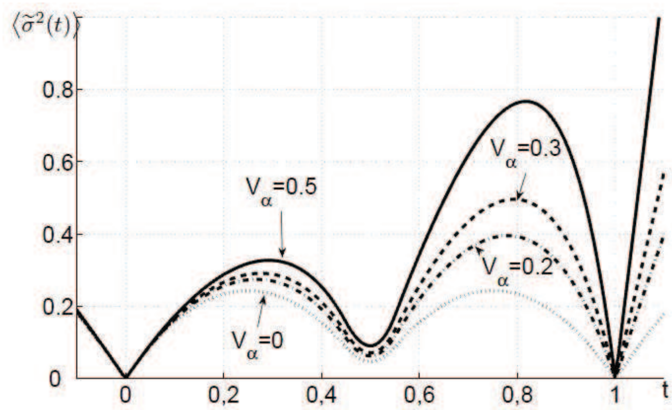

FIGURE 2. Error reconstruction function of a Gaussian Process with jitter of the second sample

and to write the corresponding Fokker-Plank-Kolmogorov equation with respect of the conditional pdf of Markov process:

$$\frac{\partial w\left(x,t\left|x_0,t_0\right.\right)}{\partial t} = -\frac{\partial}{\partial x}\left[K_1\left(x,t\right)w\left(x,t\left|x_0,t_0\right.\right)\right]+ +\frac{\partial^2}{\partial x^2}\left[K_2\left(x,t\right)w\left(x,t\left|x_0,t_0\right.\right)\right]. \tag{19}$$

The solution of the equation (19) gives us the analytic expression for the conditional pdf $w\left(x,t\left|x_0,t_0\right.\right)$. This pdf has one fixed point x_0,t_0, In order to describe the *extrapolation* reconstruction procedure it is sufficiently to know the conditional pdf $w\left(x,t\left|x_0,t_0\right.\right) = w\left(x(t)\left|x(T_1)\right.\right)$. For the *interpolation* reconstruction we need to know conditional pdf with two fixed samples $w\left(x(t)\left|x(T_1),x(T_2)\right.\right)$ $(T_1 < t < T_2)$. Using Markov's property and the conditional pdf one can write the required expression:

$$w\left(x(t)\left|x(T_1),x(T_2)\right.\right) = \frac{w\left(x(t)\left|x(T_1)\right.\right)w\left(x(T_2)\left|x(t)\right.\right)}{w\left(x(T_2)\left|x(T_1)\right.\right)}. \tag{20}$$

On the basis of (20) one can calculate the conditional moment function $\tilde{m}_p = \left\langle x^p(t)\left|x(T_1),x(T_2)\right.\right\rangle$ of an arbitrary order p. The reconstruction function is the first conditional moment function

$$\tilde{m}_1(t) = \tilde{m}(t) = \left\langle x(t)\left|x(T_1),x(T_2)\right.\right\rangle \tag{21}$$

and the error reconstruction function is the conditional variance

$$\tilde{\sigma}^2(t) = \tilde{m}_2(t) - \tilde{m}_1^2(t) \tag{22}$$

The application of the expressions (20) - (22) provides a possibility to determine the both principal SRP characteristics of any non Gaussian Markov processes. The calculation must be carried out numerically.

5. **SRP of the Markov Rayleigh process.** This is a particular case. The corresponding stochastic differential equation has the view [12], [13]

$$\dot{x} = -\kappa x + \frac{N_0}{4x} + n(t) \tag{23}$$

where κ is a constant.

The solution of the corresponding Fokker-Plank-Kolmogorov equation is known, see [12], p. 195:

$$w\left(x(t)\,|x(T_1)\,\right) = \frac{x(t)}{\sigma^2\left(1-Q^2\right)} I_0\left(\frac{Q}{1-Q^2}\frac{x(t)x(T_1)}{\sigma^2}\right) \times \times \exp\left[-\frac{x^2(t)+Q^2x^2(T_1)}{2\sigma^2\left(1-Q^2\right)}\right] \tag{24}$$

where $I_0(\cdot)$ is the Bessel function and $Q = Q(t-T_1) = \exp(-\mu\,|t-T_1|)$, μ is a constant.

Example 3. SRP of the Rayleigh process without jitter

Using (24) and changing designations into (20) one can calculate the initial moment functions $\tilde{m}_1(t)$ and $\tilde{m}_2(t)$. Then following to (21) and (22) we determine the required expressions for the reconstruction function $\tilde{m}(t)$ and for the error reconstruction function $\tilde{\sigma}^2(t)$. The calculation results are presented in Fig.3. and Fig.4. with parameters $\mu = 1$, $\Delta T = 4$. The values of samples are: $x(T_1) = 0$,$x(T_2) = 2$ and $x(T_1) = 4$, $x(T_2) = 1$. There is a very important difference between SRP of Gaussian process and SRP of Rayleigh process: In the first case the error reconstruction function does not depend on sample values but in the second case this function depends on these values (see Fig. 4). In order to characterize the SRP quality by one curve it is necessary to carry out the statistical average operation with respect of two-dimensional pdf $w\left(x(T_1)\;x(T_2)\right)$.

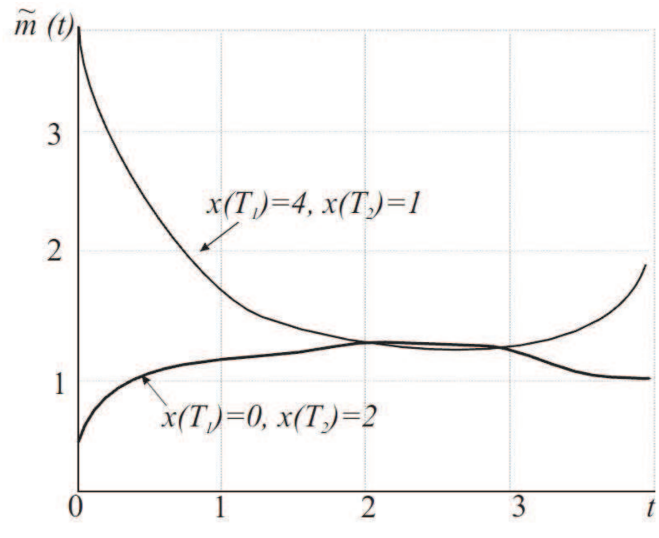

FIGURE 3. Reconstruction function of the Rayleigh process

Example 4. SRP of the Rayleigh process with jitter

In order to analyze SRP with jitter all fixed times T_j (or some of them) must be changed by $\breve{T}_j$ according to (12). After this it is necessary to fulfil the statistical

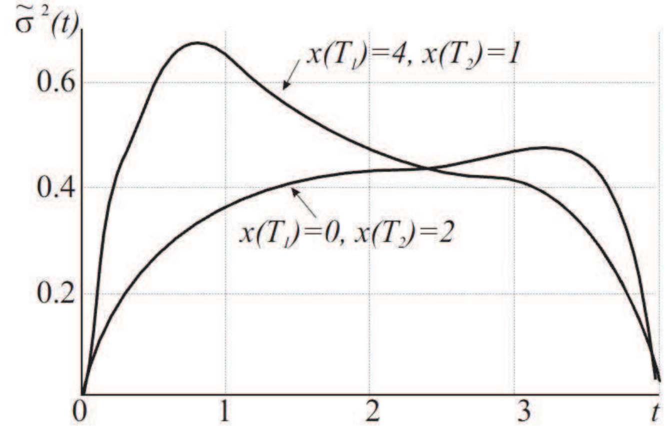

FIGURE 4. Error reconstruction function of the Rayleigh process without jitter

average operation with respect of pdf $w(\varepsilon_j)$ (see (13)). It means that instead of (22) we have to calculate the average error reconstruction function

$$\left\langle \tilde{\sigma}^2(t) \right\rangle = \left\langle \tilde{m}_2(t) \right\rangle - \left\langle \tilde{m}_1^2(t) \right\rangle \tag{25}$$

The results of calculations are presented in Fig. 5 for the following parameters $x(T_1) = 3$; $x(T_2) = 00.95$; $\Delta T = 2$. The first sample has not jitter, the second sample has jitter with parameters $\beta = \alpha = 2$ and $a = -0.2$, $b = 0.2$. One can see that the error is not equal to zero in the point of the second sample.

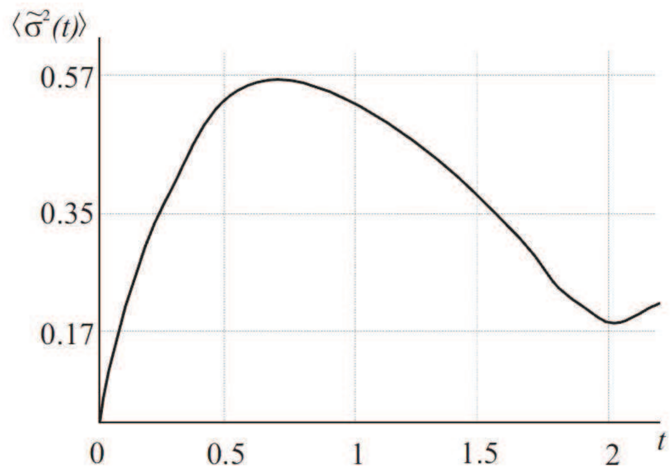

FIGURE 5. Error reconstruction function of the Rayleigh with jitter of the first sample

6. **Conclusions.** The statistical SRP description of some Markov continuous processes is given without or with a jitter effect. The general expressions for the

reconstruction function and the error reconstruction function are obtained. Some non trivial examples are given. The suggested method provides a possibility to calculate the quality of SRP when the location of samples is given or to chose the the sampling interval when the error reconstruction maximum is known.

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