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EXISTENCE AND SHARP LOCALIZATION IN VELOCITY OF SMALL-AMPLITUDE BOLTZMANN SHOCKS

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Abstract. Using a weighted H^s-contraction mapping argument based on the macro-micro decomposition of Liu and Yu, we give an elementary proof of existence, with sharp rates of decay and distance from the Chapman–Enskog approximation, of small-amplitude shock profiles of the Boltzmann equation with hard-sphere potential, recovering and slightly sharpening results obtained by Caflisch and Nicolaenko using different techniques. A key technical point in both analyses is that the linearized collision operator L is negative definite on its range, not only in the standard square-root Maxwellian weighted norm for which it is self-adjoint, but also in norms with nearby weights. Exploring this issue further, we show that L is negative definite on its range in a much wider class of norms including norms with weights asymptotic nearly to a full Maxwellian rather than its square root. This yields sharp localization in velocity at near-Maxwellian rate, rather than the square-root rate obtained in previous analyses.

Contents

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1. **Introduction.** In this paper, we study existence and structure of small-amplitude shock profiles

$$f(x,\xi,t)=\bar f(x-st,\xi), \qquad \lim_{z\to\pm\infty}\bar f(z)=f_\pm \tag{1.1}$$

of the one-dimensional Boltzman equation

$$f_t+\xi_1\partial_x f=\tau^{-1}Q(f,f), \tag{1.2}$$

x, $t\in\mathbb{R}$, where $f(x,t,\xi)\in\mathbb{R}$ denotes the distribution of velocities $\xi\in\mathbb{R}^3$ at point x, t, $\tau>0$ is the Knudsen number, and

$$Q(g,h):=\int\big(g(\xi')h(\xi'_*)-g(\xi)h(\xi_*)\big)C(\Omega,\xi-\xi_*)d\Omega d\xi_* \tag{1.3}$$

is the collision operator, with

$$\begin{aligned}&\xi\in\mathbb{R}^3,\quad \xi_*\in\mathbb{R}^3,\qquad \Omega\in S^2,\\ &\xi'=\xi+\big(\Omega\cdot(\xi_*-\xi)\big)\Omega\\ &\xi'_*=\xi_*-\big(\Omega\cdot(\xi_*-\xi)\big)\Omega.\end{aligned} \tag{1.4}$$

and various collision kernels C. Our main example is the hard sphere case, for which

$$C(\Omega,\xi)=\big|\Omega\cdot\xi\big|. \tag{1.5}$$

See, e.g., [4] for further details.

Note that Q is in this case not symmetric. Other standard examples we have in mind are associated with the class of hard cutoff potentials defined by Grad [7], as considered in [1]. By small-amplitude, we mean that the density

$$\rho(x,t) := \langle 1\rangle_f(x,t) := \int_{\mathbb{R}^3} f(x,t,\xi)d\xi$$

is confined within an ε_0-neighborhood of some fixed reference density $\rho_0 > 0$ for all x, t, for $\varepsilon_0 > 0$ sufficiently small, where, throughout our analysis, we have fixed

$$\tau \equiv 1.$$

Substituting (1.1) into (1.2), we seek, equivalently, stationary solutions of the *traveling-wave equation*

$$(\xi_1 - s)\partial_x f = Q(f,f). \tag{1.6}$$

By frame-indifference, we may without loss of generality take $s = 0$.

Recall [7, 4, 12, 1, 15] that the set of collision invariants $\langle\psi\rangle$, that is, linear forms such that

$$\int_{\mathbb{R}^3} \psi(\xi)Q(g,g)(\xi)d\xi \equiv 0,$$

is spanned by

$$Rf := \langle\Psi\rangle_f = \int \Psi(\xi)f(\xi)d\xi \in \mathbb{R}^5, \qquad \Psi(\xi) = (1,\xi_1,\xi_2,\xi_3,\frac{1}{2}|\xi|^2)^T. \tag{1.7}$$

Associated with these invariants are the macroscopic fluid-dynamical variables

$$\mathrm{u} := Rf =: (\rho,\rho v_1,\rho v_2,\rho v_3,\rho E)^T, \tag{1.8}$$

where ρ is density, $v = (v_1,v_2,v_3)$ is velocity, $E = e + \frac{1}{2}|v|^2$ is total energy density, and e is internal energy density. Here, we are assuming that $f(x,t,\cdot)$ is confined to a space $\mathbb{H}$ to be specified later such that the integral converges for $f \in \mathbb{H}$.

Taking moments of (1.2) and applying definition (1.8), we find that the fluid variables obey the one-dimensional Euler equations

$$\begin{aligned} \rho_t + \partial_x(\rho v_1) &= 0\\ (\rho v)_t + \partial_x(v_1\rho v + pe_1) &= 0\\ (\rho E)_t + \partial_x(v_1(\rho E + p)) &= 0, \end{aligned} \tag{1.9}$$

$e_1 = (1,0,0)^T$ the first standard basis element, where the new variable $p = p(f)$, denoting pressure, depends in general on higher, non-fluid-dynamical moments of f.

The set of equilibrium states $Q(f,f) = 0$ are exactly (see [4]) the Maxwellians

$$M_{\mathrm{u}}(\xi) = \frac{\rho}{\sqrt{(4\pi e/3)^3}}e^{-\frac{|\xi-v|^2}{4e/3}}. \tag{1.10}$$

Making the equilibrium assumption $f = M_{\mathrm{u}}$, we obtain a closed system of equations for the fluid-dynamical variables consisting of the one-dimensional Euler equations (1.9) with pressure $p = p(\rho,E)$ given by the monatomic ideal gas equation of state

$$p = (2/3)\rho E. \tag{1.11}$$

This corresponds to the zeroth-order approximation obtained by formal Chapman-Enskog expansion about a Maxwellian state [7, 12], where the expansion can be taken equivalently in powers of τ, or, as pointed out in [14, 17], in powers of k, where k is the frequency in x, t of perturbations. In the present context, it is the latter derivation that is relevant, since (as we shall see better in a moment) we seek slowly varying solutions near a constant, Maxwellian, state.

The next-, and presumably more accurate, first-order Chapman-Enskog approximation yields the one-dimensional Navier–Stokes equations

$$\begin{aligned} \rho_t + \partial_x(\rho v_1) &= 0 \\ (\rho v)_t + \partial_x(v_1 \rho v + p) &= (\mu v_x)_x \\ (\rho E)_t + \partial_x(v_1(\rho E + p)) &= (\mu v_1 v_x)_x + (\kappa T_x)_x, \end{aligned} \tag{1.12}$$

where temperature T is related to internal energy by $e = \frac{3}{2}RT$, R the universal gas constant, and

$$\mu = \mu(T) > 0 \quad \text{and} \quad \kappa = \kappa(T) > 0 \tag{1.13}$$

are coefficients of viscosity and heat conduction. In the hard sphere case, these may be computed explicitly as $\mu(T) = (RT)^{1/2}\mu(1/R)$, $\kappa(T) = (RT)^{1/2}\kappa(1/R)$ (Chapman's formulae). For derivations, see, e.g., [12], Section 3.

By (1.9), the fluid-dynamical variables associated with a traveling wave (1.1) must satisfy

$$\begin{aligned} -s\partial_x\rho + \partial_x(\rho v_1) &= 0 \\ -s\partial_x(\rho v) + \partial_x(v_1\rho v + pe_1) &= 0 \\ -s\partial_x(\rho E) + \partial_x(v_1(\rho E + p)) &= 0, \end{aligned} \tag{1.14}$$

hence, integrating from $x = -\infty$ to $x = +\infty$, the *Rankine–Hugoniot conditions*

$$s[\rho] = [\rho v_1], \quad s[\rho v] = [v_1\rho v + pe_1], \quad s[\rho E] = [v_1(\rho E + p)], \tag{1.15}$$

where $[h] := h(f_+) - h(f_-)$ denotes change in h across the shock.

Noting that endstates $f_\pm$ of (1.1) by (1.6) necessarily satisfy $Q(f, f)_\pm = 0$, we find that they are Maxwellians $f_\pm = M_{u_\pm}$, and so the associated pressures $p_\pm = p(f_\pm)$ are given by the ideal gas formula (1.11), recovering the standard fact that endstates of a Boltzmann shock (1.1) are Maxwellians with fluid-dynamical variables corresponding to fluid-dynamical shock waves of the Euler equations with monatomic ideal gas equation of state [7, 1].

This gives a rigorous if straightforward connection between Boltzmann shocks and their zeroth order Chapman–Enskog approximation. The following, main result of this paper gives a rigorous connection to the first-order Chapman–Enskog approximation given by the Navier–Stokes equations (1.12) in the limit as shock amplitude goes to zero.

Recall [3], for an ideal-gas equation of state (1.11) under assumptions (1.13), that for each pair of end-states $u_\pm$ satisfying the Rankine–Hugoniot conditions (1.15), the Navier–Stokes equations (1.12) admit a unique up to translation smooth traveling-wave solution

$$u(x, t) = \bar{u}_{NS}(x - st), \quad \lim_{z\to\pm\infty} \bar{u}_{NS}(z) = u_\pm,$$

or Navier–Stokes shock, if and only if the endstates satisfy the *Lax entropy condition*

$$\operatorname{sgn}(v_1^- - s)c^- < \operatorname{sgn}(v_1^+ - s)c^+, \tag{1.16}$$

where $v_1^\pm := v_1(u_\pm)$, $c^\pm := c(u_\pm)$, and $c(u) := ae^{1/2}$ denotes sound speed, $a > 0$ constant. Condition (1.16) is equivalent [31, 13] to the condition that thermodynamical entropy increase across the shock in the upstream direction. Moreover, denoting shock amplitude by $\varepsilon := |u_+ - u_-|$, we have for $\varepsilon > 0$ sufficiently small the asymptotic description [25]

$$|\partial_x^k(\bar{u}_{NS} - u_\pm)| \le C_k \varepsilon^{k+1} e^{-\theta_k \varepsilon |x|}, \quad x \gtrless 0, \quad C_k,\, \theta_k > 0, \quad \text{all } k \ge 0. \tag{1.17}$$

Up to this point in the discussion, we have made essentially no assumption on the nature of the collision kernel $C(\Omega, \xi)$. For the analysis of exact profiles, we require specific properties of C. For simplicity of exposition, we specialize hereafter to the hard-sphere case (1.5). As discussed in Section 11, the arguments extend to a more general class of kernels including the hard cutoff potentials of Grad [7]. Then, our main result is as follows.

Theorem 1.1. *In the hard-sphere case* (1.5)*, for any given fluid-dynamical reference state* u_0 *and* $\eta > 0$*, there exist* $\varepsilon_0 > 0$*,* $\delta_k > 0$*, and* $C_k > 0$ *such that for* $u_\pm$ *satisfying the Rankine-Hugoniot conditions* (1.15)*, the Lax entropy condition* (1.16)*, and* $|u_+ - u_0| \le \varepsilon_0$ *and* $\varepsilon = |u_+ - u_-| \le \varepsilon_0$*, the standing-wave equation* (1.6) *has a solution* $\bar{f}$ *satisfying for all* $k \ge 0$

$$\begin{aligned} \left|\partial_x^k(\bar{u} - \bar{u}_{NS})(x)\right| &\le C_k \varepsilon^{k+2} e^{-\delta_k \varepsilon |x|}, \\ \left|\partial_x^k(\bar{f} - f_{\bar{u}_{NS}})(x,\xi)\right| &\le C_k \varepsilon^{k+2} e^{-\delta_k \varepsilon |x|} M_{u_0}(\xi)^{1-\eta}, \\ \left|\partial_x^k(\bar{f} - f_\pm)(x,\xi)\right| &\le C_k \varepsilon^{k+1} e^{-\delta_k \varepsilon |x|} M_{u_0}(\xi)^{1-\eta}, \end{aligned} \tag{1.18}$$

where $\bar{u} := R\bar{f}$ *is the associated fluid-dynamical profile. Moreover, up to translation, this solution is unique among functions satisfying for* $0 \le k \le 2$*,* c_k *sufficiently small, the weaker estimate*

$$\left|\partial_x^k(\bar{f} - f_{\bar{u}_{NS}})(x,\xi)\right| \le c_k \varepsilon^{k+1} e^{-\delta_k \varepsilon |x|} M_{u_0}(\xi)^{\frac{1}{2}}. \tag{1.19}$$

Remark 1.2. Satisfaction of the Lax entropy condition by $\bar{u}$ follows, through the approximation by $\bar{u}_{NS}$, from strict dissipativity of the Chapman–Enskog expansion (5.6), [25]. This in turn is related to symmetry and coercivity of the linearized collision operator about a Maxwellian, properties deriving ultimately from Boltzmann's H-Theorem. See the discussion of Section 3.1.

Existence of small-amplitude Boltzmann profiles was established some time ago in [1] for the full class of hard cutoff potentials, viewing them as bifurcations from the constant Maxwellian solution $f \equiv M_{u_-}$, with the somewhat weaker existence result

$$\left|\partial_x^k(\bar{f} - f_{\bar{u}_{NS}})(x,\xi)\right| \le C_k \varepsilon^{k+2} e^{-\delta_k \varepsilon |x| - \tau_k |x|^\beta} M_{u_0}(\xi)^{\frac{1}{2}},$$

$0 \le \beta \le 1$, but also the somewhat stronger result of uniqueness among solutions satisfying

$$\left|\bar{f} - M_{u_0}(x,\xi)\right| \le C\varepsilon e^{-\delta_k \varepsilon |x| - \tau_k |x|^\beta} M_{u_0}(\xi)^{\frac{1}{2}} \tag{1.20}$$

for $C > 0$ bounded and $\varepsilon > 0$ sufficiently small. For the hard sphere potential, positivity of profiles, and the improved estimate (1.19) were shown by Liu and Yu [15] by a "macro-micro decomposition" method in which fluid (macroscopic, or equilibrium) and transient (microscopic) effects are separated and estimated by different techniques. This was used in [15] to establish time-evolutionary stability of profiles with respect to perturbations of zero fluid-dynamical mass, $\int u(x)dx = 0$, and thus, *assuming the existence result of* [1], to establish positivity of Boltzmann profiles by the positive maximum principle for the Boltzmann equation (1.2) together with convergence to the Boltzmann profile of its own Maxwellian approximation: by definition, a perturbation of zero relative mass in fluid-dynamical variables.

The purpose of the present paper is to obtain existence from first principles by an elementary argument in the spirit of [15], based on approximate Chapman–Enskog expansion combined with Kawashima type energy estimates [11] (the macro–micro decomposition of the reference), but carried out for the *stationary* (traveling-wave) rather than the time-evolutionary equations, and estimating the finite-dimensional fluid part using sharp ODE estimates in place of the sophisticated energy estimates of [15].[1] In this latter part, we are much aided by the more favorable properties of the stationary fluid equations, a rather standard boundary value ODE system, as compared to the time-evolutionary equations, a hyperbolic–parabolic system of PDE. This in a sense completes the analysis of [15], providing by a common set of techniques both existence (through the present argument) and (through the argument of [15]) positivity. At the same time it gives a truly elementary proof of existence of Boltzmann profiles.

For similar results in the general finite-dimensional relaxation case, see [23, 22]. The key new technical observations needed for the infinite-dimensional case are a way of choosing Kawashima compensators of finite rank (see Remark 4.3), and the fact that the linearized collision operator remains negative definite on its range not only in norms of square-root Maxwellian weight where it is self-adjoint, but also in norms with nearby weights; this allows coordinatization with respect to a single global Maxwellian, avoiding unbounded commutators associated with a changing local Maxwellian frame.

In passing, we obtain also the new result of sharp localization in velocity at near-Maxwellian rate (1.18), which comes from improved estimates on the linearized collision operator independent of the basic argument. A key technical point in all three analyses– [1], [15], and the present one– is that the linearized collision operator L is negative definite on its range, not only in the standard square-root Maxwellian weighted norm, but also in norms with nearby weights. Exploring this issue further, we show that L is negative definite on its range in a much wider class of norms including norms with weights asymptotic nearly to a full Maxwellian rather than its square root. This observation, of interest in its own right, yields through the same existence argument sharp localization in velocity at near-Maxwellian rate, rather than the square-root rate obtained in previous analyses.

Finally, we note that stability of small-amplitude Boltzmann shocks has been shown in [15] with respect to small H^s perturbations with zero mass in fluid variables. It would be very interesting to continue along the same lines to obtain a complete nonlinear stability result as in [30] or [17, 18], with respect to general, not necessarily zero mass, perturbations.

2. The nonlinear collision operator.

We begin by a careful study of the collision operator. Related results may be found, for example, in [2, 5].

2.1. Splitting of the collision operator.

In view of definition (1.3), we split

$$Q(g,h) = Q_+(g,h) - Q_-(g,h) \tag{2.1}$$

into gain and loss parts [7, 4], where, for Ω defined as in (1.4),

$$Q_+(g,h)(\xi) = \int Q_\Omega(g,h)d\Omega, \qquad Q_\Omega(g,h) = \int g(\xi')h(\xi'_*)C(\Omega,\xi-\xi')d\xi_* \tag{2.2}$$

[1] See also [6, 10] in the fluid-dynamical case.

and

$$Q_-(g,h) = g(\xi)\nu_h(\xi), \qquad \nu_h(\xi) = \int C(\Omega, \xi_* - \xi)h(\xi_*)d\xi_* d\Omega. \tag{2.3}$$

2.2. **Estimates for Q_-.** In the hard sphere case (1.5),

$$\nu_h(\xi) = \int C(\Omega, \xi_* - \xi)h(\xi_*)d\xi_* d\Omega = c\int |\xi - \eta)|\ h(\eta)d\eta. \tag{2.4}$$

Here and elsewhere, denote $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ following standard convention. We use the notation $f \lesssim g$ to indicate that $|f| \le C|g|$ for some constant $C > 0$.

Lemma 2.1. *In the hard-sphere case* (1.5)*, for $h > 0$ with $\langle\xi\rangle h \in L^1$, ν_h is positive, continuous and*

$$\langle\xi\rangle \lesssim \nu_h(\xi) \le \langle\xi\rangle \big\|\langle\eta\rangle h\big\|_{L^1}. \tag{2.5}$$

(Here, the constant in the lower bound depends on h.)

Proof. Evidently,

$$\nu_h(\xi) \le A|\xi| + B, \qquad A = \|h\|_{L^1}, \quad B = \||\eta|h\|_{L^1}.$$

This implies the upper bound. Next,

$$\nu_h(\xi) \ge A|\xi| - B$$

which implies the lower bound for $|\xi| > 2(B+1)/A$. For ξ bounded, the integral is continuous and bounded from below. □

2.3. **Estimates for Q_+.** Consider the Maxwellians

$$\omega_s(\xi) = e^{-s|\xi|^2} \tag{2.6}$$

and the weighted L^2 spaces $\mathbb{H}^s = \omega_s L^2(\mathbb{R}^3)$ with norm

$$\big\|f\big\|_{\mathbb{H}^s} = \Big(\int e^{2s|\xi|^2}|f(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \tag{2.7}$$

Proposition 2.2. *In the hard-sphere case* (1.5)*, for any $s > 0$,*

$$\big\|\langle\xi\rangle^{-\frac{1}{2}}Q_+(g,h)\big\|_{\mathbb{H}^s} \lesssim \big\|g\big\|_{\mathbb{H}^s}\big\|\langle\xi\rangle^{\frac{1}{2}}h\big\|_{\mathbb{H}^s}. \tag{2.8}$$

$$\big\|Q_+(g,h)\big\|_{\mathbb{H}^s} \lesssim \big\|g\big\|_{\mathbb{H}^s}\big\|\langle\xi\rangle h\big\|_{\mathbb{H}^s}. \tag{2.9}$$

We first estimate Q_Ω (see definition (2.2)) for a fixed $\Omega \in S^2$.

Lemma 2.3. *In the hard-sphere case* (1.5)*, for $s > 0$,*

$$\big\|\langle\xi\rangle^{-\frac{1}{2}}Q_\Omega(g,h)\big\|_{\mathbb{H}^s} \le C_s\big\|g\big\|_{\mathbb{H}^s}\big\|\langle\xi\rangle^{\frac{1}{2}}h\big\|_{\mathbb{H}^s} \tag{2.10}$$

and

$$\big\|Q_\Omega(g,h)\big\|_{\mathbb{H}^s} \le C_s\big\|g\big\|_{\mathbb{H}^s}\big\|\langle\xi\rangle h\big\|_{\mathbb{H}^s} \tag{2.11}$$

Proof. Choose an orthonormal basis such that $\Omega = (1,0,0)$, in which case

$$\xi' = (\xi_{*1}, \xi_2, \xi_3), \quad \xi'_* = (\xi_1, \xi_{*2}, \xi_{*3}), \quad C(\Omega, \xi_* - \xi) = |\xi_1 - \xi_{*1}|.$$

and Q_Ω has the more explicit form

$$Q_1(g,h)(\xi) = \int g(\eta_1, \xi_2, \xi_3)h(\xi_1, \eta_2, \eta_3)|\xi_1 - \eta_1|d\eta. \tag{2.12}$$

Introducing $G = e^{s|\xi|^2}|g(\xi)|$, $H = e^{s'|\xi|^2}|h(\xi)|$, and

$$q_1 := e^{s|\xi|^2}|Q_\Omega(g,h)(\xi)|,$$

we thus have

$$q_1(\xi) = \int G(\eta_1, \xi_2, \xi_3) H(\xi_1, \eta_2, \eta_3) |\xi_1 - \eta_1| e^{-s|\eta|^2} d\eta.$$

By the Cauchy–Schwarz inequality, therefore,

$$|q_1(\xi)|^2 \lesssim \langle \xi_1 \rangle^2 \int |G(\eta_1, \xi_2, \xi_3)|^2 \; |H(\xi_1, \eta_2, \eta_3)|^2 d\eta,$$

where we have used that

$$\int |\xi_1 - \eta_1|^2 e^{-2s|\eta|^2} d\eta \lesssim \langle \xi_1 \rangle^2.$$

Integrating in ξ, we obtain

$$\|q_1\|_{L^2} \leq \|G\|_{L^2} \|\langle \xi_1 \rangle H\|_{L^2},$$

yielding the first estimate. The proof of the second estimate is similar. □

Integrating over Ω on the unit sphere, implies Proposition 2.2. Combining the estimates for Q_- and Q_+ implies:

Corollary 2.4. *In the hard-sphere case (1.5), for all s, there is C_s such that*

$$\big\| \langle \xi \rangle^{-\frac{1}{2}} Q(g,h) \big\|_{\mathbb{H}^s} \leq C_s \big\| \langle \xi \rangle^{\frac{1}{2}} g \big\|_{\mathbb{H}^s} \big\| \langle \xi \rangle^{\frac{1}{2}} h \big\|_{\mathbb{H}^s} \tag{2.13}$$

2.4. **Further estimates.**

Proposition 2.5. *In the hard sphere case (1.5), suppose that $0 < s < s'$, and $g \in \mathbb{H}^s$ and $h \in \mathbb{H}^{s'}$. Then*

$$\big\| Q_+(g,h) \big\|_{\mathbb{H}^s} \leq C_{s,s'} \big\| g \big\|_{\mathbb{H}^s} \big\| h \big\|_{\mathbb{H}^{s'}}. \tag{2.14}$$

Proof. This follows by (2.9) and $\big\| \langle \xi \rangle h \big\|_{\mathbb{H}^s} \leq C_{s,s'} \big\| h \big\|_{\mathbb{H}^{s'}}$, where

$$C_{s,s'} := \sup_{\xi} \langle \xi \rangle e^{(s-s')|\xi_1|^2}.$$

□

Proposition 2.6. *In the hard-sphere case (1.5), suppose that $0 < s < s'$, and $g \in \mathbb{H}^{s'}$ and $h \in \mathbb{H}^s$. Then*

$$\big\| Q_+(g,h) \big\|_{\mathbb{H}^s} \leq C_{s,s'} \big\| g \big\|_{\mathbb{H}^{s'}} \big\| h \big\|_{\mathbb{H}^s}. \tag{2.15}$$

Proof. Introduce $G = e^{s'|\xi|^2} |g(\xi)|$ and $H = e^{s|\xi|^2} |h(\xi)|$. Then

$$q_\Omega := e^{s|\xi|^2} |Q_\Omega(g,h)(\xi)|$$

satisfies

$$q_\Omega(\xi) \leq e^{(s-s')(|\xi|^2 - (\xi\cdot\Omega)^2)} \int \Phi_\Omega(\xi,\eta) |(\xi - \eta)\cdot \Omega| e^{-s|\eta|^2} d\eta$$

where for all $\Omega \in S^2$:

$$\big\| \Phi_\Omega \big\|_{L^2(\mathbb{R}^3 \times \mathbb{R}^3)} \leq \big\| G \big\|_{L^2} \big\| H \big\|_{L^2}$$

Integrate over Ω and use the Cauchy–Schwarz inequality to get

$$\big| e^{2s|\xi|^2} Q_+(g,h)(\xi) \big|^2 \leq \Big| \int q_\Omega(\xi) d\Omega \Big|^2 \leq A \int |\Phi_\Omega(\xi,\eta)|^2 d\eta d\Omega$$

with

$$A := \int e^{2(s-s')(|\xi|^2 - (\xi\cdot\Omega)|^2)} |(\xi - \eta)\cdot \Omega|^2 e^{-s|\eta|^2} d\eta d\Omega.$$

Thus

$$A \lesssim 1 + \int e^{2(s-s')(|\xi|^2-(\xi\cdot\Omega)|^2)}|\xi\cdot\Omega|^2 d\Omega.$$

To compute the integral (for large ξ), we can choose coordinates such that $\xi = (0,0,t)$, $t>0$, and parametrize the sphere with angular coordinates $\theta\in[0,2\pi]$ and $\varphi\in[-\frac{1}{2}\pi,\frac{1}{2}\pi]$, so that $\Omega = (\cos\varphi\cos\theta,\cos\varphi\sin\theta,\sin\varphi)$ In this case the integral becomes

$$2\pi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2(s-s')t^2\cos^2\varphi}t^2\sin^2\varphi\cos\varphi\, d\varphi$$

which is smaller than

$$2\pi\int_0^{\frac{1}{2}\pi} e^{2(s-s')t^2\cos^2\varphi}2t^2\sin\varphi\cos\varphi\, d\varphi =$$

$$2\pi\int_0^1 e^{2(s-s')t^2u}t^2du = 2\pi\int_0^{t^2} e^{2(s-s')u}du \le \frac{2\pi}{s'-s}.$$

Therefore A is uniformly bounded and

$$\int \left|e^{2s|\xi|^2}Q_+(g,h)(\xi)\right|^2 d\xi \lesssim \int |\Phi_\Omega(\xi,\eta)|^2 d\eta d\Omega d\xi \lesssim \left\|G\right\|_{L^2}^2\left\|H\right\|_{L^2}^2.$$

□

Corollary 2.7. *In the hard-sphere case (1.5), suppose that $0<s<s'$. and $f\in\mathbb{H}^s$. Then for $a\in\mathbb{H}^{s'}$ the mappings $f\mapsto Q_-(a,f)$, $f\mapsto Q_+(a,f)$ and $f\mapsto Q_+(f,a)$ are bounded from $\mathbb{H}^s$ to $\mathbb{H}^s$, with norm controlled by a constant times $\|a\|_{\mathbb{H}^{s'}}$.*

Remark 2.8. The estimates above were proved for convenience for the Gaussian weights $\omega = e^{|\xi|^2}$ and ω^s. They immediately extend to any Maxwellian weight $M_{\rm u}$ and $M_{\rm u}^s$.

3. **The linearized collision operator.** We next study the linearized collision operator about a Maxwellian or nearby velocity distribution. Fix a reference state $\underline{\rm u}$. The associated Maxwellian $M_{\underline{\rm u}}$ is denoted by $\underline{M}$.

For $s\in]0,1]$, let $\mathbb{H}^s$ denote the space of functions f on $\mathbb{R}^3$ such that

$$\left\|f\right\|_{\mathbb{H}^s}^2 = \int \underline{M}(\xi)^{-2s}|f(\xi)|^2 d\xi < +\infty. \tag{3.1}$$

Note that $\underline{M}\in\mathbb{H}^s$ for all $s<1$. The space $\mathbb{H}^{\frac{1}{2}}$ plays a particular role as will be clear below.

Proposition 3.1. *The quadratic mapping $f\mapsto Q(f,f)$ is continuous from $\langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^s$ to $\ker R\cap(\langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s)$, where R is the operator (1.7) defining the thermodynamical variables. .*

Proof. The action from $\langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^s$ to $\langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s$ is a consequence of Corollary 2.4 and Remark 2.8. That the image is contained in $\ker R$ follows from the known properties of the collision operator:

$$\forall f\in\mathbb{H}^s: \qquad RQ(f,f) = 0. \tag{3.2}$$

□

Given a function a, the linearized collision operator at a is

$$L_a g = Q(a,g) + Q(g,a). \tag{3.3}$$

In particular, we consider first the linearized operator at $a = \underline{M}$:

$$\underline{L} g = Q'_{\underline{M}} g = Q(\underline{M}, g) + Q(g, \underline{M}). \tag{3.4}$$

Corollary 2.7 implies the following result:

Lemma 3.2. *For all $s \in [\frac{1}{2}, 1[$, $\underline{L}$ is a bounded linear operator from $\langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^s$ to $\ker R \cap \langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s$ and from $\langle\xi\rangle^{-1}\mathbb{H}^s$ to $\ker R \cap \mathbb{H}^s$.*

3.1. **Symmetry and coercivity on $\mathbb{H}^{\frac{1}{2}}$.** Let $\mathbb{V} = \ker R \cap \mathbb{H}^{\frac{1}{2}}$ and let $\mathbb{U}$ denote the orthogonal complement of $\mathbb{V}$ in $\mathbb{H}$. It has dimension 5. Noting that

$$\big(\chi, f\big)_{L^2} = \big(\chi \underline{M}, f\big)_{\mathbb{H}} \tag{3.5}$$

we see that $\mathbb{U}$ is spanned by the functions $\psi_j \underline{M}$. An orthogonal basis is

$$\phi_j(\xi) = \chi_j(\xi)\underline{M}(\xi), \qquad j = 0, \dots, 4, \tag{3.6}$$

with

$$\begin{aligned} &\chi_0(\xi) = 1, \qquad \chi_j(\xi) = \frac{\xi_j - \underline{v}_j}{\sqrt{\gamma \underline{T}}} \quad \text{for } j = 1,2,3, \\ &\chi_4(\xi) = \frac{1}{\sqrt{6}}\Big(\frac{|\xi - \underline{v}|^2}{\gamma \underline{T}} - 3\Big). \end{aligned} \tag{3.7}$$

We denote by $\mathbb{P}_{\mathbb{U}}$ and $\mathbb{P}_{\mathbb{V}}$ the orthogonal projection from $\mathbb{H}^{\frac{1}{2}}$ to $\mathbb{U}$ and $\mathbb{V}$ respectively. In the language of [15], $\mathbb{U}$ is the macroscopic part of f and $\mathbb{V}$ is the microscopic part.

Note that $\mathbb{U} \subset \langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^{\frac{1}{2}}$ and $\mathbb{U} \subset \mathbb{H}^s$ for all $s < 1$. Therefore

Lemma 3.3. *For $s \in [\frac{1}{2}, 1[$, $\mathbb{P}_V$ maps $\mathbb{H}^s$ onto $\mathbb{V}^s := \mathbb{V} \cap \mathbb{H}^s = \ker R \cap \mathbb{H}^s$ and $\langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s$ onto $\mathbb{V} \cap (\langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s)$.*

Remark 3.4. The projections $\mathbb{P}_{\mathbb{U}}$ and $\mathbb{P}_{\mathbb{V}}$ do not commute with the operator of multiplication by ξ_1. They are not orthogonal in $\mathbb{H}^s$ for $s > \frac{1}{2}$, but still produce a continuous decomposition $\mathbb{H}^s = \mathbb{U} \oplus \mathbb{V}^s$.

Proposition 3.5. *$\underline{L}$ is (formally) self adjoint and nonpositive in $\mathbb{H}^{\frac{1}{2}}$ and negative definite on $\mathbb{V}$. More precisely,*

i) for all f and g in $f \in \langle\xi\rangle^{-1}\mathbb{H}^{\frac{1}{2}}$,

$$\big(\underline{L} f, g\big)_{\mathbb{H}^{\frac{1}{2}}} = \big(f, \underline{L} g\big)_{\mathbb{H}^{\frac{1}{2}}}. \tag{3.8}$$

ii) there is $\delta > 0$ such that for all $f \in \langle\xi\rangle^{-1}\mathbb{H}$:

$$\delta \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{V}} f \big\|^2_{\mathbb{H}^{\frac{1}{2}}} \le -\mathrm{Re}\,\big(\underline{L} f, f\big)_{\mathbb{H}^{\frac{1}{2}}}. \tag{3.9}$$

Notes on the proof. This is a classical result in the theory of Boltzmann equation in the hard sphere case and more generally in the case of hard cut off potentials (see e.g. [2, 7, 4, 1])

1. The analysis of section 2 splits $\underline{L}$ into $\underline{L} = -\nu(\xi) + K$, with

$$\begin{aligned} Kg(\xi) = &- \int k_1(\xi, \xi_*) g(\xi_*) d\xi_* \\ &+ \int k_2(\xi, \Omega, \xi_*) g(\xi') d\xi_* d\Omega + \int k_3(\xi, \Omega, \xi_*) g(\xi'_*) d\xi_* d\Omega \end{aligned}$$

with

$$k_1(\xi,\xi_*) = \underline{M}(\xi)\int C(\Omega,\xi_*-\xi)d\Omega = \underline{M}c_0(\xi-\xi_*)$$
$$k_2(\xi,\Omega,\xi_*) = \underline{M}(\xi'_*)C(\Omega,\xi_*-\xi)$$
$$k_3(\xi,\Omega,\xi_*) = \underline{M}(\xi')C(\Omega,\xi_*-\xi)$$

Using the conservations

$$\xi+\xi_* = \xi'+\xi'_*, \qquad |\xi|^2+|\xi_*|^2 = |\xi'|^2+|\xi'_*|^2, \qquad d\xi d\xi_* = d\xi' d\xi'_*, \tag{3.10}$$

which imply that

$$\underline{M}(\xi)\underline{M}(\xi_*) = \underline{M}(\xi')\underline{M}(\xi'_*), \tag{3.11}$$

one shows that

$$\big(\underline{M}^{-1}K_j g, h\big)_{L^2} = \big(g, \underline{M}^{-1}K_j h\big)_{L^2}$$

for $j=1,2,3$, implying the symmetry of $\underline{L}$ in $\mathbb{H}^{\frac12}$.

2. One can also argue as follows. By Boltzmann's H-theorem,

$$\int Q(f,f)\log f d\xi \le 0$$

for all f with enough decay at infinity. Hence, Taylor expanding about the Maxwellian $\underline{M}$, a minimizer of $\int Q(f,f)\log f d\xi$, we obtain symmetry and nonnegativity of the Hessian,

$$\int \frac{(\underline{L}h)h}{\underline{M}} d\xi \le 0,$$

giving nonnegativity of $\underline{L}$ on $\mathbb{H}^{\frac12}$ and also formal self-adjointness.

3. It is known that K is compact in $\mathbb{H}^{\frac12}$ and that $\ker \underline{L} = \mathbb{U}$ (this can be proved using the formulas above). By self-adjointness of $\underline{L}$ on $\mathbb{H}^{\frac12}$, to establish strict negativity on $\mathbb{V}$, it is sufficient to establish a spectral gap between the eigenvalue zero and the essential spectrum of $\underline{L}$. But, this follows from Weyl's Lemma by comparison of $\underline{L} = -\nu + K$ with the multiplication operator by $-\nu(\xi) \le -c_0 < 0$. □

In a more explicit form, the inequality (3.9) reads

$$\delta\int \langle\xi\rangle \underline{M}(\xi)^{-1}|\mathbb{P}_{\mathbb{V}}f(\xi)|^2 d\xi \le -\mathrm{Re}\int \underline{M}(\xi)^{-1}\underline{L}f(\xi)f(\xi)d\xi. \tag{3.12}$$

We also point out the following properties which are freely used below and which follow from the symmetry of $\underline{L}$ in $\mathbb{H}^{\frac12}$:

$$\underline{L} = \mathbb{P}_{\mathbb{V}}\underline{L} = \underline{L}\mathbb{P}_{\mathbb{V}}, \qquad \mathbb{P}_{\mathbb{U}}\underline{L} = \underline{L}\mathbb{P}_{\mathbb{U}} = 0. \tag{3.13}$$

3.2. **Coercivity on $\mathbb{H}^s$.** With $\lambda \ge 0$ to be determined later on, introduce the equivalent norms:

$$\|f\|^2_{\widetilde{\mathbb{H}}^s} := \|f\|_{\mathbb{H}^s} + \lambda\|f\|^2_{\mathbb{H}^{\frac12}}. \tag{3.14}$$

Proposition 3.6. *For $\frac12 \le s < 1$, the operator $\underline{L}$ is continuous from $\langle\xi\rangle^{-\frac12}\mathbb{H}^s$ to $(\langle\xi\rangle^{\frac12}\mathbb{H}^s)\cap\mathbb{V}$ and from $\langle\xi\rangle^{-1}\mathbb{H}^s$ to $\mathbb{V}^s$, and formally coercive on $\mathbb{V}^s$ for the norm (3.14). More precisely, there are $\lambda\ge 0$ and $\delta>0$ such that for all $f\in\langle\xi\rangle^{-1}\mathbb{V}^s$:*

$$\delta\big\|\langle\xi\rangle^{\frac12}\mathbb{P}_{\mathbb{V}}f\big\|^2_{\widetilde{\mathbb{H}}^s} \le -\mathrm{Re}\,\big(\underline{L}f,f\big)_{\widetilde{\mathbb{H}}^s} \tag{3.15}$$

Proof. We want to prove that

$$\begin{aligned}&\delta \int \langle\xi\rangle \big(\underline{M}(\xi)^{-2s} + \lambda \underline{M}(\xi)^{-1}\big) |f(\xi)|^2 d\xi \\ &\quad\leq -\mathrm{Re} \int \big(\underline{M}(\xi)^{-2s} + \lambda \underline{M}(\xi)^{-1}\big) \underline{L} f(\xi) f(\xi) d\xi,\end{aligned} \tag{3.16}$$

Following the analysis of Section 2,

$$\underline{L} = -\nu_0 \mathrm{Id} + K$$

where $\nu_0(\xi) \approx \langle\xi\rangle$ and K is bounded from $\mathbb{H}^s$ to $\mathbb{H}^s$, since the Maxwellian $\underline{M} \in \mathbb{H}^{s'}$ for $s < s' < 1$. Hence there is $\delta_1 > 0$ such that

$$\delta_1 \int \langle\xi\rangle \underline{M}(\xi)^{-2s} |f(\xi)|^2 d\xi \leq -\mathrm{Re} \int \underline{M}(\xi)^{-2s} \underline{L} f(\xi) f(\xi) d\xi + C \int \underline{M}(\xi)^{-2s} |f(\xi)|^2 d\xi.$$

Moreover, by (3.12), there is δ_0 such that

$$\delta_0 \int \langle\xi\rangle \underline{M}(\xi)^{-1} |f(\xi)|^2 d\xi \leq -\mathrm{Re} \int \underline{M}(\xi)^{-1} \underline{L} f(\xi) f(\xi) d\xi.$$

Hence:

$$\begin{aligned}&\int \langle\xi\rangle \big(\delta_1 \underline{M}(\xi)^{-2s} + \lambda \delta_0 \underline{M}(\xi)^{-1}\big) |f(\xi)|^2 d\xi \\ &\qquad \leq -\mathrm{Re} \big(\underline{L} f, f\big)_{\widetilde{\mathbb{H}}^s} + C \int \underline{M}(\xi)^{-2s} |f(\xi)|^2 d\xi.\end{aligned}$$

We choose λ such that for all ξ

$$C M(\xi)^{-2s} \leq \frac{1}{2} \langle\xi\rangle \big(\delta_1 \underline{M}(\xi)^{-2s} + \lambda \delta_0 \underline{M}(\xi)^{-1}\big)$$

implying the inequality (3.16). □

Remark 3.7. Included in the bound (3.15) is the observation that both the first-order Chapman–Enskog approximation $\bar{U}_{NS}$ and the entire hierarchy of higher-order Chapman–Enskog correctors lie in $\mathbb{H}^s$, any $0 < s < 1$, something that is not immediately obvious. Indeed, looking closely at the inversion of L_a, we see that they in fact decay at successively higher polynomial multiples of the full Maxwellian rate.

3.3. **Comparison.** We consider the linearized operator $L_a g = Q(a, g) + Q(g, a)$ at a, not necessarily nonnegative, close to $\underline{M}$.

Proposition 3.8. *For $s \in [\frac{1}{2}, 1[$ and $a \in \langle\xi\rangle^{-\frac{1}{2}} \mathbb{H}^s$, L_a is bounded from $\langle\xi\rangle^{-\frac{1}{2}} \mathbb{H}^s$ to $\langle\xi\rangle^{\frac{1}{2}} \mathbb{H}^s$. Moreover, there are constants $\delta > 0$, $C > 0$, $\lambda \geq 0$ and $\varepsilon_0 > 0$ such that for all $f \in \langle\xi\rangle^{-\frac{1}{2}} \mathbb{H}^s$*

$$\big\| \langle\xi\rangle^{-\frac{1}{2}} L_a f \big\|_{\widetilde{\mathbb{H}}^s} \leq C \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{V}} f \big\|_{\widetilde{\mathbb{H}}^s} + C\varepsilon \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{U}} f \big\|_{\widetilde{\mathbb{H}}^s}, \tag{3.17}$$

$$\big| \big(L_a f, f\big)_{\widetilde{\mathbb{H}}^s} \big| \leq C \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{V}} f \big\|^2_{\widetilde{\mathbb{H}}^s} + C\varepsilon^2 \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{U}} f \big\|^2_{\widetilde{\mathbb{H}}^s}, \tag{3.18}$$

and

$$\delta \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{V}} f \big\|^2_{\widetilde{\mathbb{H}}^s} \leq -\mathrm{Re} \big(L_a f, f\big)_{\widetilde{\mathbb{H}}^s} + C\varepsilon^2 \big\| \langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{U}} f \big\|^2_{\widetilde{\mathbb{H}}^s}, \tag{3.19}$$

with

$$\varepsilon = \varepsilon(a) := \big\| \langle\xi\rangle^{\frac{1}{2}} (a - \underline{M}) \big\|_{\widetilde{\mathbb{H}}^s}. \tag{3.20}$$

In (3.19) the $\widetilde{\mathbb{H}}^s$ scalar product has to be understood as the integral

$$\big(L_a f, f\big)_{\widetilde{\mathbb{H}}^s} = \int \big(\underline{M}(\xi)^{-2s} + \lambda \underline{M}(\xi)^{-1}\big) L_a f(\xi) f(\xi) d\xi, \tag{3.21}$$

which is well defined since $f \in \langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^s$ and $L_a f \in \langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s$.

Proof. That L_a is bounded from $\langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^s$ to $\langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s$ for all $s \in [\frac{1}{2}, 1[$, follows directly from Section 2. Moreover,

$$L_a f - \underline{L} f = Q(a - \underline{M}, f) + Q(f, a - \underline{M})$$

and

$$\big\|\langle\xi\rangle^{-\frac{1}{2}}(L_a f - \underline{L} f)\big\|_{\widetilde{\mathbb{H}}^s} \le C\varepsilon \big\|\langle\xi\rangle^{\frac{1}{2}} f\big\|_{\widetilde{\mathbb{H}}^s}.$$

Since $\underline{L} f = \underline{L}\mathbb{P}_{\mathbb{V}} f$, this implies (3.17). Since $L_a f$ and $\underline{L} f$ belong to $\mathbb{V}$ and thanks to the definition of the modified scalar product $\widetilde{\mathbb{H}}^s$,

$$\big(L_a f - \underline{L} f, f\big)_{\widetilde{\mathbb{H}}^s} = \big(L_a f - \underline{L} f, \mathbb{P}_{\mathbb{V}} f\big)_{\widetilde{\mathbb{H}}^s} = O\Big(\varepsilon \big\|\langle\xi\rangle^{\frac{1}{2}} f\big\|_{\mathbb{H}^s} \big\|\langle\xi\rangle^{\frac{1}{2}} \mathbb{P}_{\mathbb{V}} f\big\|_{\mathbb{H}^s}\Big).$$

With (3.17), this implies (3.18) with a new constant C. With Proposition 3.6, this implies (3.19). □

Remark 3.9. Since $\mathbb{U}$ is finite dimensional, one can use any norm for $\mathbb{P}_{\mathbb{U}} f$ in the estimates (3.17) (3.18) and (3.19) above.

Remark 3.10. We have in mind that $\varepsilon(a)$ can be taken arbitrarily small. This holds if $a = M_{\mathrm{u}}$ and u is close to $\underline{\mathrm{u}}$ since , when $s < 1$,

$$\big\|\langle\xi\rangle^{\frac{1}{2}}(M_{\mathrm{u}} - \underline{M})\big\|^2_{\mathbb{H}^s} = \int \langle\xi\rangle \Big|M_{\mathrm{u}} - \underline{M}\Big|^2 \underline{M}^{-2s} d\xi \;\to\; 0$$

as $|\mathrm{u} - \underline{\mathrm{u}}| \to 0$ by Lebesgue's dominated convergence theorem.

4. Abstract formulation.

We now rephrase the problem in a general framework, for the square-root Maxwellian norm $\mathbb{H} = \mathbb{H}^{\frac{1}{2}}$ in which we carry out the main analysis. We treat general weights in Section 10.2, by a bootstrap argument. Taking the shock speed equal to 0 by frame-indifference, we consider (1.6) as the abstract standing-wave ODE

$$AU' = Q(U, U). \tag{4.1}$$

with

$$Af(\xi) = \xi_1 f(\xi) \tag{4.2}$$

independent of U (semilinearity of the Boltzmann equation), and Q as in (1.3), (1.5).

4.1. Bounds on the transport operator.

The collision operator has been studied acting in spaces $\mathbb{H}^s$ associated to our reference Maxwellian $\underline{M}$. We have the following evident facts regarding the transport operator A.

Proposition 4.1. *For $s \in [\frac{1}{2}, 1[$, the operator A is bounded from $\langle\xi\rangle^{-\frac{1}{2}}\mathbb{H}^s$ to $\langle\xi\rangle^{\frac{1}{2}}\mathbb{H}^s$ and (formally) self adjoint in $\mathbb{H}^s$ as well as in $\widetilde{\mathbb{H}}^s$.*

4.2. **Kawashima multiplier.** We next construct a Kawashima compensator as in [11, 23, 22], but taking special care that the operator remains bounded in this infinite-dimensional setting. Introducing the weighted norms

$$\langle f, g\rangle_{\mathbb{H}_j} := \langle \langle\xi\rangle^j f, \langle\xi\rangle^j g\rangle_{\mathbb{H}},$$

denote by $\mathbb{H}_j = \langle\xi\rangle^{-j}\mathbb{H}$ the space of functions with bounded $\mathbb{H}_j$-norm, $\mathbb{H}_1 \subset \mathbb{H} \subset \mathbb{H}_{-1}$.

Proposition 4.2. *There are C, $\delta > 0$, $\lambda \geq 0$ and there is a finite rank operator $K \in \mathscr{L}(\mathbb{H}_{-1}, \mathbb{H}_1)$ such that such K is skew symmetric in $\widetilde{\mathbb{H}}^s$ and satisfies*

$$\mathrm{Re}\ (KA - \underline{L}) \geq \delta\langle\xi\rangle \mathrm{Id}. \tag{4.3}$$

meaning that

$$c\big\|\langle\xi\rangle^{\frac12} f\big\|^2_{\widetilde{\mathbb{H}}^s} \leq \mathrm{Re}\,\big((KA - L_a)f, f\big)_{\widetilde{\mathbb{H}}^s} \leq C\big\|\langle\xi\rangle^{\frac12} f\big\|^2_{\widetilde{\mathbb{H}}^s}. \tag{4.4}$$

Proof. **a)** We first check that the genuine coupling condition is satisfied, i.e. that there is no eigenvector of A in $\ker \underline{L} = \mathbb{U}$. Indeed, using the basis ϕ_j of $\mathbb{U}$ given in (3.6), an eigenvector of A with eigenvalue τ in $\mathbb{U}$ is a linear combination $\sum \alpha_j\phi_j$ such that the polynomial

$$(\xi_1 - \tau)\sum_{j=0}^{4} \alpha_j\chi_j(\xi)$$

is identically zero. Equating to zero the term of degree 3 implies that $\alpha_4 = 0$. Equating to zero the coefficient of the terms of degree 2 implies that $\alpha_j = 0$ for $j = 1, 2, 3$, and finally $\alpha_0 = 0$. Thus the property is satisfied.

b) We look for K as

$$K = \theta\big(K_{11} + K_{12} + K_{21}\big) \tag{4.5}$$

with $\theta > 0$ a parameter to be chosen and

$$K_{12} = A_{12} := \mathbb{P}_{\mathbb{U}} A \mathbb{P}_{\mathbb{V}} = A^*_{21}, \qquad K_{21} = -K^*_{12} = -\mathbb{P}_{\mathbb{V}} A \mathbb{P}_{\mathbb{U}} := -A_{21},$$

and

$$K_{11} = \mathbb{P}_{\mathbb{U}} K_{11} \mathbb{P}_{\mathbb{U}} = -K^*_{11}.$$

Here $*$ means the adjoint with respect to the scalar product in $\mathbb{H}$. We have used Proposition 4.1.

Thus, with $A_{11} := \mathbb{P}_{\mathbb{U}} A \mathbb{P}_{\mathbb{U}}$,

$$\mathrm{Re}\ \mathbb{P}_{\mathbb{U}} K A \mathbb{P}_{\mathbb{U}} = \frac12 [K_{11}, A_{11}] + A^*_{21} A_{21}, \tag{4.6}$$

The condition a) means that A_{11} (restricted to $\mathbb{U}$) has no eigenvector in $\ker A_{21} = \ker A^*_{21}A_{21}$, with $A^*_{21}A_{21}$ symmetric positive semidefinite and A_{11} symmetric. Since $\dim \mathbb{U}$ is finite (equal to 5), this implies by the standard, finite-dimensional construction of Kawashima et al [11] that one can choose K_{11} such that $\mathrm{Re}\ \mathbb{P}_{\mathbb{U}} K A \mathbb{P}_{\mathbb{U}}$ is positive definite on $\mathbb{U}$: there is $c_1 > 0$ such that

$$\big(\mathrm{Re}\ \mathbb{P}_{\mathbb{U}} K A \mathbb{P}_{\mathbb{U}} f, \mathbb{P}_{\mathbb{U}} f\big)_{\mathbb{H}} \geq c_1 \big\|\mathbb{P}_{\mathbb{U}} f\big\|^2_{\mathbb{H}}. \tag{4.7}$$

Moreover, since $\dim \mathbb{U}$ is finite, there is another $c_1 > 0$ such that

$$\big(\mathrm{Re}\ \mathbb{P}_{\mathbb{U}} K A \mathbb{P}_{\mathbb{U}} f, \mathbb{P}_{\mathbb{U}} f\big)_{\mathbb{H}} \geq c_1 \big\|\mathbb{P}_{\mathbb{U}} f\big\|^2_{\mathbb{H}_1}. \tag{4.8}$$

Thus, using Proposition 3.9 for $a = \underline{M}$:

$$\mathrm{Re}\ \big((KA - \underline{L})f, f\big)_{\mathbb{H}} \geq \theta c_1 \big\|\mathbb{P}_{\mathbb{U}} f\big\|^2_{\mathbb{H}_1} + c\big\|\mathbb{P}_{\mathbb{V}} f\big\|^2_{\mathbb{H}_1} - \theta C \big\|f\big\|_{\mathbb{H}_1}\big\|\mathbb{P}_{\mathbb{V}} f\big\|_{\mathbb{H}_1}$$

with

$$C = \big\|K_{11}\mathbb{P}_{\mathbb{U}} A \mathbb{P}_{\mathbb{V}}\big\| + \big\|K_{12}\mathbb{P}_{\mathbb{V}} A \mathbb{P}_{\mathbb{V}}\big\| + \big\|K_{21}\mathbb{P}_{\mathbb{U}} A \mathbb{P}_{\mathbb{U}}\big\| + \big\|K_{21}\mathbb{P}_{\mathbb{U}} A \mathbb{P}_{\mathbb{V}}\big\|$$

where the norms are taken in $\mathscr{L}(\mathbb{H}_1; \mathbb{H}_{-1})$. All these operators have finite rank $\leq n$ and are bounded. Thus if θ is small enough, this shows that Re $(KA - \underline{L})$ is positive definite in the sense of (4.4). Using the perturbation Lemma 3.8 implies that the estimate remains true for a satisfying (3.20). □

Remark 4.3. The construction above, by reduction to the equilibrium manifold, is essentially different from the original proof of [11] in the finite-dimensional case, which would yield a symmetrizer of infinite rank. The advantage of finite rank is that we need not worry about boundedness of the operator. We note that this is related to methods in the Boltzmann literature in which the Kawashima compensator is replaced by estimates on a reduced Chapman-Enskog approximation such as the Grad 13-moments model or the Navier–Stokes approximation, again to avoid possible boundedness issues; see, e.g., [7, 15].

See also the related construction of [8] in the case that u is scalar, for which K_{11} may be taken equal to zero. We note that we could apply the same reduction argument to the reduced problem and proceed by iteration to this scalar case, thus obtaining an alternative proof in the finite-dimensional case as well.

4.3. **Reduction to bounded operators.** In the hard-sphere case (1.5), we may rescale the equations to obtain a problem involving only bounded operators. We have $\mathbb{H}_1 \subset \mathbb{H} \subset \mathbb{H}_{-1}$, bounded operators from $\mathbb{H}_1$ to $\mathbb{H}_{-1}$ and we work with the scalar product of $\mathbb{H}$. We can multiply the equations on the left by $\langle\xi\rangle^{-1}$: if $A \in \mathscr{L}(\mathbb{H}_1; \mathbb{H}_{-1})$ then

$$\hat{A} := \langle\xi\rangle^{-1} A \in \mathscr{L}(\mathbb{H}_1; \mathbb{H}_1)$$

and

$$\big(\hat{A} f, f\big)_{\mathbb{H}_1} = \big(Af, f\big)_{\mathbb{H}}$$

so that if A is symmetric in $\mathbb{H}$, $\hat{A}$ is symmetric in $\mathbb{H}_1$.

Equivalently, we can make the change of variable $f \mapsto \tilde{f} = \langle\xi\rangle^{\frac{1}{2}} f$ from $\mathbb{H}_1$ to $\mathbb{H}$ and define

$$\tilde{A}\tilde{f} := \langle\xi\rangle^{-\frac{1}{2}} A f = \langle\xi\rangle^{-\frac{1}{2}} A \langle\xi\rangle^{-\frac{1}{2}} f.$$

Then $\tilde{A} \in \mathscr{L}(\mathbb{H}; \mathbb{H})$ and $\tilde{A}$ is symmetric in $\mathbb{H}$ if A is.

By Corollary 2.4, the corresponding collision operators

$$\hat{Q}(f, f) := \langle\xi\rangle^{-1} Q(f, f)$$

and

$$\tilde{Q}(\tilde{f}, \tilde{f}) := \langle\xi\rangle^{-\frac{1}{2}} Q(\langle\xi\rangle^{-\frac{1}{2}} f, \langle\xi\rangle^{-\frac{1}{2}} f)$$

by Corollary 2.4 are bounded as well: $\hat{Q} \in \mathscr{B}(\mathbb{H}_1; \mathbb{H}_1)$ and $\tilde{Q} \in \mathscr{B}(\mathbb{H}; \mathbb{H})$, where $\mathscr{B}(H; H')$ denotes the space of continuous bilinear forms from $H \to H'$, i.e., $B \in \mathscr{B}(H; H')$ if and only if

$$\big\|B(g, h)\big\|_{H'} \leq C_s \big\|g\big\|_H \big\|h\big\|_H. \tag{4.9}$$

4.4. **The framework.** At this point, we have reduced to the following abstract problem, with semilinear structure quite similar to that treated in the finite-dimensional analysis of [23]. Working in $\mathbb{H}$ with operators $\tilde{A}$ and $\tilde{Q}$ and dropping tildes, we study the standing-wave ODE

$$AU' = Q(U,U), \tag{4.10}$$

with U taking its values in an infinite dimensional space $\mathbb{H}$.

4.4.1. *Assumptions on the full system.* We make the following assumptions, verified above for the Boltzmann equation in the hard-sphere case with A, Q replaced by $\tilde{A}$, $\tilde{Q}$.

Assumption 4.4. *(i) A is a bounded self adjoint operator in a (real) Hilbert space $\mathbb{H}$;*

(ii) There is an orthogonal splitting $\mathbb{H} = \mathbb{U} \oplus \mathbb{V}$ with $\mathbb{U}$ finite dimensional

(iii) Q is bilinear, continuous (in sense (4.9)) and symmetric (in U) from $\mathbb{H}\times\mathbb{H}$ to $\mathbb{V}$.

For $U \in \mathbb{H}$, we denote by L_U the bounded operator $V \mapsto 2Q(U,V)$, that is the differential of $Q(U,U)$. We denote by $\mathbb{P}_\mathbb{U}$ and $\mathbb{P}_\mathbb{V}$ the orthogonal projectors on $\mathbb{U}$ and $\mathbb{V}$ respectively. We use the notations $U = u + v$, with $u = \mathbb{P}_\mathbb{U} U$ and $v = \mathbb{P}_\mathbb{V} U$.

Assumption 4.5. *We are given a reference state $\underline{M}$ (in a smaller space $\mathbb{M} \subset \mathbb{H}$) such that $Q(\underline{M},\underline{M}) = 0$, $\underline{L} = L_{\underline{M}}$ is self adjoint with kernel $\mathbb{U}$, and $\underline{L}$ is negative definite on $\mathbb{V}$.*

Lemma 4.6. *There are $\delta > 0$, ε_0 and $C \geq$ such that for $a \in \mathbb{M}$ and $U \in \mathbb{H}$:*

$$-\operatorname{Re}\big(L_a U, U\big)_\mathbb{H} \geq \delta \big\|\mathbb{P}_\mathbb{V} U\big\|_\mathbb{H}^2 - C\varepsilon \big\|\mathbb{P}_\mathbb{U} U\big\|_\mathbb{H} \big\|\mathbb{P}_\mathbb{V} U\big\|_\mathbb{H} \tag{4.11}$$

provided that

$$\big\|a - \underline{M}\big\|_\mathbb{H} \leq \varepsilon \leq \varepsilon_0. \tag{4.12}$$

Proof. By continuity of Q, (4.9), there is C such that

$$\big\|L_a U - \underline{L} U\big\|_\mathbb{H} \leq C \big\|a - \underline{M}\big\|_\mathbb{H} \big\|U\big\|_\mathbb{H}. \tag{4.13}$$

Moreover, there is $\delta > 0$ such that

$$-\big(\underline{L}U, U\big)_\mathbb{H} = -\big(\underline{L}U, \mathbb{P}_\mathbb{V} U\big)_\mathbb{H} = -\big(\underline{L}\mathbb{P}_\mathbb{V} U, \mathbb{P}_\mathbb{V} U\big)_\mathbb{H} \geq \delta \big\|\mathbb{P}_\mathbb{V} U\big\|_\mathbb{H}^2.$$

Since

$$\operatorname{Re}\big(L_a U, U\big)_\mathbb{H} = \operatorname{Re}\big(L_a U, \mathbb{P}_\mathbb{V} U\big)_\mathbb{H}$$

the lemma follows. ☐

Lemma 4.7. *In an $\mathbb{H}$-neighborhood of $\underline{M}$, the zero set of Q is given by a smooth (indeed C^∞) manifold $M = \{U :\ v = v_*(u)\}$ with $v_* : \mathbb{U} \to \mathbb{V}$ smooth.*

Proof. Assumption 4.5 and the Implicit Function Theorem, together with the observation that Q as a continuous biinear form (in sense (4.9)) is C^∞ in the Frechet sense. ☐

We further assume the Kawashima condition established in Proposition 4.3.

Assumption 4.8. *There is a skew symmetric bounded operator $K \in \mathscr{L}(\mathbb{H})$ and a constant $\gamma > 0$ such that*

$$\operatorname{Re} KA - \underline{L} \geq \gamma \mathrm{Id}. \tag{4.14}$$

Using (4.13), this implies

Lemma 4.9. *There are $\gamma > 0$ and $\varepsilon_0 > 0$ such that for $a \in \mathbb{H}$ satisfying (4.12) and $U \in \mathbb{H}$:*

$$\operatorname{Re}\big((KA - L_a)U, U\big)_{\mathbb{H}} \geq \gamma \big\|U\big\|_{\mathbb{H}}^2. \tag{4.15}$$

4.4.2. *Assumptions on the reduced system.* Coordinatizing $U \in \mathbb{H}$ as

$$U = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P_{\mathbb{U}} U \\ P_{\mathbb{V}} U \end{pmatrix}, \tag{4.16}$$

we have

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ q(U,U) \end{pmatrix} \tag{4.17}$$

with $A_{11} \in \mathscr{L}(\mathbb{U};\mathbb{U})$, $A_{12} \in \mathscr{L}(\mathbb{V};\mathbb{U})$ etc. We use the notation

$$h(u,v) := A_{11}u + A_{12}v \in \mathbb{U}. \tag{4.18}$$

Finally, the equilibria are parametrized by u:

$$M(u) = \begin{pmatrix} u \\ v_*(u) \end{pmatrix}, \tag{4.19}$$

where v_* is the smooth mapping from a neighborhood of $\underline{u}$ to a neighborhood of $\underline{v} = v_*(\underline{u})$ in $\mathbb{V}$, as described in Lemma 4.7.

Recall from [32] that the reduced, Navier–Stokes type equations obtained by Chapman–Enskog expansions are

$$h_*(u)' = (b_*(u)u')', \tag{4.20}$$

where

$$h_*(u) \quad := h(u, v_*(u)) = A_{11}u + A_{12}v_*(u), \tag{4.21}$$

$$b_*(u) \quad := -A_{12}c_*(u) \tag{4.22}$$

with

$$\begin{aligned} c_*(u) := \partial_v q^{-1}(u, v_*(u)) & \\ \Big(A_{21} + A_{22}dv_*(u) - dv_*(u)(A_{11} + A_{12}dv_*(u)\Big). \end{aligned} \tag{4.23}$$

Note also, by the Implicit Function Theorem, that $dv_*(u) = -\partial_v q^{-1}\partial_u q(u, v_*(u))$.

An important property of the Chapman-Enskog approximation, following either by direct computation or by coordinate-independence of the physical derivation, is that the form (4.21)–(4.23) of the equations is coordinate invariant, changing tensorially with respect to constant linear coordinate changes; moreover, the change in functions h_*, b_* due to a constant linear coordinate change may be computed directly from (4.20) using the coordinate change in u alone. From this we find in the Boltzmann case that (4.20) is equivalent through a constant linear coordinate change to the Navier–Stokes equations (1.12) with monatomic ideal gas equation of state and viscosity and heat conduction coefficients satisfying (1.13). We make the following assumptions on the reduced system, verified for the Navier–Stokes equations (hence satisfied for the Boltzmann equation) in [19].

Assumption 4.10. *(i) There exists $s(u)$ symmetric positive definite such that $s\,dh_*$ is symmetric and sb_* is symmetric positive semidefinite.*

(ii) There is no eigenvector of dh_* in $\ker b_*$.

(iii) The matrix $b_*(u)$ has constant left kernel.

(iv) For all values of u, $\ker \pi_* dh_*(u) \cap \ker b_*(u) = \{0\}$, where $\pi_*(u)$ is the zero eigenprojection associated with $b_*(u)$.

Finally, we assume that the classical theory of weak shocks can be applied to (4.20), requiring that the flux f_* have a genuinely nonlinear eigenvalue near 0.

Assumption 4.11. *In a neighborhood* $\mathcal{U}_*$ *of a given base state* u_0*,* dh_* *has a simple eigenvalue* α *near zero, with* $\alpha(u_0) = 0$*, and such that the associated hyperbolic characteristic field is genuinely nonlinear, i.e., after a choice of orientation,* $\nabla\alpha \cdot r(u_0) < 0$*, where* r *denotes the eigendirection associated with* α*.*

Remark 4.12. As discussed in [32], Assumptions 4.10(i)–(ii) hold in great generality. Assumptions 4.10(iii)-(iv) must be checked in individual cases.

4.5. **The basic estimate.** With these preparations, we can establish existence by an argument almost identical to that used in [23] to treat the finite-dimensional case: indeed, somewhat simpler. The single difference is that in carrying out the basic symmetric energy estimates controlling microscopic variables we do not attempt to exactly symmetrize L_a at each x value as was done in [23], but only use the fact that each L_a is approximately symmetric by construction. This is important in the infinite-dimensional case, since exact symmetrization can (and does in the Boltzmann case) introduce unbounded commutator terms that wreck the argument. To isolate this important technical point, we carry out the key estimate here, before describing the rest of the argument.

We consider the equation

$$A\partial_x U - L_a U = F \tag{4.24}$$

with $a = a(x)$ satisfying

$$\big\|\partial_x^k\big(a(x) - \underline{M}\big)\big\|_{\mathbb{H}} \leq C_k \varepsilon^{k+1} e^{-\varepsilon\theta\langle x\rangle}. \tag{4.25}$$

We assume that

$$\mathbb{P}_{\mathbb{U}} F = \varepsilon h + \partial_x f. \tag{4.26}$$

Lemma 4.13. *There is a constant* C *such that for* ε *sufficiently small, one has*

$$\|U'\|_{L^2} + \|\mathbb{P}_{\mathbb{V}} U\|_{L^2} \leq C\big(\|f\|_{H^2} + \|h\|_{H^1} + \|g\|_{H^1} + \varepsilon\|\mathbb{P}_{\mathbb{U}} U\|_{L^2}\big). \tag{4.27}$$

Here, the norms L^2, H^1 etc denote the norms in $L^2(\mathbb{R};\mathbb{H})$, $H^1(\mathbb{R};\mathbb{H})$ etc.

Proof. Introduce the symmetrizer

$$\mathcal{S} = \partial_x^2 + \partial_x \circ K - \lambda \mathrm{Id}. \tag{4.28}$$

One has

$$\begin{aligned}
&\mathrm{Re}\,\partial_x^2(A\partial_x - L_a) = -\mathrm{Re}\,\partial_x \circ L_a \circ \partial_x - \mathrm{Re}\,\partial_x \circ L_{\partial_x a}\\
&\mathrm{Re}\ \partial_x \circ K(A\partial_x - L_a) = \partial_x \circ \mathrm{Re}\ KA \circ \partial_x - \mathrm{Re}\,\partial_x \circ KL_a\\
&\mathrm{Re}\ (A\partial_x - L_a) = -\mathrm{Re}\ L_a
\end{aligned}$$

where $\mathrm{Re}\,T = \frac{1}{2}(T + T^*)$ and the adjoint is taken in $L^2(\mathbb{R};\mathbb{H})$. We have used that $[\partial_x, L_a] = L_{\partial_x a}$ by linearity of L with respect to a. Thus

$$\begin{aligned}
\mathrm{Re}\ \mathcal{S} \circ (A\partial_x - L_a) =&\partial_x \circ (\mathrm{Re}\ AK - L_a) \circ \partial_x - \lambda \mathrm{Re}\,L_a\\
&- \mathrm{Re}\ \partial_x \circ L_{\partial_x a} - \mathrm{Re}\ \partial_x \circ KL_a.
\end{aligned}$$

Therefore, for $U \in H^2(\mathbb{R})$, (4.11), (4.15) and the continuity of K and Q imply that

$$\begin{aligned}
\mathrm{Re}\ (\mathcal{S}F, U)_{L^2} \geq\ &\gamma\|\partial_x U\|^2_{L^2} + \lambda\big(\delta\|\mathbb{P}_{\mathbb{V}} U\|^2_{L^2} - C\varepsilon\|\mathbb{P}_{\mathbb{U}} U\|_{L^2}\|\mathbb{P}_V VU\|_{L^2}\big)\\
&- C\|\partial_x a\|_{L^\infty}\|U\|_{L^2}\|\partial_x U\|_{L^2} - C\|\partial_x U\|_{L^2}\|L_a U\|_{L^2}.
\end{aligned}$$

We note that

$$L_a U = \underline{L}\mathbb{P}_{\mathbb{V}}U + (L_a - \underline{L})U \tag{4.29}$$

Therefore,

$$\big\|L_a U\big\|_{HH} \lesssim \big\|\mathbb{P}_{\mathbb{V}}U\big\|_{HH} + \varepsilon\big\|\mathbb{P}_{\mathbb{U}}U\big\|_{HH} \tag{4.30}$$

Taking λ large enough and using (4.25) yields

$$\|U'\|^2_{L^2} + \|\mathbb{P}_{\mathbb{V}}U\|^2_{L^2} \lesssim \mathrm{Re}\,(\mathcal{S}F, U)_{L^2} + \varepsilon\|\mathbb{P}_{\mathbb{U}}U\|_{L^2}\big(\|\mathbb{P}_{\mathbb{V}}U\|_{L^2} + \|U'\|_{L^2}\big).$$

In the opposite direction,

$$\begin{aligned}\mathrm{Re}\,(\mathcal{S}F, U)_{L^2} \le &\|\partial_x U\|_{L^2}\big(\|\partial_x F\|_{L^2} + \|K\|\|F\|_{L^2}\big)\\ &+ \lambda\big(\varepsilon\|h\|_{L^2}\|\mathbb{P}_{\mathbb{U}}u\|_{L^2} + \|f\|_{L^2}\|\mathbb{P}_U U\partial_x U\|_{L^2}\\ &\qquad + \|\mathbb{P}_{\mathbb{V}}F\|_{L^2}\|\mathbb{P}_{\mathbb{V}}U\|_{L^2}\big).\end{aligned}$$

The estimate (4.27) follows provided that ε is small enough.

This proves the lemma under the additional assumption that $U \in H^2$. When $U \in H^1$, the estimate follows using Friedrichs mollifiers. □

5. **Basic L^2 result.** We now describe a simpler version of our main result, carried out in the L^2 norm $\mathbb{H}$. For clarity of exposition, we carry out the entire argument in this more transparent context, indicating afterward in Section 10 how to extend to the general (pointwise, higher weight) norms described in Theorem 1.1.

5.1. **Chapman–Enskog approximation.** Integrating the first equation of (4.10) and noticing that the end states $(u_\pm, v_\pm)$ must be equilibria and thus satisfy $v_\pm = v_*(u_\pm)$, we obtain

$$\begin{aligned}A_{11}u + A_{12}v &= f_*(u_\pm),\\ A_{21}u' + A_{22}v' &= q(u,v).\end{aligned} \tag{5.1}$$

Because f is linear, the first equation reads

$$f_*(u) + A_{12}(v - v_*(u)) = f_*(u_\pm). \tag{5.2}$$

The idea of Chapman–Enskog approximation is that $v - v_*(u)$ is small (compared to the fluctuations $u - u_\pm$). Taylor expanding the second equation, we obtain

$$\begin{aligned}(A_{21} + A_{22}dv_*(u))u' + A_{22}(v - v_*(u))' = &\partial_v q(u, v_*(u))(v - v_*(u))\\ &+ O(|v - v_*(u)|^2),\end{aligned}$$

or inverting $\partial_v q$

$$\begin{aligned}v - v_*(u) = &\partial_v q^{-1}(u, v_*(u))\big(A_{21} + A_{22}dv_*(u)\big)u'\\ &+ O(|v - v_*(u)|^2) + O(|(v - v_*(u))'|).\end{aligned} \tag{5.3}$$

The derivative of (5.2) implies that

$$\big(A_{11}u + A_{12}dv_*(u)\big)u' = O(|(v - v_*(u))'|).$$

Therefore, (5.3) can be replaced by

$$v - v_*(u) = c_*(u)u' + O(|v - v_*(u)|^2) + O(|(v - v_*(u))'|), \tag{5.4}$$

where c_* is defined at (4.23). Substituting in (5.2), we thus obtain the approximate viscous profile ODE

$$b_*(u)u' = f_*(u) - f_*(u_\pm) + O(|v - v_*(u)|^2) + O(|(v - v_*(u))'|), \tag{5.5}$$

where b_* is as defined in (4.22).

Motivated by (5.3)–(5.5), we define an approximate solution $(\bar u_{NS}, \bar v_{NS})$ of (5.1) by choosing $\bar u_{NS}$ as a solution of

$$b_*(\bar u_{NS})\bar u'_{NS} = f_*(\bar u_{NS}) - f_*(u_\pm), \tag{5.6}$$

and $\bar v_{NS}$ as the first approximation given by (5.3)

$$\bar v_{NS} - v_*(\bar u_{NS}) = c_*(\bar u_{NS})\bar u'_{NS}. \tag{5.7}$$

Small amplitude shock profiles solutions of (5.6) are constructed using the center manifold analysis of [25] under conditions (i)-(iv) of Assumption 4.10; see discussion in [21].

Proposition 5.1 ([21])**.** *Under Assumptions 4.10 and 4.11, in a neighborhood of (u_0, u_0) in $\mathbb{R}^n \times \mathbb{R}^n$, there is a smooth manifold $\mathcal{S}$ of dimension n passing through (u_0, u_0), such that for $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small, and direction $(u_+ - u_-)/\varepsilon$ sufficiently close to $r(u_0)$, the zero speed shock profile equation (5.6) has a unique (up to translation) solution $\bar u_{NS}$ in $\mathcal{U}_*$. The shock profile is necessarily of* Lax type*: i.e., with dimensions of the unstable subspace of $dh_*(u_-)$ and the stable subspace of $dh_*(u_+)$ summing to one plus the dimension of u, that is $n+1$.*

Moreover, there is $\theta > 0$ and for all k there is C_k independent of (u_-, u_+) and ε, such that

$$|\partial_x^k(\bar u_{NS} - u_\pm)| \le C_k \varepsilon^{k+1} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0. \tag{5.8}$$

We denote by $\mathcal{S}_+$ the set of $(u_-, u_+) \in \mathcal{S}$ with amplitude $\varepsilon := |u_+ - u_-| > 0$ sufficiently small and direction $(u_+ - u_-)/\varepsilon$ sufficiently close to $r(u_0)$ such that the profile $\bar u_{NS}$ exists. Given $(u_-, u_+) \in \mathcal{S}_+$ with associated profile $\bar u_{NS}$, we define $\bar v_{NS}$ by (5.7) and

$$\bar U_{NS} := (\bar u_{NS}, \bar v_{NS}). \tag{5.9}$$

It is an approximate solution of (5.1) in the following sense:

Corollary 5.2. *For $(u_-, u_+) \in \mathcal{S}_+$,*

$$A_{11}\bar u_{NS} + A_{12}\bar v_{NS} - f_*(u\pm) = 0 \tag{5.10}$$

and

$$\mathcal{R}_v := A_{21}\bar u'_{NS} + A_{22}\bar v'_{NS} - q(\bar u_{NS}, \bar v_{NS})$$

satisfies

$$|\partial_x^k \mathcal{R}_v(x)| \le C_k \varepsilon^{k+3} e^{-\theta\varepsilon|x|}, \quad x \gtrless 0 \tag{5.11}$$

where C_k is independent of (u_-, u_+) and $\varepsilon = |u_+ - u_-|$.

Proof. Given the choice of $\bar v_{NS}$, the first equation is a rewriting of the profile equation (5.6).

Next, note that

$$\bar v_{NS} - v_*(\bar u_{NS}) = O(|\bar u'_{NS}|), \quad \big(\bar v_{NS} - v_*(\bar u_{NS})\big)' = O(|\bar u''_{NS}|) + O(|\bar u'_{NS}|^2),$$

where here $O(\cdot)$ denote smooth functions of $\bar u_{NS}$ and its derivatives, which vanish as indicated. With similar notations, the Taylor expansion of q and the definition of $\bar v_{NS}$ show that

$$\begin{aligned}\mathcal{R}_v =& O(|\bar v_{NS} - v_*(\bar u_{NS})|^2) + O(|(\bar v_{NS} - v_*(\bar u_{NS}))'|) \\ &+ dv_*(\bar u_{NS}\big(A_{11} + A_{12} dv_*(\bar u_{NS})\big)\bar u'_{NS}.\end{aligned}$$

Moreover,

$$\big(A_{11}+A_{12}dv_*(\bar u_{NS})\big)\bar u'_{NS} = \big(f_*(\bar u_{NS})\big)' = \big(b_*(\bar u_{NS})\bar u'_{NS}\big)' \\ = O(|\bar u'_{NS}|^2)+O(|\bar u''_{NS}|).$$

This implies that

$$\mathcal{R}_v = O(|\bar u'_{NS}|^2)+O(|\bar u''_{NS}|).$$

satisfies the estimates stated in (5.11). □

Remark 5.3. One may check that if we did not include the correction from equilibrium on the righthand side of (5.7), taking instead the simpler prescription $\bar v_{NS} = v_*(\bar u_{NS})$ as in [15], then the residual error that would result in (5.10) would be too large for our later iteration scheme to close. This is a crucial difference between our analysis and the analysis of [15]. The prescription $\bar U_{NS}$ corresponds to the first-order Chapman–Enskog approximation in both variables, u and v together.

5.2. **Basic L^2 result.** We are now ready to state the basic L^2 version of our main result. Define a base state $U_0=(u_0,v_*(u_0))$ and a neighborhood $\mathcal{U}=\mathcal{U}_*\times\mathcal{V}$.

Proposition 5.4. *Let Assumptions (SS), (GC), and 4.10 hold on the neighborhood $\mathcal{U}$ of U_0, with $Q\in C^\infty$. Then, there are $\varepsilon_0>0$ and $\delta>0$ such that for $(u_-,u_+)\in \mathcal{S}+$ with amplitude $\varepsilon := |u_+-u_-|\le\varepsilon_0$, the standing-wave equation (4.10) has a solution $\bar U$ in $\mathcal{U}$, with associated Lax-type equilibrium shock (u_-,u_+), satisfying for all k:*

$$\begin{aligned} \big|\partial_x^k(\bar U-\bar U_{NS})\big| &\le C_k\varepsilon^{k+2}e^{-\delta\varepsilon|x|},\\ |\partial_x^k(\bar u-u_\pm)| &\le C_k\varepsilon^{k+1}e^{-\delta\varepsilon|x|}, \quad x\gtrless 0,\\ \big|\partial_x^k(\bar v-v_*(\bar u)\big| &\le C_k\varepsilon^{k+2}e^{-\delta\varepsilon|x|}, \end{aligned} \tag{5.12}$$

where $\bar U_{NS}=(\bar u_{NS},\bar v_{NS})$ is the approximating Chapman–Enskog profile defined in (5.9), and C_k is independent of ε. Moreover, up to translation, this solution is unique within a ball of radius $c\varepsilon$ about $\bar U_{NS}$ in norm $\varepsilon^{-1/2}\|\cdot\|_{L^2}+\varepsilon^{-3/2}\|\partial_x\cdot\|_{L^2}+\varepsilon^{-5/2}|\partial_x\cdot\|_{L^2}$, for $c>0$ sufficiently small. (For comparison, $\bar U_{NS}-U_\pm$ is order ε in this norm, by (5.8).

6. **Outline of the proof.** We describe now the main steps in the proof of Proposition 5.4, exactly following the finite-dimensional analysis of [23].

6.1. **Nonlinear perturbation equations.** Defining the perturbation variable

$$U := \bar U-\bar U_{NS},$$

and expanding about $\bar U_{NS}$, we obtain from (5.1) the nonlinear perturbation equations

$$A_{11}u+A_{12}v = 0 \tag{6.1}$$

$$A_{21}u'+A_{22}v'-dq(\bar U_{NS})U = -\mathcal{R}_v+N(U) \tag{6.2}$$

where the remainder $N(U)$ is a smooth function of U_{NS} and U, vanishing at second order at $U=0$:

$$N(U)=\mathcal{N}(\bar U_{NS},U)=O(|U|^2). \tag{6.3}$$

We push the reduction a little further, using that

$$M := dq(\bar u_{NS},\bar v_{NS})-dq(\bar u_{NS},v_*(u_{NS}))=O(|\bar v_{NS}-v_*(\bar u_{NS})|). \tag{6.4}$$

Therefore the equation reads

$$\begin{aligned}\mathcal{L}_*^\varepsilon U &:= \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix} U' + \begin{pmatrix} A_{11} & A_{12} \\ -Q_{21} & -Q_{22} \end{pmatrix} U \\ &= \begin{pmatrix} 0 \\ -\mathcal{R}_v + MU + N(U) \end{pmatrix}\end{aligned} \tag{6.5}$$

where

$$Q_{21} = \partial_u q(\bar u_{NS}, v_*(\bar u_{NS})), \quad Q_{22} = \partial_v q(\bar u_{NS}, v_*(\bar u_{NS})). \tag{6.6}$$

Differentiating the first line, it implies that

$$L_*^\varepsilon U := AU' - dQ(\bar u_{NS}, v_*(\bar u_{NS}))U = \begin{pmatrix} 0 \\ -\mathcal{R}_v + MU + N(U) \end{pmatrix}. \tag{6.7}$$

The linearized operator $A\partial_x - dQ(\bar U)$ about an exact solution $\bar U$ of the profile equations has kernel $\bar U'$, by translation invariance, so is not invertible. Thus, the linear operators L_*^ε and $\mathcal{L}_*^\varepsilon$ are not expected to be invertible, and we shall see later that they are not. Nonetheless, one can check that $\mathcal{L}_*^\varepsilon$ is surjective in Sobolev spaces and define a right inverse $\mathcal{L}_*^\varepsilon(\mathcal{L}_*^\varepsilon)^\dagger \equiv I$, or solution operator $(\mathcal{L}_*^\varepsilon)^\dagger$ of the equation

$$\mathcal{L}_*^\varepsilon U = F := \begin{pmatrix} f \\ g \end{pmatrix}, \tag{6.8}$$

as recorded by Proposition 6.2 below. Note that L_*^ε is not surjective because the first equation requires a zero mass condition on the source term. This is why we solve the integrated equation (6.5) and not (6.7).

To define the partial inverse $(\mathcal{L}_*^\varepsilon)^\dagger$, we specify one solution of (6.8) by adding the co-dimension one internal condition:

$$\ell_\varepsilon \cdot u(0) = 0 \tag{6.9}$$

where ℓ_ε is a certain unit vector to be specified below.

Remark 6.1. There is a large flexibility in the choice of ℓ_ε. Conditions like (6.9) are known to fix the indeterminacy in the resolution of the linearized profile equation from (5.6) and it remains well adapted in the present context, see section 8 below. A possible choice, would be to choose ℓ_ε independent of ε and parallel to the left eigenvector of $dh_*(u_0)$ for the eigenvalue 0 (see Assumption 4.11), which, together with the asymptotics of Proposition 5.1, gives

$$\ell_\varepsilon \cdot \bar U'_{NS}(0) \sim \varepsilon^2 \neq 0. \tag{6.10}$$

6.2. **Fixed-point iteration scheme.** The coefficients and the error term $\mathcal{R}_v$ are smooth functions of $\bar u_{NS}$ and its derivative, thus behave like smooth functions of εx. Thus, it is natural to solve the equations in spaces which reflect this scaling. We do not introduce explicitly the change of variables $\tilde x = \varepsilon x$, but introduce norms which correspond to the usual H^s norms in the $\tilde x$ variable :

$$\|f\|_{H^s_\varepsilon} = \varepsilon^{\frac12}\|f\|_{L^2} + \varepsilon^{-\frac12}\|\partial_x f\|_{L^2} + \cdots + \varepsilon^{\frac12 - s}\|\partial_x^s f\|_{L^2}. \tag{6.11}$$

We also introduce weighted spaces and norms, which encounter for the exponential decay of the source and solution: introduce the notations.

$$< x > := (x^2+1)^{1/2} \tag{6.12}$$

For $\delta \geq 0$ (sufficiently small), we denote by $H^s_{\varepsilon,\delta}$ the space of functions f such that $e^{\delta\varepsilon<x>}f \in H^s$ equipped with the norm

$$\|f\|_{H^s_{\varepsilon,\delta}} = \varepsilon^{\frac{1}{2}} \sum_{k\leq s} \varepsilon^{-k} \|e^{\delta\varepsilon<x>}\partial_x^k f\|_{L^2}. \tag{6.13}$$

Note that for $\delta \leq 1$, this norm is equivalent, with constants independent of ε and δ, to the norm

$$\|e^{\delta\varepsilon<x>}f\|_{H^s_\varepsilon}.$$

Proposition 6.2. *Under the assumptions of Theorem 5.4, there are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ and for all $\varepsilon \in]0, \varepsilon_0]$, there is a unit vector ℓ_ε such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $f \in H^3_{\varepsilon,\delta}$, $g \in H^2_{\varepsilon,\delta}$ the operator equations (6.8) (6.9) has a unique solution $U \in H^2_{\varepsilon,\delta}$, denoted by $U = (\mathcal{L}^\varepsilon_*)^\dagger F$, which satisfies*

$$\big\|(\mathcal{L}^\varepsilon_*)^\dagger F\big\|_{H^2_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\big(\|f\|_{H^3_{\varepsilon,\delta}} + \|g\|_{H^2_{\varepsilon,\delta}}\big). \tag{6.14}$$

Moreover, for $s \geq 3$, there is a constant C_s such that for $\varepsilon \in]0, \varepsilon_0]$ and $f \in H^{s+1}_{\varepsilon,\delta}$, $g \in H^s_{\varepsilon,\delta}$ the solution $U = (\mathcal{L}^\varepsilon_*)^\dagger F \in H^s_{\varepsilon,\delta}$ and

$$\big\|(\mathcal{L}^\varepsilon_*)^\dagger F\big\|_{H^s_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\big(\|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}}\big) + C_s\big\|(\mathcal{L}^\varepsilon_*)^\dagger F\big\|_{H^{s-1}_{\varepsilon,\delta}}. \tag{6.15}$$

The proof of this proposition comprises most of the work of the paper. Once it is established, existence follows by a straightforward application of the Contraction-Mapping Theorem. Defining

$$\mathcal{T} := (\mathcal{L}^\varepsilon_*)^\dagger \begin{pmatrix} 0 \\ -\mathcal{R}_v + MU + N(U)) \end{pmatrix}, \tag{6.16}$$

we reduce (6.7) to the fixed-point equation

$$\mathcal{T}U := U. \tag{6.17}$$

6.3. **Proof of the basic result.**

Proof of Theorem 5.4. The profile $\bar{u}_{NS}$ exists if ε is small enough. The estimates (5.8) imply that

$$\|\bar{u}_{NS} - u_\pm\|_{H^s_{\varepsilon,\delta}} \leq C_s\varepsilon \tag{6.18}$$

with C_s independent of ε and δ, provided that $\delta \leq \theta/2$. Similarly, (5.11) implies that

$$\|\mathcal{R}_v\|_{H^s_{\varepsilon,\delta}} \leq C_s\varepsilon^3, \tag{6.19}$$

and (6.4) implies that

$$\|M\|_{H^s_{\varepsilon,\delta}} \leq C_s\varepsilon^2. \tag{6.20}$$

Moreover, with the choice of norms (6.11), the Sobolev inequality reads

$$\|u\|_{L^\infty} \leq C\|u\|_{H^1_\varepsilon} \leq C\|u\|_{H^1_{\varepsilon,\delta}} \tag{6.21}$$

with C independent of ε. Moreover, for smooth functions Φ, there are nonlinear estimates

$$\|\Phi(u)\|_{H^s_\varepsilon} \leq C\big(\|u\|_{L^\infty}\big)\ \|u\|_{H^s_\varepsilon}. \tag{6.22}$$

which also extend to weighted spaces, for $\delta \leq 1$:

$$\|\Phi(u)\|_{H^s_{\varepsilon,\delta}} \leq C\big(\|u\|_{L^\infty}\big)\ \|u\|_{H^s_{\varepsilon,\delta}}. \tag{6.23}$$

In particular, this implies that for $s \geq 1$, $\delta \leq \min\{1, \theta/2\}$ and ε small enough:

$$\begin{aligned} \|MU\|_{H^s_{\varepsilon,\delta}} &\leq C\big(\|M\|_{H^1_{\varepsilon,\delta}}\|U\|_{H^s_{\varepsilon,\delta}} + \|M\|_{H^s_{\varepsilon,\delta}}\|U\|_{H^1_{\varepsilon,\delta}}\big) \\ &\leq \varepsilon^2\big(C\|U\|_{H^s_{\varepsilon,\delta}} + C_s\|U\|_{H^1_{\varepsilon,\delta}}\big) \end{aligned} \tag{6.24}$$

where the first constant C is independent of s. Similarly,

$$\|N(U)\|_{H^s_{\varepsilon,\delta}} \leq C\big(\|U\|_{L^\infty}\big)\|U\|_{H^1_{\varepsilon,\delta}}\|U\|_{H^s_{\varepsilon,\delta}}. \tag{6.25}$$

Combining these estimates, we find that

$$\|\mathcal{T}U\|_{H^s_{\varepsilon,\delta}} \leq \varepsilon^{-1}\big(C_s\varepsilon^3 + C\varepsilon^2\|U\|_{H^s_{\varepsilon,\delta}} + C_s\varepsilon^2\|U\|_{H^1_{\varepsilon,\delta}} + C\|U\|_{H^1_{\varepsilon,\delta}}\|U\|_{H^s_{\varepsilon,\delta}}\big),$$

that is

$$\|\mathcal{T}U\|_{H^s_{\varepsilon,\delta}} \leq C_s\varepsilon^2 + C(\varepsilon + \varepsilon^{-1}\|U\|_{H^1_{\varepsilon,\delta}})\|U\|_{H^s_{\varepsilon,\delta}} + C_s\varepsilon\|U\|_{H^1_{\varepsilon,\delta}} \tag{6.26}$$

provided that $\varepsilon \leq \varepsilon_0$, $\delta \leq \min\{1, \theta/2\}$ and $\|U\|_{L^\infty} \leq 1$.

Consider first the case $s = 2$. Then, $\mathcal{T}$ maps the ball $\mathcal{B}_{\varepsilon,\delta} = \{\|U\|_{H^2_{\varepsilon,\delta}} \leq \varepsilon^{1+\frac{1}{2}}\}$ to itself, if $\varepsilon \leq \varepsilon_1$ where $\varepsilon_1 > 0$ is small enough. Similarly,

$$\|\mathcal{T}U - \mathcal{T}V\|_{H^2_{\varepsilon,\delta}} \leq C\varepsilon^{-1}\big(\varepsilon^2 + \|U\|_{H^2_\varepsilon} + \|V\|_{H^2_\varepsilon}\big)\|U - V\|_{H^2_{\varepsilon,\delta}}, \tag{6.27}$$

provided that $\|U\|_{L^\infty} \leq 1$ and $\|V\|_{L^\infty} \leq 1$, from which we readily find that, for $\varepsilon > 0$ sufficiently small, $\mathcal{T}$ is contractive on $\mathcal{B}_{\varepsilon,\delta}$, whence, by the Contraction-Mapping Theorem, there exists a unique solution U^ε of (6.17) in $\mathcal{B}_{\varepsilon,\delta}$ for ε sufficiently small.

Moreover, from the contraction property

$$\|\bar{U}^\varepsilon - \mathcal{T}(0)\|_{H^2_\varepsilon} = \|\mathcal{T}(\bar{U}^\varepsilon) - \mathcal{T}(0)\|_{H^2_\varepsilon} \leq c\|\bar{U}^\varepsilon\|_{H^2_\varepsilon},$$

with $c < 1$, we obtain as usual that $\|U^{\varepsilon,\delta}\|_{H^2_{\varepsilon,\delta}} \leq C\|\mathcal{T}(0)\|_{H^2_{\varepsilon,\delta}}$, whence

$$\|U^\varepsilon\|_{H^2_{\varepsilon,\delta}} \leq C\varepsilon^2. \tag{6.28}$$

by (6.26). In particular, $e^{\varepsilon\delta\langle x\rangle}U^\varepsilon = O(\varepsilon^2)$ in H^2_ε and by the Sobolev embedding

$$\|e^{\varepsilon\delta\langle x\rangle}U^\varepsilon\|_{L^\infty} = O(\varepsilon^2), \quad \|e^{\varepsilon\delta\langle x\rangle}\partial_x U^\varepsilon\|_{L^\infty} = O(\varepsilon^3). \tag{6.29}$$

For $s \geq 3$, the estimates (6.26) show that for $\varepsilon \leq \varepsilon_1$ independent of s, the iterates $\mathcal{T}^n(0)$ are bounded in $H^s_{\varepsilon,\delta}$, and similarly that $\mathcal{T}^n(0) - \mathcal{T}(0) = O(\varepsilon^2)$ in $H^s_{\varepsilon,\delta}$, implying that the limit U belongs to $H^s_{\varepsilon,\delta}$ with norm $O(\varepsilon^2)$. Together with the Sobolev inequality (6.21), this implies the pointwise estimates (1.18).

Finally, the assertion about uniqueness follows by uniqueness in $\mathcal{B}_{c\varepsilon,\delta}$ under the additional phase condition (6.9) for the choice $\delta = 0$ and $c > 0$ sufficiently small (noting by our argument that also $\mathcal{B}_{c\varepsilon,\delta}$ is mapped to itself for ε sufficiently small, for any $c > 0$), together with the observation that phase condition (6.9) may be achieved for any solution $\bar{U} = \bar{U}_{NS} + U$ with

$$\|U'\|_{L^\infty} \leq c\varepsilon^2 << \bar{U}'_{NS}(0) \sim \varepsilon^2$$

by translation in x, yielding $\bar{U}_a(x) := \bar{U}(x+a) = \bar{U}_{NS}(x) + U_a(x)$ with

$$U_a(x) := \bar{U}_{NS}(x+a) - \bar{U}_{NS}(x) + U(x+a)$$

so that $\partial_a(\ell_\varepsilon \cdot u_a(0)) = \ell_\varepsilon \cdot \big(\bar{u}'_{NS}(a) + u'(a)\big) \sim \ell_\varepsilon \cdot \bar{u}'_{NS}(0)$ and so (by the Implicit Function Theorem applied to $h(a) := \varepsilon^{-2}\big(\ell_\varepsilon \cdot u_a\big)$, together with $\ell_\varepsilon \cdot u_0 = o(\varepsilon)$ and the assumed property that $\ell_a \cdot \bar{u}'_{NS}(0) \sim \varepsilon^2$ coming from our choice of ℓ_ε; see (6.10), Remark 6.1) the inner product $\ell_a \cdot \bar{u}'_{NS}(0)$ may be set to zero by appropriate choice of $a = o(\varepsilon^{-1})$ leaving U_a in the same $o(\varepsilon)$ neighborhood, by the computation $U_a - U_0 \sim \partial_a U \cdot a \sim o(\varepsilon^{-1})\varepsilon^2$. □

It remains to prove existence of the linearized solution operator and the linearized bounds (6.15), which tasks will be the work of most of the rest of the paper. We concentrate first on estimates, and prove the existence next, using a viscosity method combined with (the single new step in treating the infinite-dimensional case) discretization in velocity.

7. **Internal and high frequency estimates.**

We begin by establishing a priori estimates on solutions of the equation (6.8) This will be done in two stages. In the first stage, carried out in this section, we establish energy estimates showing that "microscopic", or "internal", variables consisting of v and derivatives of (u, v) are controlled by and small with respect to the "macroscopic", or "fluid" variable, u. As discussed in Section 4.5, this is the main new aspect in the infinite-dimensional case.

In the second stage, carried out in Section 8, we estimate the macroscopic variable u by Chapman–Enskog approximation combined with finite-dimensional ODE techniques such as have been used in the study of fluid-dynamical shocks [20, 21, 33, 34, 8], exactly as in the finite-dimensional analysis of [23].

7.1. **The basic H^1 estimate.**

We consider the equation

$$\mathcal{L}_*^\varepsilon U := \begin{pmatrix} A_{11}u + A_{12}v \\ A_{21}u' + A_{22}v' - dq(\bar u_{NS}, v_*(\bar u_{NS}))U \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{7.1}$$

and its differentiated form:

$$AU' - dQ(\bar u_{NS}, v_*(\bar u_{NS}))U = \begin{pmatrix} f' \\ g \end{pmatrix}. \tag{7.2}$$

The internal variables are $U' = (u', v')$ and $\tilde v$ where

$$\tilde v := v + pu, \qquad p = \partial_v q^{-1}\partial_u q(\bar u_{NS}, v_*(\bar u_{NS})) = -dv_*(\bar u_{NS}) \tag{7.3}$$

is the linearization about $(\bar u_{NS}, \bar v_{NS})$ of the key variable $v - v_*(u)$ arising in the Chapman–Enskog expansion of Section 5.1. Noting that $pu = 0$ at the reference point $\underline{U}$ by Assumption 4.5, we have the important fact that

$$\|pu\|_{\mathbb{H}} = O(\varepsilon)\|u\|_{\mathbb{H}} \tag{7.4}$$

on the set of U we consider (ε^2 close to $\bar U_{NS}$, so ε close to $\underline{U}$), so that v and $\tilde v$ are nearly equivalent.

Proposition 7.1. *Under the assumptions of Theorem 5.4, for there are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ and $0 \le \delta \le \delta_0$, $f \in H^2_{\varepsilon,\delta}$, $g \in H^1_{\varepsilon,\delta}$ and $U = (u, v) \in H^1_{\varepsilon,\delta}$ of (7.1) satisfies*

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde v\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}\big). \tag{7.5}$$

Proof. For $\delta = 0$, the result follows by Lemma 4.13 together with (7.4).

For $\delta > 0$ small, consider $U^w = e^{\varepsilon\delta\langle x\rangle} U$. Then, U^w satisfies

$$\mathcal{L}_*^\varepsilon U^w = \begin{pmatrix} f^w \\ g^w \end{pmatrix}, \tag{7.6}$$

with $f^w = e^{\varepsilon\delta\langle x\rangle} f$ and $g^w = e^{\varepsilon\delta\langle x\rangle} g + \varepsilon\delta\langle x\rangle'(A_{21}u^w + A_{22}v^w)$. We note that,

$$\|U'\|_{L^2_{\varepsilon,\delta}} \le \|(U^w)'\|_{L^2_\varepsilon} + \varepsilon\|U^w\|_{L^2_\varepsilon}, , \quad \|\tilde v\|_{L^2_{\varepsilon,\delta}} \lesssim \|\tilde v^w\|_{L^2_\varepsilon},$$
$$\|f^w, (f^w)', (f^w)''\|_{L^2_\varepsilon} \lesssim \|(f, f', f'')\|_{L^2_{\varepsilon,\delta}},$$
$$\|g^w, (g^w)'\|_{L^2_\varepsilon} \lesssim \|(g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon\delta\|(U, U')\|_{L^2_{\varepsilon,\delta}}.$$

We use the estimate (7.5) with $\delta = 0$ for U^w, and the Proposition follows provided that δ is small enough. □

7.2. Higher order estimates.

Proposition 7.2. *There are constants C, $\varepsilon_0 > 0$, $\delta_0 > 0$ and for all $k \ge 2$, there is C_k, such that $0 < \varepsilon \le \varepsilon_0$, $\delta \le \delta_0$, $U \in H^s_{\varepsilon,\delta}$, $f \in H^{s+1}_{\varepsilon,\delta}$ and $g \in H^s_{\varepsilon,\delta}$ satisfying (7.2) satisfies:*

$$\begin{aligned} \|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k \tilde v\|_{L^2_{\varepsilon,\delta}} &\le C\|\partial_x^k(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} \\ &\quad + \varepsilon^k C_k\big(\|U'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|\tilde v\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|u\|_{L^2_{\varepsilon,\delta}}\big) \end{aligned} \tag{7.7}$$

Proof. Differentiating (7.1) k times, yields

$$A\partial_x U^k - dQ(\bar u_{NS}, v_*(\bar u_{NS}))\partial_x U^k = \begin{pmatrix} \partial_x^k f' \\ \partial_x^k g + r_k \end{pmatrix}, \tag{7.8}$$

where

$$r_k = -\sum_{l=0}^{k-1} \partial_x^{k-l} Q_{22}\, \partial_x^l \tilde v.$$

Here we have used that $dq(\bar u_{NS}, v_*(\bar u_{NS})U = Q_{22}\tilde v$. The H^1 estimate yields

$$\begin{aligned} \|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k v + p\partial_x^k u\|_{L^2_{\varepsilon,\delta}} &\le C\big(\|\partial_x^k(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} \\ &\quad + \varepsilon\|\partial_x^k u\|_{L^2_{\varepsilon,\delta}} + \|\partial_x r_k\|_{L^2_{\varepsilon,\delta}} + \|r_k\|_{L^2_{\varepsilon,\delta}}\big), \end{aligned}$$

for $0 \le k \le s$, with $r_0 = 0$ when $k = 0$. Since Q is a function of $\bar u_{NS}$, its $k - l$-th derivative is $O(\varepsilon^{k-l+1})$ when $k - l > 0$. Therefore:

$$\|\partial_x r_k\|_{L^2_{\varepsilon,\delta}} + \|r_k\|_{L^2_{\varepsilon,\delta}} \le C_k\varepsilon^k\big(\|\tilde v'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon\|\tilde v\|_{L^2_{\varepsilon,\delta}}\big).$$

Similarly, for $k = 1$

$$\|\partial_x \tilde v_k\|_{L^2_{\varepsilon,\delta}} \le \|\partial_x v + p\partial_x u\|_{L^2_{\varepsilon,\delta}} + C\varepsilon^2\|u\|_{L^2_{\varepsilon,\delta}}$$

and for $k \ge 2$:

$$\|\partial_x^k \tilde v_k\|_{L^2_{\varepsilon,\delta}} \le \|\partial_x^k v + p\partial_x^k u\|_{L^2_{\varepsilon,\delta}} + C_k(\varepsilon^k\|u'\|_{H^{k-2}_{\varepsilon,\delta}} + \varepsilon^{k+1}\|\tilde u\|_{L^2_{\varepsilon,\delta}}).$$

□

8. **Linearized Chapman–Enskog estimate.**

8.1. **The approximate equations.**

It remains only to estimate $\|u\|_{L^2_{\varepsilon,\delta}}$ in order to close the estimates and establish (7.5). To this end, we work with the first equation in (7.1) and estimate it by comparison with the Chapman-Enskog approximation (see the computations Section 5.1), exactly as in the finite-dimensional case [23].

From the second equation

$$A_{21}u' + A_{22}v' - g = \partial_u q u + \partial_v q v = \partial_v q \tilde{v},$$

where we use the notations $\tilde{v}$ of Proposition 7.1, we find

$$\tilde{v} = \partial_v q^{-1}\Big((A_{21} + A_{22}\partial_v dv_*(\bar{u}_{NS}))u' + A_{22}\tilde{v}' - g \Big). \tag{8.1}$$

Introducing $\tilde{v}$ in the first equation, yields

$$(A_{11} + A_{12} dv_*(\bar{u}_{NS}))u + A_{12}\tilde{v} = f,$$

thus

$$(A_{11} + A_{12} dv_*(\bar{u}_{NS}))u' = f' - A_{12}\tilde{v}' - d^2 v_*(\bar{u}_{NS})(\bar{u}'_{NS}, u).$$

Therefore, (8.1) can be modified to

$$\tilde{v} = c_*(\bar{u}_{NS})u' + r \tag{8.2}$$

with

$$r = d_v^{-1} q(\bar{u}_{NS}, v_*(\bar{u}_{NS}))\Big(A_{22}(\tilde{v})' - g + dv_*(\bar{u}_{NS})\big(f' - A_{12}\tilde{v}' - d^2 v_*(\bar{u}_{NS})(\bar{u}'_{NS}, u)\big)\Big).$$

This implies that u satisfies the linearized profile equation

$$\bar{b}_* u' - d\bar{h}_* u = A_{12} r - f \tag{8.3}$$

where $\bar{b}_* = b_*(\bar{u}_{NS})$ and $d\bar{h}_* := dh_*(\bar{u}_{NS}) = A_{11} + A_{12} dv_*(\bar{u}_{NS})$.

8.2. **L^2 estimates and proof of the main estimates.**

The following estimate was established in [23] using standard finite-dimensional ODE techniques; for completeness, we recall the proof here as well, in Section 8.3 below.

Proposition 8.1 ([23])**.** *The operator $\bar{b}_*\partial_x - d\bar{h}_*$ has a right inverse $(b_*\partial_x - dh^*)^\dagger$ satisfying*

$$\|(\bar{b}_*\partial_x - d\bar{h}_*)^\dagger h\|_{L^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}\|h\|_{L^2_{\varepsilon,\delta}}, \tag{8.4}$$

uniquely specified by the property that the solution $u = (b_*\partial_x - dh^*)^\dagger h$ satisfies

$$\ell_\varepsilon \cdot u(0) = 0. \tag{8.5}$$

for a certain unit vector ℓ_ε.

Taking this proposition for granted, we finish the proof of the main estimates in Proposition 6.2.

Proposition 8.2. *There are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $f \in H^3_{\varepsilon,\delta}$, $g \in H^2_{\varepsilon,\delta}$ and $U \in H^2_{\varepsilon,\delta}$ satisfying (6.8) and (6.9)*

$$\big\|U\big\|_{H^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}\big(\|f\|_{H^3_{\varepsilon,\delta}} + \big\|g\big\|_{H^2_{\varepsilon,\delta}}\big). \tag{8.6}$$

Proof. Going back now to (8.3), u satisfies

$$\bar{b}_* u' - \bar{d}h_* u = O(|\tilde{v}'| + |g| + |f'| + \varepsilon^2 |u|) - f,$$

If in addition u satisfies the condition (8.5) then

$$\|u\|_{L^2_{\varepsilon,\delta}} \le C\varepsilon^{-1}(\|\tilde{v}'\|_{L^2_{\varepsilon,\delta}} + \|(f, f', g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}). \tag{8.7}$$

By Proposition 7.1 and Proposition 7.2 for $k = 1$, we have

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde{v}\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}}\big). \tag{8.8}$$

$$\begin{aligned} \|U''\|_{L^2_{\varepsilon,\delta}} + \|\tilde{v}'\|_{L^2_{\varepsilon,\delta}} \le \\ C\big(\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|U'\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}\big). \end{aligned} \tag{8.9}$$

Combining these estimates, this implies

$$\begin{aligned} \|\tilde{v}'\|_{L^2_{\varepsilon,\delta}} &\le C\big(\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}\big) \\ &\le C\big(\varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}\big). \end{aligned}$$

Substituting in (8.7), yields

$$\varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon,\delta}} + \varepsilon^2 \|u\|_{L^2_{\varepsilon,\delta}}\big).$$

Hence for ε small,

$$\varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \le C\big(\|(f, f', g)\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon,\delta}}\big). \tag{8.10}$$

Plugging this estimate in (8.8)

$$\|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde{v}\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \le C\|(f, f', f'', g, g')\|_{H^1_{\varepsilon,\delta}} + \big). \tag{8.11}$$

Hence, with (8.9), one has

$$\begin{aligned} \|U''\|_{L^2_{\varepsilon,\delta}} + \|\tilde{v}'\|_{L^2_{\varepsilon,\delta}} \le \\ C\big(\|(f', f'', f''', g', g'')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|(f, f', f'', g, g')\|_{H^1_{\varepsilon,\delta}}\big). \end{aligned} \tag{8.12}$$

Therefore,

$$\|U'\|_{H^1_{\varepsilon,\delta}} + \|\tilde{v}\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \le C\|f, f', f'', g, g'\|_{H^1_{\varepsilon,\delta}} \tag{8.13}$$

The left hand side dominates

$$\|U'\|_{H^1_{\varepsilon,\delta}} + \varepsilon \|U'\|_{L^2_{\varepsilon,\delta}} = \varepsilon \|U'\|_{H^2_{\varepsilon,\delta}}$$

and the right hand side is smaller than or equal to $\|f\|_{H^2_{\varepsilon,\delta}} + \|g\|_{H^1_{\varepsilon,\delta}}$. The estimate (8.6) follows. □

Knowing a bound for $\|u\|_{L^2_{\varepsilon,\delta}}$, Proposition 7.2 immediately implies

Proposition 8.3. *There are constants C, $\varepsilon_0 > 0$ and $\delta_0 > 0$ and for $s \ge 3$ there is a constant C_s such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $f \in H^{s+1}_{\varepsilon,\delta}$, $g \in H^s_{\varepsilon,\delta}$ and $U \in H^s_{\varepsilon,\delta}$ satisfying (6.8) and (6.9), one has*

$$\|U\|_{H^s_{\varepsilon,\delta}} \le C\varepsilon^{-1}\big(\|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}}\big) + C_s \|U\|_{H^{s-1}_{\varepsilon,\delta}}. \tag{8.14}$$

Remark 8.4. The estimate of Proposition 8.1 may be recognized as somewhat similar to the estimates of Goodman [6] obtained by energy methods in the time-evolutionary case, the same ones used by Liu and Yu [15] to control the macroscopic variable u. More precisely, the argument is a simplified version of the one used by Plaza and Zumbrun [26] to show time-evolutionary stability of general small-amplitude waves.

8.3. **Proof of Proposition 8.1.** By Assumption 4.10(i), we may assume that there are linear coordinates $u = (u_1, u_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $h = (h_1, h_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, with $n_2 = \operatorname{rank} b_*(\bar u)$ such that

$$b_*(\bar u) = \begin{pmatrix} 0 & 0 \\ b_{21}(\bar u) & b_{22}(\bar u) \end{pmatrix} \tag{8.15}$$

and $b_{22}(\bar u)$ is uniformly invertible on $\mathcal{U}_*$. Introducing the new variable

$$\tilde u_2 = u_2 + \bar V u_1, \quad \bar V = (b^{22})^{-1} b_{21}(\bar u_{NS}), \tag{8.16}$$

the equation $\bar b_* u' - \bar{d}h_* u = h$ has the form:

$$\begin{aligned} \bar a^{11} u_1 + \bar a^{12} \tilde u_2 &= h_1, \\ \bar b^{22} \tilde u_2' - \bar a^{21} u_1 - \bar a^{22} \tilde u_2 &= h_2 \end{aligned} \tag{8.17}$$

where

$$\bar a := \bar{d}h_* \begin{pmatrix} \mathrm{Id} & 0 \\ -\bar V & \mathrm{Id} \end{pmatrix} + \bar b * \begin{pmatrix} 0 & 0 \\ \bar V' & 0 \end{pmatrix}.$$

Assumption 4.10(ii) implies that the left upper corner block $\bar a^{11}$ is uniformly invertible. Solving the first equation for u_1, we obtain the reduced nondegenerate ordinary differential equation

$$\bar b_*^{22} \tilde u_2' + \bar a^{21} (\bar a^{11})^{-1} \bar a^{12} \tilde u_2 - \bar a^{22} \tilde u_2 = h_2 + \bar a^{21} (\bar a^{11})^{-1} h_1$$

or

$$\check b u_2' - \check a u_2 = \check h = O(|h_1| + |h_2|). \tag{8.18}$$

Note that $\det \bar{d}h_* = \det \bar a^{11} \det \check a$ by standard block determinant identities, so that $\det \check a \sim \det \bar{d}h_*$ by Assumption 4.10(ii). Moreover, as established in [20], by Assumption 4.11 and the construction of the profile $\bar u_{NS}$ we find that $m := (\check b)^{-1} \check a$ has the following properties:

i) with $m_\pm$ denoting the end points values of m, there is $\theta > 0$ such that for all k :

$$|\partial_x^k (m(x) - m_\pm)| \lesssim \varepsilon^{k+1} e^{-\varepsilon \theta |x|}; \tag{8.19}$$

ii) $m(x)$ has a single simple eigenvalue of order ε, dented by $\varepsilon \mu(x)$, and there is $c > 0$ such that for all x and ε the other eigenvalues λ satisfy $|\mathrm{Re}\ \lambda| \geq c$;

iii) the end point values $\mu_\pm$ of μ satisfy

$$\mu_- \geq \alpha \qquad \mu_+ \leq -\alpha \tag{8.20}$$

for some $\alpha > 0$ independent of ε.

In the strictly parabolic case $\det b_* \neq 0$, this follows by a lemma of Majda and Pego [16].

At this point, we have reduced to the case

$$u_2' - m(x) u_2 = O(|h_1| + |h_2|), \tag{8.21}$$

with m having the properties listed above. The important feature is that $m' = O(\varepsilon^2) << \varepsilon$, the spectral gap between stable, unstable, and ε-order subspaces of m. The conditions above imply that there is a matrix ω such that

$$p := \omega^{-1} m \omega = \text{blockdiag}\{p^+, \varepsilon\mu, p-\},$$

where the spectrum of $p_\pm$ lies in $\pm\text{Re}\ \lambda \geq c$. Moreover, ω and p satisfies estimates similar to (8.19). The change of variables $u_2 = \omega z$ reduces (8.21) to

$$z' - pz = \omega^{-1}\omega' z + O(|h_1| + |h_2|). \tag{8.22}$$

The equations $(z^+)' - p^+ z^+ = h^+$ and $(z^-)' - p^- z^- = h^-$ either by standard linear theory [9] or by symmetrizer estimates as in [8], admit unique solutions in weighted L^2 spaces, satisfying

$$\|e^{\delta|x|} z^\pm\|_{L^2} \leq C \|e^{\delta|x|} h^\pm\|_{L^2},$$

provided that δ remains small, typically $\delta < |\text{Re}\ p^\pm|$.

The equation $z_0' - \varepsilon\mu z_0 = h_0$ may be converted by the change of coordinates $x \to \tilde{x} := \varepsilon x$ to

$$\partial_{\tilde{x}} \tilde{z}_0 - \tilde{\mu}(\tilde{x}) z_0 = \tilde{h}_0(\tilde{x}) = \varepsilon^{-1} h_0(\tilde{x}/\varepsilon), \tag{8.23}$$

where $\tilde{z}_0(\tilde{x}) = z_0(\tilde{x}/\varepsilon)$ and $\tilde{\mu}(\tilde{x}) := \mu(\tilde{x}/\varepsilon)$. By (8.19)

$$|\tilde{\mu}(\tilde{x}) - \mu_\pm| \leq C e^{-\theta|\tilde{x}|}$$

with $\mu_\pm$ satisfying (8.20). This equation is underdetermined with index one, reflecting the translation-invariance of the underlying equations. However, the operator $\partial_{\tilde{x}} - \tilde{\mu}$ has a bounded L^2 right inverse $(\partial_{\tilde{x}} - \tilde{\mu})^{-1}$, as may be seen by adjoining an additional artificial constraint

$$\tilde{z}_0(0) = 0 \tag{8.24}$$

fixing the phase. This can be seen by solving explicitly the equation or applying the gap lemma of [24] to reduce the problem to two constant-coefficient equations on $\tilde{x} \gtrless 0$, with boundary conditions at $z = 0$. We obtain as a result that

$$\|e^{\delta|\tilde{x}|} \tilde{z}_0\|_{L^2} \leq C \|e^{\delta|\tilde{x}|} \tilde{h}_0\|_{L^2}$$

if $\delta < \min\{\alpha, \theta\}$, which yields by rescaling the estimate

$$\|e^{\varepsilon\delta|x|} z_0\|_{L^2} \leq C\varepsilon^{-1} \|e^{\varepsilon\delta|x|} h_0\|_{L^2}$$

Together with the (better) previous estimates, this gives existence and uniqueness for the equation

$$z' - pz = h, \qquad z_0(0) = 0$$

with the estimate $\|e^{\varepsilon\delta|x|} z\|_{L^2} \leq C\varepsilon^{-1} \|e^{\varepsilon\delta|x|} h\|_{L^2}$. Because $\omega^{-1}\omega' = O(\varepsilon^2)$, this implies that for ε small enough, the equation (8.22) with $z_0(0) = 0$ has a unique solution. Tracing back to the original variables u, the condition $z_0(0) = 0$ translates into a condition of the form $\ell_\varepsilon \cdot u(0) = 0$. Therefore, the equation $\bar{b}_* u' - \bar{d} f_* u = h$ has a unique solution such u that $\ell_\varepsilon \cdot u(0) = 0$, which satisfies

$$\|e^{\varepsilon\delta|x|} u\|_{L^2} \leq C\varepsilon^{-1} \|e^{\varepsilon\delta|x|} h\|_{L^2}$$

for δ and ε small enough, finishing the proof of Proposition 8.1.

9. **Existence for the linearized problem.** The desired estimates (6.14) and (6.15) are given by Propositions 8.2 and 8.3. It remains to prove existence for the linearized problem with phase condition $u(0) \cdot r(\varepsilon) = 0$. This we carry out using a vanishing viscosity argument.

Fixing ε, consider in place of $\mathcal{L}_*^\varepsilon U = F$ the family of modified equations

$$\mathcal{L}_*^{\varepsilon,\eta} U := \mathcal{L}_*^\varepsilon U - \eta \begin{pmatrix} u' \\ v'' \end{pmatrix} = F := \begin{pmatrix} f \\ g \end{pmatrix}, \quad \ell_\varepsilon \cdot u(0) = 0. \tag{9.1}$$

Differentiating the first equation yields

$$AU' - dQ(x)U - U'' = \begin{pmatrix} f' \\ g \end{pmatrix}, \quad \ell_\varepsilon \cdot u(0) = 0. \tag{9.2}$$

where $dQ(x)$ denotes here the matrix $dQ(\bar{u}_{NS}, v_*(\bar{u}_{NS}))$.

9.1. **Uniform estimates.** We first prove uniform a-priori estimates. We denote by $\mathscr{S}$ the Schwartz space and for $\delta \geq 0$, by $\mathscr{S}_{\varepsilon\delta}$ the space of functions u such that $e^{\varepsilon\delta\langle x\rangle} u \in \mathscr{S}$, with $\langle x \rangle = \sqrt{1+x^2}$ as in (6.12).

Proposition 9.1. *There are constants $\varepsilon_0 > 0$, $\delta_0 > 0$ and $\eta_0 > 0$, and for all $s \geq 2$ a constant C_s, such that for $\varepsilon \in]0, \varepsilon_0]$, $\delta \in [0, \delta_0]$, $\eta \in]0, \eta_0]$, and U and F in $\mathscr{S}_{\varepsilon\delta}(\mathbb{R})$, satisfying (9.1)*

$$\|U\|_{H^s_{\varepsilon,\delta}} \leq C_s \varepsilon^{-1} \big(\|f\|_{H^{s+1}_{\varepsilon,\delta}} + \|g\|_{H^s_{\varepsilon,\delta}} \big). \tag{9.3}$$

Proof. The argument of Proposition 7.1 goes through essentially unchanged, with new η terms providing additional favorable higher-derivative terms sufficient to absorb new higher-derivative errors coming from the Kawashima part.

Thus we are led to equations of the form (7.2) with the additional term $-\eta U''$ in the left hand side. Using the symmetrizer $\mathcal{S}$ (4.28), one gains $\eta \|U''\|^2_{L^2} + \lambda \|U'\|^2_{L^2}$ in the minorization of Re $(\mathcal{S}F, U)$ and loses commutator terms which are dominated by

$$\eta \|S''\|_{L^\infty} (\|U'\|^2_{L^2} + \|U\|_{L^2} \|U'\|_{L^2}) + \eta \|K\|_{L^\infty} (\|U'\|_{L^2} + \|U\|_{L^2}) \|U''\|_{L^2},$$

which can be absorbed by the left hand side yielding uniform estimates

$$\sqrt{\eta} \|\widetilde{U}''\|_{L^2} + \|\widetilde{U}'\|_{L^2} + \|\tilde{v}\|_{L^2} \leq C \big(\|f\|_{H^2} + \|h\|_{H^1} + \|\tilde{g}\|_{H^1} + \varepsilon \|u\|_{L^2} \big). \tag{9.4}$$

Going back to (9.2), this implies uniform estimates of the form

$$\sqrt{\eta} \|U''|_{L^2_{\varepsilon,\delta}} + \|U'\|_{L^2_{\varepsilon,\delta}} + \|\tilde{v}\|_{L^2_{\varepsilon,\delta}} \leq C \big(\|(f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \big). \tag{9.5}$$

for $\delta = 0$, and next for $\delta \in [0, \delta_0]$ with $\delta_0 > 0$ small, as in the proof of Proposition 7.1.

When commuting derivatives to the equation, the additional term $\eta \partial_x^2$ brings no new term and the proof of Proposition 7.2 can be repeated without changes, yielding estimates of the form

$$\begin{aligned} \sqrt{\eta} \|D_x^k U''\|_{L^2_{\varepsilon,\delta}} + &\|\partial_x^k U'\|_{L^2_{\varepsilon,\delta}} + \|\partial_x^k \tilde{v}\|_{L^2_{\varepsilon,\delta}} \\ &\leq C \|\partial_x^k (f, f', f'', g, g')\|_{L^2_{\varepsilon,\delta}} \\ &\quad + \varepsilon^k C_k \big(\|U'\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon \|\tilde{v}\|_{H^{k-1}_{\varepsilon,\delta}} + \varepsilon \|u\|_{L^2_{\varepsilon,\delta}} \big). \end{aligned} \tag{9.6}$$

Next, applying the Chapman–Enskog argument of Section 8 to the viscous system, we obtain in place of (8.3) the equation

$$\bar{b}_* u' - d\bar{h}_* u = f + O(|\tilde{v}'| + |g| + |f'|) + \varepsilon^2 O(|u|) + \eta O(|u'| + |U''|), \tag{9.7}$$

where the final η term coming from artificial viscosity is treated as a source. One applies Proposition 8.1 to estimate $\varepsilon\|u\|_{L^2_{\varepsilon,\delta}}$ by the $L^2_{\varepsilon,\delta}$-norm of the right hand side, and continuing as in the proof of Proposition 8.2, the estimate (8.13) is now replaced by

$$\begin{aligned}\sqrt{\eta}\|U'''\|_{L^2_{\varepsilon,\delta}} + \big\|U'\big\|_{H^1_{\varepsilon,\delta}} + \big\|\tilde{v}\big\|_{L^2_{\varepsilon,\delta}} + \varepsilon\big\|u\big\|_{L^2_{\varepsilon,\delta}} \\ \leq C\big(\big\|f, f', f'', g, g'\big\|_{H^1_{\varepsilon,\delta}} + \eta(\|U'\|_{L^2_{\varepsilon,\delta}} + \|U''\|_{L^2_{\varepsilon,\delta}})\big).\end{aligned} \tag{9.8}$$

Therefore, for η small, the new $O(\eta)$ terms can be absorbed, and (9.3) for $s = 2$ follows as before. The higher order estimates follow from (9.6). ☐

9.2. **Existence.** We now prove existence and uniqueness for (9.1). First, recast the the problem as a first-order system

$$\mathcal{U}' - \mathbb{A}\mathcal{U} = \mathcal{F} \tag{9.9}$$

with

$$\mathcal{U} = \begin{pmatrix} u \\ v \\ v' \end{pmatrix}', \qquad \mathcal{F} = \begin{pmatrix} f \\ 0 \\ g \end{pmatrix},$$

and

$$\mathbb{A} := \eta^{-1}\begin{pmatrix} A_{11} & A_{12} & 0 \\ 0 & 0 & \eta I \\ \eta^{-1}A_{21}A_{11} - Q_{21} & \eta^{-1}A_{21}A_{12} - Q_{22} & A_{22} \end{pmatrix}. \tag{9.10}$$

Next, consider this as a transmission problem or a doubled boundary value problem on $x \gtrless 0$, with boundary condtitions given by the $n + 2r$ matching conditions $\mathcal{U}(0^-) = \mathcal{U}(0^+)$ at $x = 0$ together with the phase condition $\ell_\varepsilon \cdot u(0) = 0$, that is $n + 2r + 1$ conditions in all:

$$\mathcal{U}(0^-) = \mathcal{U}(0^+), \qquad \ell_\varepsilon \cdot u(0) = 0. \tag{9.11}$$

Note that the operator-valued coefficient matrix $\mathbb{A}$ converges exponentially to its endstates at $\pm\infty$, by exponential convergence of $\bar{U}_{NS}$ and boundedness of A, Q.

Lemma 9.2. *There is $\theta_1 > 0$ such that for ε small enough , the limiting coefficient matrices $\mathbb{A}_\pm$ have no eigenvalue in the strip $|\mathrm{Re}\, z| \leq \varepsilon\delta_0$.*

Proof. The proof is parallel to the proof of the estimates. Dropping the $\pm$, suppose that $i\tau$ is an eigenvalue of $\mathbb{A}$, or equivalently that there is a constant vector $U \neq 0$ such that $e^{i\tau x}U$ is a solution of of equations (9.1) Thus

$$\begin{aligned} &A_{11}u + A_{12}v = i\tau\eta u, \\ &(i\tau A - Q + \tau^2\eta)U = 0. \end{aligned} \tag{9.12}$$

In the first equation, introduce once again the variable $\tilde{v} = v + Q_{22}^{-1}Q_{21}u$, so that the equations are transformed to

$$\begin{aligned} &A_{11}^*u + A_{12}\tilde{v} = i\tau\eta u, \\ &(i\tau A - Q + \tau^2\eta)U = 0. \end{aligned} \tag{9.13}$$

Denoting by K the end point values of the Kawashima multipliers associated to A and Q, consider the symmetrizer

$$\Sigma = |\tau|^2 - i\overline{\tau}K - \lambda.$$

Multiplying the second equation in (9.13) by Σ and taking the real part of the scalar product with U yields

$$\begin{aligned}|\tau|^2\text{Re }(KA-QU,U)+\lambda(QU,U)+\eta|\tau|^4(U,U)\\ \le C\big(|\text{Im}\,\tau|(|\tau|^2+\lambda)\big)|U|^2+C|\tau||QU||U|\\ +\eta(|\tau|^2|\text{Im}\,\tau|^2+|\tau|^3+\lambda|\tau|^2)\big)|U|^2.\end{aligned}$$

Therefore, choosing appropriately λ, for η and $|\text{Im}\,\tau|$ sufficiently small, one has

$$(\eta|\tau|^4+|\tau|^2)|U|^2+|v|^2\le C(|\text{Im}\,\tau|+\varepsilon)|u|^2 \tag{9.14}$$

In particular, $|\tau|$ must be small if $\text{Im}\,\tau$, ε are small.

From the equation $i\tau A_{21}u+A_{22}v-Q_{21}u-Q_{22}v+\eta\tau^2v=0$ and the fact that $|Q_{21}|=O(\varepsilon)$ by (7.4), one deduces that

$$\tilde v-i\tau(Q_{22})^{-1}A_{21}u=O(|\tau|+\eta|\tau|^2)|\tilde v|.$$

Substituting in the first equation of (9.13), we obtain the Chapman-Enskog approximation

$$(A^*_{11}-i\tau\bar b_*)u=O(\eta|\tau|+|\tau|+\eta|\tau|^2)|\text{Im}\,\tau|^{\frac12}))|u|$$

where $\bar b_*$ denotes the end point value of the function (4.22). Therefore,

$$|(\bar b_*)^{-1}A^*_{11}u-i\tau u|\le C|\text{Im}\,\tau|^{\frac12}|\tau||u| \tag{9.15}$$

with arbitrarily small $c>0$. We know from Assumption 4.11 that for ε small, $(\bar b_*)^{-1}A^*_{11}$ has a unique small eigenvalue, of order $O(\varepsilon)$, real. Let us denote it by $\varepsilon\mu$. Then we know that $|\mu|$ is bounded from below, see (8.20). Then (9.15) implies that there is a constant C such that for $|\text{Im}\,\tau|$ small enough, and thus $|\tau|$ small, $|i\tau-\varepsilon\mu|\le C|\text{Im}\,\tau|^{\frac12}|\tau|$. Therefore, $l\text{Im}\,\tau+\varepsilon\mu|\le\frac12\varepsilon|\mu|$ if ε is small enough.

Summing up, we have proved that if ε is small enough, $\mathbb{A}$ has at most one eigenvalue z in the strip $|\text{Re}\,z\le\varepsilon 2|\mu|$, such that $|z-\varepsilon\mu|\le\frac12\varepsilon|\mu|$. This implies the lemma. □

Remark 9.3. The same reasoning can be applied to prove that $\mathbb{A}$ actually has a simple eigenvalue such that $|z-\varepsilon\mu|\le\frac12\varepsilon|\mu|$.

9.2.1. *Finite-dimensional case.* We first review the case that U is finite-dimensional, recalling for completeness the analysis of [23].

Proposition 9.4 ([23])**.** *There are constants $\varepsilon_0>0$, $\delta_0>0$ and $\eta_0>0$ such that for $\varepsilon\in]0,\varepsilon_0]$, $\delta\in[0,\delta_0]$, $\eta\in]0,\eta_0]$, and F in $\mathscr{S}_{\varepsilon\delta}(\mathbb{R})$, (9.1) admits a unique solution $U\in\mathscr{S}_{\varepsilon\delta}(\mathbb{R})$.*

Proof. Noting that the coefficient matrix $\mathbb{A}$ converges exponentially to $\mathbb{A}_\pm$ at $\pm\infty$, we may apply the conjugation lemma of [23] to convert the equation (9.9) by an asymptotically trivial change of coordinates $\mathcal{U}=T(x)Z$ to a constant-coefficient problems

$$Z'_--\mathbb{A}_-Z_-=F_-,\quad Z'_+-\mathbb{A}_+Z_+=F_+, \tag{9.16}$$

on $\{\pm x\ge 0\}$, with $n+2r+1$ modified boundary conditions determined by the value of the transformation T at $x=0$, where $\mathbb{A}_\pm:=\mathbb{A}(\pm\infty)$, and $Z_\pm(x):=Z(x)$ for $\pm x>0$.

By standard boundary-value theory (see, e.g., [9]), to prove existence and uniqueness in the Schwartz space for the problem (9.9) on $\{x<0\}$ and $\{x>0\}$ with transmission conditions (9.11), it is sufficient to show that

(i) the limiting coefficient matrices $\mathbb{A}_\pm$ are hyperbolic, i.e., have no pure imaginary eigenvalues,

(ii) the number of boundary conditions is equal to the number of stable (i.e., negative real part) eigenvalues of $\mathbb{A}_+$ plus the number of unstable eigenvalues (i.e., positive real part) of $\mathbb{A}_-$, and

(iii) there exists no nontrivial solution of the homogeneous equation $f = 0$, $g = 0$.

Moreover, since the eigenvalues of $\mathbb{A}_\pm$ are located in $\{|\mathrm{Re}\, z| \geq \theta_1\varepsilon$, the conjugated form (9.16) of the equation show that if the source term f has an exponential decay $e^{-\varepsilon\delta\langle x\rangle}$ at infinity, then the bounded solution also has the same exponential decay, provided that $\delta < \theta_1$. Therefore, the three conditions above are also sufficient to prove existence and uniqueness in $\mathscr{S}_{\varepsilon\delta}$ if ε and δ are small.

Note that (i) is a consequence of Lemma 9.2, while (iii) follows from the estimate (9.3). To verify (ii), it is enough to establish the formulae

$$\begin{aligned} \dim \mathcal{S}(\mathbb{A}_\pm) &= r + \dim \mathcal{S}(A_{11}^{*\pm}), \\ \dim \mathcal{U}(\mathbb{A}_\pm) &= r + \dim \mathcal{U}(A_{11}^{*\pm}), \end{aligned} \tag{9.17}$$

where $A_{11}^{*\pm} = dh_*(u_\pm) = A_{11} + A_{12}dv_*(u_\pm)$ and $\mathcal{S}(M)$ and $\mathcal{U}(M)$ denote the stable and unstable subspaces of a matrix M. We note that $A_{11}^{*\pm} = dh_*(u_\pm)$ are invertible, with dimensions of the stable subspace of A_{11}^{*+} and the unstable subspace of A_{11}^{*-} summing to $n+1$, by Proposition 5.1. Thus, (9.17) implies that

$$\dim \mathcal{S}(\mathbb{A}_+) + \dim \mathcal{U}(\mathbb{A}_-) = 2r + \dim \mathcal{S}(A_{11}^{*+}) + \dim \mathcal{U}(A_{11}^{*-}) = 2r + n + 1$$

as claimed.

To establish (9.17), introduce the variable $\tilde{v} = v + Q_{22}^{-1}Q_{21}u$, and the variable corresponding to $\tilde{v}'$ scaled by a factor $\eta^{\frac{1}{2}}$, that is $\tilde{w} = \eta^{\frac{1}{2}}w + \eta^{-\frac{1}{2}}Q_{22}^{-1}Q_{21}(A_{11}u + A_{12}v)$. After this change of variables, the matrix $\mathbb{A}$ it conjugated to $\widetilde{\mathbb{A}}$ with

$$\eta^{\frac{1}{2}}\widetilde{\mathbb{A}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & -Q_{22} & 0 \end{pmatrix} + \eta^{-\frac{1}{2}} \begin{pmatrix} A_{11}^* & A_{12} & 0 \\ 0 & 0 & 0 \\ O(\eta^{-\frac{1}{2}}) & O(\eta^{-\frac{1}{2}}) & A_{22} \end{pmatrix}. \tag{9.18}$$

From (i), the matrix $\eta^{\frac{1}{2}}\widetilde{\mathbb{A}}$ has no eigenvelue on the imaginary axis, and the number of eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$ is independent of η, and thus can be determined taking η to infinity. The limiting matrix has r eigenvalues in $\{\mathrm{Re}\,\lambda > 0\}$, r eigenvalues in $\{\mathrm{Re}\,\lambda < 0\}$ and the eigenvalue 0 with multiplicity n, since $-Q_{22}$ has its spectrum in $\{\mathrm{Re}\,\lambda > 0\}$. The classical perturbation theory as in [17] shows that for $\eta^{-\frac{1}{2}}$ small, $\eta^{\frac{1}{2}}\widetilde{\mathbb{A}}$ has n eigenvalues of order $\eta^{-\frac{1}{2}}$, close to the spectrum of A_{11}^* with error $O(\eta^{-1})$. Thus, for $\eta > 0$ large, $\eta^{\frac{1}{2}}\widetilde{\mathbb{A}}$ has $r + \dim \mathcal{S}(A_{11}^*)$ eigenvalue in $\{\mathrm{Re}\,\lambda < 0\}$, proving (9.17).

The proof of the Proposition is now complete. □

9.3. **Finite dimensional approximations.** To treat the infinite-dimensional case, we proceed by finite-dimensional approximations. Let $\underline{Q} = Q_{\underline{M}}$ and $\underline{K} = K_{\underline{M}}$ denote the operators Q_U and K_U evaluated at the equilibrium $\underline{M} = M(\underline{u})$, so that

$$\underline{A} = (\underline{A})^*, \quad \underline{Q} = (\underline{Q})^* = \begin{pmatrix} 0 & 0 \\ 0 & \underline{Q}_{22} \end{pmatrix}, \tag{9.19}$$

with

$$\underline{Q}_{22} \geq c\mathrm{Id}, \quad c > 0. \tag{9.20}$$

Moreover, the Kawashima multiplier has the form

$$\underline{K} = -(\underline{K})^* = \theta \begin{pmatrix} \underline{K}_{11} & \underline{K}_{12} \\ \underline{K}_{21} & 0 \end{pmatrix}. \tag{9.21}$$

Thanks to (9.20) the condition (4.14) is satisfied for θ small enough as soon as

$$\text{Re}\,\big(\underline{K}_{11}\underline{A}_{11} + \underline{K}_{12}\underline{A}_{21}\big) \geq c\text{Id}, \quad c > 0. \tag{9.22}$$

Consider an increasing sequence of finite dimensional subspaces

$$\mathbb{V}_r \subset \mathbb{V}_{r+1}, \qquad \cup \mathbb{V}_r \text{ dense in } \mathbb{V}. \tag{9.23}$$

Similarly, let $\mathbb{H}_r = \mathbb{U} \oplus \mathbb{V}_r$. Let $\Pi_r(U)$ denote the orthogonal projector onto $\mathbb{H}_r$, so that

$$\Pi_r = \Pi_r^*, \tag{9.24}$$

and define

$$A_r(U) = \Pi_r(U) A \Pi_r(U), \qquad Q_r(U) = \Pi_r(U) Q_U \Pi_r(U). \tag{9.25}$$

Lemma 9.5. *(i) $A_r(U)$ is symmetric.*

(ii) $Q_r(U)$ is uniformly negative definite on $\{0\} \oplus \mathbb{V}_r$.

(iii) With $K_r : \Pi_r(U) K_U \Pi_r(U)$, the Kawashima condition

$$\text{Re}\, K_r A_r - Q_r \geq \gamma \text{Id} \tag{9.26}$$

is uniformly satisfied for U in a neighborhood of $\underline{M}$ and r large enough.

Proof. (i) follows by (9.24) and symmetry of A on $\mathbb{H}$.

On $\{0\} \oplus \mathbb{V}_r$, $Q_r(U)$ is a perturbation of $\pi_r \underline{Q}_{22} \pi_r$ where π_r is the (usual) orthogonal projection onto $\mathbb{V}_r$, with

$$\underline{\Pi}_r \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ \pi_r v \end{pmatrix}.$$

To prove (9.26), is is sufficient to prove the property at $U = \underline{M}$. Restricting to $\mathbb{V}_r$ by π_r, one has

$$\underline{A}_r = \begin{pmatrix} \underline{A}_{11} & \underline{A}_{12}\pi_r \\ \pi_r \underline{A}_{21} & \pi_r \underline{A}_{22} \pi_r \end{pmatrix}, \qquad \underline{K}_r = \theta \begin{pmatrix} \underline{K}_{11} & \underline{K}_{12}\pi_r \\ \pi_r \underline{K}_{21} & 0 \end{pmatrix}.$$

Note that

$$\underline{K}_{11}\underline{A}_{11} + \underline{K}_{12}\pi_r \underline{A}_{21}$$

is an n-dimensional perturbation of $\underline{K}_{11}\underline{A}_{11} + \underline{K}_{12}\underline{A}_{21}$ whose real part is positive definite. Therefore, *for r large enough*,

$$\text{Re}\,\big(\underline{K}_r \underline{A}_r\big)_{11} = \text{Re}\,\big(\underline{K}_{11}\underline{A}_{11} + \underline{K}_{12}\pi_r \underline{A}_{21}\big) \geq c\text{Id}$$

with c independent of r. Since

$$\pi_r \underline{Q}_{22} \pi_r \geq c_1 \text{Id} \qquad \text{on } \mathbb{V}_r$$

uniformly in r, and since the other blocks of $K_r A_r$ are uniformly $O(\theta)$, the condition (9.26) is satisfied for r large enough and θ sufficiently small. □

Corollary 9.6. *On $\mathbb{H}_r$ the equation*

$$A_r(\bar{u}_{NS}) \partial_x U - Q_r(\bar{u}_{NS}) U = \begin{pmatrix} f' \\ g \end{pmatrix}, \qquad \ell \cdot u(0) = 0 \tag{9.27}$$

is well posed, and there are uniform estimates in r, for r sufficiently large.

Corollary 9.7. *On $\mathbb{H}$ the equation*

$$A\partial_x U - Q(\bar{u}_{NS})U = \begin{pmatrix} f' \\ g \end{pmatrix}, \qquad \ell \cdot u(0) = 0 \tag{9.28}$$

is well posed.

9.4. **Proof of Proposition 6.2.** Let $(\mathcal{L}_*^{\varepsilon,\eta})^\dagger$ denote the inverse operator of $\mathcal{L}_*^{\varepsilon,\eta}$ defined by (9.1), for sufficiently small $\eta > 0$. The uniform bound (9.3), and weak compactness of the unit ball in H^2, for $F \in \mathscr{S}$, we obtain existence of a weak solution $U \in H^2$ of

$$\mathcal{L}_*^\varepsilon U = F := \begin{pmatrix} f \\ g \end{pmatrix}, \qquad \ell_\varepsilon \cdot u(0) = 0, \tag{9.29}$$

along some weakly convergent subsequence. Proposition 8.2 implies uniqueness in H^2 for this problem, therefore the full family converges, giving sense to the definition

$$(\mathcal{L}_*^\varepsilon)^\dagger = \lim_{\eta \to 0} (\mathcal{L}_*^{\varepsilon,\eta})^\dagger \tag{9.30}$$

acting from $\mathscr{S}$ to H^2.

For $F \in \mathcal{S}_{\varepsilon\delta}$, the uniform bounds (9.3) imply that the limit $(\mathcal{L}_*^\varepsilon)^\dagger U \in H^s_{\varepsilon,\delta}$ and satisfies same estimate. By density, the operator $(\mathcal{L}_*^\varepsilon)^\dagger$ extends to $f \in H^{s+1}_{\varepsilon,\delta}$ and $g \in H^1_{\varepsilon,\delta}$, with $(\mathcal{L}_*^\varepsilon)^\dagger F \in H^s_{\varepsilon,\delta}$.

The sharp bound (6.14) and (6.15) now follow immediately from Propositions 8.2 and 8.3. The proof of Proposition 6.2 is now complete.

10. **Other norms.** We now briefly discuss the modifications needed to obtain the full result of Theorem 1.1.

10.1. **Pointwise velocity estimates.**

Proof of Theorem 1.1 ($\mathbb{H}^{1/2}$). To obtain pointwise bounds with respect to velocity, we carry out the same argument as in the proof of Proposition 5.4, substituting in place of the L^2 norm $|\cdot|$ in ξ, the weighted H^s (Sobolev) norm

$$|f|_s := \sum_{k=0}^{s} C^{-k} |\partial_\xi^k f|_2,$$

$C > 0$ sufficiently large, similarly as we did for the x-variable in order to get pointwise bounds in x.

We have only to observe that differentiating the linearized equations in ξ gives the same principal part applied to the ξ-derivative of U, plus commutator terms. Since commutator terms, both for the linearized collision operator L and the transport operator A are of one lower derivative in ξ and also one lower factor in $\langle \xi \rangle$ (straightforward computation differentiating $|\xi - \xi'|$, ξ_1, respectively) for the hard-sphere case, we easily find that commutator terms are absorbable for $C > 0$ sufficiently large by lower order estimates already carried out.

Thus, we obtain all the same estimates as before and the argument closes to give the same result in the stronger norm $|\cdot|_s$. (Note: a detail is to observe that truncation errors of the approximate solution are of the same order in the Sobolev norm, which follows by the corresponding property of the Maxwellian.) Applying the Sobolev embedding estimate in ξ, we obtain (1.18) for $\eta = 1/2$, which evidently implies the same estimate for $\eta \geq 1/2$. ∎

10.2. **Higher weights.**

Proof of Theorem 1.1 ($\mathbb{H}^s$). To extend our results from $\mathbb{H}^{\frac{1}{2}}$ to $\mathbb{H}^s$, we use a simple bootstrap argument together with the key observation that the $\mathbb{H}^s$ norm of $\mathbb{P}_{\mathbb{U}} f$ is controlled (by equivalence of finite-dimensional norms) by the $\mathbb{H}^{\frac{1}{2}}$ estimates already obtained. Namely, starting similarly as in (4.24) with the equation $A\partial_x - L_a U = F$, $\mathbb{P}_{\mathbb{U}} F = f$, $\mathbb{P}_{\mathbb{V}} F = g$, we find, taking the $\mathbb{H}^s$-inner product of U against this equation and applying the result of Proposition 3.6 and recalling that A is formally self-adjoint in $\mathbb{H}^s$, we obtain the estimate

$$\|\mathbb{P}_{\mathbb{V}} U\|_{L^2} \le C\big(\|f\|_{L^2} + \|g\|_{L^2} + \|\mathbb{P}_{\mathbb{U}} U\|_{L^2}\big). \tag{10.1}$$

Differentiating the equations k times and taking the inner product with $\partial_x^k U$, we find, similarly, the higher-derivative estimate

$$\|\mathbb{P}_{\mathbb{V}} U\|_{H^k_{\varepsilon,\delta}} \le C\big(\|f\|_{H^k_{\varepsilon,\delta}} + \|g\|_{H^k_{\varepsilon,\delta}} + \|\mathbb{P}_{\mathbb{U}} U\|_{H^k_{\varepsilon,\delta}}\big). \tag{10.2}$$

Specializing now to the case (6.8), (6.9), and bounding the $\mathbb{H}^s$ norm of $\mathbb{P}_{\mathbb{U}} U$ by a constant times the $\mathbb{H}^{\frac{1}{2}}$ bound obtained already in our previous analysis, we recover the key bounds (6.14)–(6.15) of Proposition 6.2 in the general space $\mathbb{H}^s$. With this bound, the entire contraction mapping argument goes through in $\mathbb{H}^s$, since this relies only on boundedness estimates on A, Q already obtained, the estimate (5.2) (still valid in $\mathbb{H}^s$), and the linearized estimates (6.14)–(6.15), yielding (1.18)(i) and (ii) as claimed, for any $\eta > 0$.

The estimate (1.18)(iii) then follows by Remark 3.7 estimating decay in velocity ξ of the approximating profile $\bar f_{NS}$. □

Remark 10.1. We emphasize that L is *not* approximately self-adjoint with respect to $\mathbb{H}^s$, $s >> 1/2$, and, likewise, the splitting $\mathbb{H}^s = \mathbb{U} \oplus \mathbb{V}^s$ using projectors $\mathbb{P}_U$ and $\mathbb{P}_V$ is not orthogonal in this norm. For this reason, we obtain term $\|\mathbb{P}_{\mathbb{U}} U\|_{H^k_{\varepsilon,\delta}}$ in the righthand side of (10.2) and not $\varepsilon\|\mathbb{P}_{\mathbb{U}} U\|_{H^k_{\varepsilon,\delta}}$ as in the $\mathbb{H}^{1/2}$ computations. The missing ε factor was crucial in closing the argument in $\mathbb{H}^{1/2}$ and estimating $\mathbb{P}_{\mathbb{U}} U$. However, with $\mathbb{P}_{\mathbb{U}} U$ already bounded it is no longer needed, since our final estimates make no distinction between $\mathbb{P}_{\mathbb{U}} U$ and $\mathbb{P}_{\mathbb{V}} U$ components; that is, the lost ε factor is needed only to close the loop between microscopic and macroscopic estimates, and not to bound $\mathbb{P}_{\mathbb{V}} U$ in terms of $\mathbb{P}_{\mathbb{U}} U$.

11. **Other potentials.** Finally, we briefly indicate the changes needed to accomodate general hard cutoff potentials. Recall [1, 5] that these give structure $L = -\nu(\xi) + \mathcal{K}$, where $\nu \sim \langle \xi \rangle^\beta$, $0 < \beta < 1$, and $\mathcal{K}$ is compact, and similarly for Q. Dividing by $\nu \sim \langle \xi \rangle^\beta$ as before, we can thus obtain Q, L bounded, but this leaves A unbounded. Nonetheless, a closer look shows that the Kawashima compensator K as constructed is still bounded, the key point. For, examining A_{12}, we see that it decays as a polynomial in ξ times full Maxwellian, so is clearly bounded in $\mathbb{H}^s$ for $s < 1$.

Since the norm of A does not enter except through the good term KA, our basic micro-estimates therefore still survive. Of course, the macro-estimates, since finite-dimensional, survive as well. (This follows by the same estimate that shows that K as constructed is bounded; that is, one has only to check that A_{12} and A_{21} entries remain bounded, thanks to Maxwellian rate decay.) Thus, the argument goes through as before, also for these more general potentials.

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REFERENCES

[1] R. Caflisch and B. Nicolaenko, *Shock profile solutions of the Boltzmann equation,* Comm. Math. Phys., **86** (1982), 161–194.

[2] T. Carleman, *Sur la theorie des equations integrales et ses applications*, Verhandl. des Internat. Math. Kong., I, Zurich, (1932), 138–151.

[3] D. Gilbarg, *The existence and limit behavior of the one-dimensional shock layer,* Amer. J. Math., **73** (1951), 256–274.

[4] R. Glassey, "The Cauchy Problem in Kinetic Theory," Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[5] F. Golse, B. Perthame and C. Sulem, *On a boundary layer problem for the nonlinear Boltzmann equation,* Arch. Rational Mech. Anal., **103** (1988), 81–96.

[6] J. Goodman, *Remarks on the stability of viscous shock waves*, in: "Viscous Profiles and Numerical Methods for Shock Waves," (Raleigh, NC, 1990), 66–72, SIAM, Philadelphia, PA, (1991).

[7] H. Grad, *Asymptotic theory of the Boltzmann equation. II,* 1963, in: "Rarefied Gas Dynamics" (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. **I**, 26–59, Academic Press, New York.

[8] O. Gues, G. Métivier, M. Williams and K. Zumbrun, *Navier-Stokes regularization of multidimensional Euler shocks*, Ann. Sci. École Norm. Sup. (4), **39** (2006), 75–175.

[9] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, Springer–Verlag, Berlin (1981), iv + 348 pp.

[10] J. Humpherys and K. Zumbrun, *Spectral stability of small-amplitude shock profiles for dissipative symmetric hyperbolic-parabolic systems,* Z. Angew. Math. Phys., **53** (2002), 20–34.

[11] S. Kawashima, "Systems of a Hyperbolic–Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics," thesis, Kyoto University, 1983.

[12] S. Kawashima, A. Matsumura and T. Nishida, *On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier–Stokes equation*, Comm. Math. Phys., **70** (1979), 97–124.

[13] P. D. Lax, *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. v+48 pp.

[14] T. P. Liu, *Hyperbolic conservation laws with relaxation,* Comm. Math. Phys., **108** (1987), 153–175.

[15] T.-P. Liu and S.-H. Yu, *Boltzmann equation: micro-macro decompositions and positivity of shock profiles*, Comm. Math. Phys., **246** (2004), 133–179.

[16] A. Majda and R. Pego, *Stable viscosity matrices for systems of conservation laws*, J. Diff. Eqs., **56** (1985), 229–262.

[17] C. Mascia and K. Zumbrun, *Pointwise Green's function bounds and stability of relaxation shocks*, Indiana Univ. Math. J., **51** (2002), 773–904.

[18] C. Mascia and K. Zumbrun, *Stability of large-amplitude shock profiles of general relaxation systems*, SIAM J. Math. Anal., **37** (2005), 889–913.

[19] C. Mascia and K. Zumbrun, *Spectral stability of weak relaxation shock profiles*, Comm. Partial Differential Equations, **34** (2009), 119–136.

[20] C. Mascia and K. Zumbrun, *Pointwise Green function bounds for shock profiles of systems with real viscosity,* Arch. Rational Mech. Anal., **169** (2003), 177–263.

[21] C. Mascia and K. Zumbrun, *Stability of small–amplitude shock profiles of symmetric hyperbolic–parabolic systems,* Comm. Pure Appl. Math., **57** (2004), 841–876.

[22] G. Métivier, B. Texier and K. Zumbrun, *Existence of quasilinear relaxation shocks*, preprint, arXiv:math/0908.4048.

[23] G. Métivier and K. Zumbrun, *Existence of semilinear relaxation shocks,* to appear, J. Mat. Pure. Appl.

[24] G. Métivier and K. Zumbrun, *Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems*, Memoirs AMS, **175** (2005).

[25] R. L. Pego, *Stable viscosities and shock profiles for systems of conservation laws,* Trans. Amer. Math. Soc., **282** (1984), 749–763.

[26] R. Plaza and K. Zumbrun, *An Evans function approach to spectral stability of small-amplitude shock profiles*, Discrete Contin. Dyn. Syst., **10** (2004), 885–924.

[27] J. Smoller, "Shock Waves and Reaction-Diffusion Equations," Springer-Verlag, New York, second edition, 1994.

[28] D. Serre, "Systems of Conservation Laws. 1," Cambridge University Press, Cambridge, 1999, Hyperbolicity, entropies, shock waves, Translated from the 1996 French original by I. N. Sneddon.

[29] D. Serre, "Systems of Conservation Laws. 2," Cambridge University Press, Cambridge, 2000, Geometric structures, oscillations, and initial-boundary value problems, Translated from the 1996 French original by I. N. Sneddon.

[30] A. Szepessy and Z. P. Xin, *Nonlinear stability of viscous shock waves,* Arch. Rational Mech. Anal., **122** (1993), 53–103.

[31] H. Weyl, *Shock waves in arbitrary fluids,* Comm. Pure Appl. Math., **2** (1949), 103–122.

[32] W.-A. Yong, *Basic structures of hyperbolic relaxation systems*, Proc. Roy. Soc. Edinburgh Sect. A, **132** (2002), 1259–1274.

[33] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, in: "Advances in the Theory of Shock Waves," 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, 2001.

[34] K. Zumbrun, *Stability of large-amplitude viscous shock profiles of the equations of fluid dynamics*, With an appendix by Helge Kristian Jenssen and Gregory Lyng, in: "Handbook of Mathematical Fluid Dynamics," Vol. **III**, 311–533, North-Holland, Amsterdam, 2004.

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