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NONEXISTENCE OF WEAK SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, we are concerned with the following quasilinear elliptic equations:

$$\text{(E)}\quad \begin{cases} -\text{div}\left\{a(x)|\nabla u|^{p-2}\nabla u\right\} &= b(x)|u|^{q-2}u \quad \text{in } \Omega,\\ u(x) &= 0 \quad \text{on } \partial\Omega,\end{cases}$$

where Ω is a domain in $\mathbf{R}^N$ $(N \geq 1)$ with smooth boundary.

When a and b are positive constants, there are many results on the nonexistence of nontrivial solutions for the equation (E). The main purpose of this paper is to discuss the nonexistence results for (E) with a class of weak solutions under some assumptions on a and b.

1. **Introduction.** In this paper, we are concerned with the following boundary value problem of quasilinear elliptic equation:

$$\text{(E)}\begin{cases} -\text{div}\{a(x)|\nabla u|^{p-2}\nabla u\} &= b(x)|u|^{q-2}u \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^N$ $(N \geq 1)$, $1 < p, q < \infty$, $a, b > 0, a, b \in L^\infty(\Omega)$. When $a \equiv 1$, a differential operator in left-hand side is a p-Laplacian. In this case, there are many studies for the equations (E).

In this paper, we are going to discuss the nonexistence of nontrivial solutions. Since it is not guaranteed the C^2 regularity of solutions for the elliptic equations with principal part consists of degenerate operator such as p-Laplacian, we need approximation arguments to prove the nonexistence of solutions by using Pohozaev's type identity. We can cite the results of Guedda-Veron [4] and Ôtani [7] to prove the nonexistence of nontrivial solutions in this viewpoint. In the beginning, we cite the result of Guedda-Veron.

Proposition 1. *Let $a \equiv 1$ in* (E)*. Assume $b \in C^1(\overline{\Omega})$, $q \leq p^*$, $0 \in \Omega$ and $u \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a solution of* (E)*. Then the following relation holds.*

$$\left(\frac{N}{q} - \frac{N-p}{p}\right)\int_\Omega b|u|^q\,dx + \frac{1}{q}\int_\Omega (x\cdot\nabla b)|u|^q\,dx = \left(1 - \frac{1}{p}\right)\int_{\partial\Omega}|\nabla u|^p(x\cdot\vec{n})dS,$$

where $p^* := \dfrac{Np}{N-p}$ if $p < N$ and $:= \infty$ if $p \geq N$ and $\vec{n}$ denotes the outward normal unit vector.

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They prove the nonexistence of positive solutions of (E) with this identity and the $C^{1,\alpha}$-estimate of the solution in the case $q = p^*$.

On the other hand, we consider the case the operator in left-hand side is a p-Laplacian with a variable coefficient. Gilbert-Shi [3] shows that the existence of weak solutions for the elliptic system involving this operator. Fang-Gilbert [1] state the result of the nonexistence of nontrivial solutions of (E) by using the following type of the Pohozaev-identity.

Proposition 2. *Let $b \in C(\overline{\Omega})$. Assume $0 \in \Omega$ and $u \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ is a solution of* (E)*, then the following relation holds:*

$$\left(\frac{N}{q} - \frac{N-p}{p}\right)\int_\Omega b|u|^q\,dx + \frac{1}{q}\int_\Omega (x\cdot\nabla b)|u|^q\,dx$$
$$-\frac{p-N}{p}\int_\Omega (a|\nabla u|^p)^{\frac{p-2}{2}} u(\nabla u\cdot\nabla a)\,dx - \frac{1}{2}\int_\Omega (x\cdot\nabla a)(a|\nabla u|^2)^{\frac{p-2}{2}}\,dx$$
$$+\int_\Omega (\nabla u\cdot\nabla a)(x\cdot\nabla u)(a|\nabla u|^2)^{\frac{p-2}{2}}\,dx = \left(1-\frac{1}{p}\right)\int_{\partial\Omega}(a|\nabla u|^2)^{\frac{p-2}{2}}(x\cdot\vec{n})dS$$

They conclude that (E) has no solutions when b is a constant and Ω is a star-shaped domain under some conditions.

Remark 1. The result of Fang-Gilbert [1] is relied on the argument of Guedda-Veron [4]. Their argument is valid only for the case $q \le p^*$.

We here introduce another method for proving nonexistence of nontrivial weak solutions. This method is invented by Ôtani [7] and does not rely on the L^∞-estimate of weak solutions. Therefore it is valid for the case q is super critical. Pohozaev-type formula which is constructed in [7] is not equality but inequality because the class of weak solution is wider.

The main concern of this paper is to discuss the nonexistence result from the viewpoint of [7] and [5] which followed it.

2. **Pohozaev-type inequality.**

As we stated, since we construct the Pohozaev-type inequality for weak solutions of elliptic equations (E), we define a class of weak solutions as follows:

$$\mathcal{P} = \{u \in W_0^{1,p}(\Omega)\cap L^q(\Omega); x_i|u|^{q-2}u \in L^{\frac{p}{p-1}}(\Omega), i = 1,\dots,N\}$$

Also we assume,

(H1) Ω is a bounded star-shaped domain.
(H2) $\exists a_0, a_1 > 0$ such that $a_0 \le a(x) \le a_1$.
(H3) a and b are positive functions in $C^1(\overline{\Omega})$.

Under those assumptions, our Pohozaev-type inequality is described as follows.

Theorem 2.1. *Under the conditions* (H1)-(H3)*, the solution u in $\mathcal{P}$ of* (E) *satisfies the following inequality:*

$$\left(\frac{N-p}{p} - \frac{N}{q}\right)\int_\Omega b|u|^q dx + \frac{1}{p}\int_\Omega (x\cdot\nabla a)|\nabla u|^p dx - \frac{1}{q}\int_\Omega (x\cdot\nabla b)|u|^q dx + \mathcal{R} \le 0, \tag{1}$$

where

$$\mathcal{R} = \limsup_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{\bar{p}-1}{p}\int_{\partial\Omega} a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}(x\cdot\vec{n})dS \quad \text{and} \quad \bar{p} = \min(p,2).$$

Now let me give the proof of our Pohozaev-type inequality. Let u be a weak solution of (E) belonging to $\mathcal{P}$.

In order to establish this Pohozaev-type inequality, we need some approximation procedures. At first, we add the term $|u|^{q-2}u$ to both sides of the equation (E) to get the equivalent equation (E)$'$:

$$(\mathrm{E})' \begin{cases} |u|^{q-2}u - \mathrm{div}\{a(x)|\nabla u|^{p-2}\nabla u\} = & (1+b(x))|u|^{q-2}u \quad \text{in } \Omega, \\ u = & 0 \quad \text{on } \partial\Omega. \end{cases}$$

We prepare cut-off functions g_n ($n \in \mathbb{N}$). They are C^1-functions such that

$$0 \le g_n'(s) \le 1 \quad (s \in \mathbb{R}), \quad g_n(s) = \begin{cases} s & |s| \le n, \\ (n+1)\mathrm{sign}\, s & |s| \ge n+1. \end{cases} \tag{2}$$

Put $u_n = g_n(u)$, then we consider the following approximate equations $(\mathrm{E})_n$ for (E)$'$:

$$(\mathrm{E})_n \begin{cases} |w_n|^{q-2}w_n - \mathrm{div}\,\{a(x)|\nabla w_n|^{p-2}\nabla w_n\} &= (1+b(x))|u_n|^{q-2}u_n \quad \text{in } \Omega, \\ w_n &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

Since $u_n \in L^\infty(\Omega)$, we can take a sequence v_n^ε in $C_0^\infty(\Omega)$ satisfying

$$\|v_n^\varepsilon\|_{L^\infty} \le C_0 \qquad \text{for all } \varepsilon \in (0,1), \tag{4}$$

$$v_n^\varepsilon \to (1+b)|u_n|^{q-2}u_n \quad \text{strongly in} \quad L^r(\Omega) \text{ as } \varepsilon \to 0 \text{ for all } r \in [1,\infty). \tag{5}$$

We further need other approximate equations $(\mathrm{E})_n^\varepsilon$ for $(\mathrm{E})_n$ of the form:

$$(\mathrm{E})_n^\varepsilon \begin{cases} b|w_n^\varepsilon|^{q-2}w_n^\varepsilon + A_\varepsilon w_n^\varepsilon &= v_n^\varepsilon \quad \text{in } \Omega, \\ w_n^\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $A_\varepsilon u(x) = -\mathrm{div}\{a(x)(|\nabla u(x)|^2 + \varepsilon)^{(p-2)/2}\nabla u(x)\}$ and $\varepsilon > 0$.

Since $a \ge a_0$, A_ε does not have any singularity at the point where $\nabla u = 0$. Therefore the classical result such as in the book of Gilbarg-Trudinger [2] assures the existence of classical solutions w_n^ε of $(\mathrm{E})_n^\varepsilon$ and by variational method we can show the existence of solutions w_n of $(\mathrm{E})_n$.

Lemma 2.2. (i) *For each* $\varepsilon > 0$ *and* $\forall n \in \mathbb{N}$, *there exists a unique solution* $w_n^\varepsilon \in C^2(\overline{\Omega})$ *of* $(\mathrm{E})_n^\varepsilon$.

(ii) $\forall n \in \mathbb{N}$, *there exists a unique solution* $w_n \in C^1(\overline{\Omega})$ *of* $(\mathrm{E})_n$.

Furthermore, by virtue of a priori estimates and the standard arguments in convex analysis, we can show that the following convergence results.

Lemma 2.3. *Let* w_n^ε *be a solution of* $(\mathrm{E})_n^\varepsilon$ *and* w_n *be a solution of* $(\mathrm{E})_n$. *Then the following hold.*

(i) w_n^ε *converges to* w_n *in the following sense.*

$$D_i w_n^\varepsilon \to D_i w_n \quad \text{in } L^p(\Omega) \quad (1 \le i \le N) \tag{7}$$

$$w_n^\varepsilon \to w_n \quad \text{in } L^t(\Omega) \quad (\forall t \ge 1) \tag{8}$$

where $D_i u := \dfrac{\partial u}{\partial x_i}$.

(ii) w_n *converges to* u *in the following sense.*

$$D_i w_n \to D_i u \quad \text{in } L^p(\Omega) \quad (1 \le i \le N), \tag{9}$$

$$w_n \to u \quad \text{in } L^q(\Omega). \tag{10}$$

Proof. (i) Multiplying $(\mathrm{E})_n^\varepsilon$ by $|w_n^\varepsilon|^{r-2}w_n^\varepsilon$ and integration over Ω to get,

$$\int_\Omega |w_n^\varepsilon|^{q+r-2}dx + (r-1)\int_\Omega a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla w_n^\varepsilon|^2|w_n^\varepsilon|^{r-2}\,dx$$
$$= \int_\Omega v_n^\varepsilon|w_n^\varepsilon|^{r-2}w_n^\varepsilon dx. \tag{11}$$

Then we get,

$$\int_\Omega |w_n^\varepsilon|^{q+r-2}dx \le C_0\int_\Omega |w_n^\varepsilon|^{r-1}dx \le C_0'\left(\int_\Omega |w_n|^{q+r-2}dx\right)^{\frac{r-1}{q+r-2}}$$

for some constant $C_0' > 0$. Hence we easily get a priori bound for $\|w_n^\varepsilon\|_{L^{q+r-2}(\Omega)}$ independent of r, then letting $r \to +\infty$ we derive L^∞-estimate for w_n^ε. By the same argument as (iii) of Lemma 3.1 of [5] with $a(x) \ge a_0$, there exists a sequence $\{w_n^{\varepsilon_k}\}$ such that

$$w_n^{\varepsilon_k} \to w \quad \text{strongly in } L^r(\Omega) \quad \text{for all } r\in[1,\infty), \tag{12}$$
$$D_i w_n^{\varepsilon_k} \rightharpoonup D_i w \quad \text{weakly in } L^p(\Omega), \tag{13}$$
$$|w_n^{\varepsilon_k}|^{q-2}w_n^{\varepsilon_k} \rightharpoonup |w|^{q-2}w \quad \text{weakly in } L^2(\Omega) \tag{14}$$

for some w. Put

$$\phi_\varepsilon(z) = \frac{1}{p}\int_\Omega a(|\nabla z|^2+\varepsilon)^{p/2}dx \quad \text{with} \quad D(\phi_\varepsilon) = W_0^{1,p}(\Omega),$$

then its subdifferential $\partial\phi_\varepsilon$ coincides with A_ε. Hence $w_n^{\varepsilon_k}$ satisfies

$$\phi_{\varepsilon_k}(v) - \phi_{\varepsilon_k}(w_n^{\varepsilon_k}) \ge (-|w_n^{\varepsilon_k}|^{q-2}w_n^{\varepsilon_k} + v_n^{\varepsilon_k}, v - w_n^{\varepsilon_k})_{L^2} \quad \text{for all } v\in W_0^{1,p}(\Omega). \tag{15}$$

Since $\phi_\varepsilon(v) \to \phi_0(v)$ for all $v \in W_0^{1,p}(\Omega)$ and

$$\liminf_{k\to\infty}\phi_{\varepsilon_k}(w_n^{\varepsilon_k}) \ge \phi_0(w), \tag{16}$$

if follows from (5), (12), (14) that

$$\phi_0(v) - \phi_0(w) \ge (-|w|^{q-2}w + (1+b)|u_n|^{q-2}u_n, v-w)_{L^2} \quad \text{for all } v\in W_0^{1,p}(\Omega),$$

which is equivalent to $-\mathrm{div}\,(a|\nabla w|^{p-2}\nabla w) = -|w|^{q-2}w + (1+b)|u_n|^{q-2}u_n$, i.e., $w = w_n$.

Multiply (3) by w_n and (6) with $\varepsilon = \varepsilon_k$ by $w_n^{\varepsilon_k}$, then we get

$$(\partial\phi_0(w_n), w_n)_{L^2} = \int_\Omega |\nabla w_n|^p\,dx = -\int_\Omega |w_n|^q\,dx + \int_\Omega (1+b)|u_n|^{q-2}u_n w_n\,dx,$$
$$(\partial\phi_{\varepsilon_k}(w_n^{\varepsilon_k}), w_n^{\varepsilon_k})_{L^2} = \int_\Omega (|\nabla w_n^{\varepsilon_k}|^2+\varepsilon_k)^{(p-2)/2}|\nabla w_n^{\varepsilon_k}|^2\,dx$$
$$= -\int_\Omega |w_n^{\varepsilon_k}|^q\,dx + \int_\Omega v_n^{\varepsilon_k}w_n^{\varepsilon_k}\,dx.$$

Then these identities together with (5) and (12) give

$$\int_\Omega (|\nabla w_n^{\varepsilon_k}|^2+\varepsilon_k)^{(p-2)/2}|\nabla w_n^{\varepsilon_k}|^2dx \to \int_\Omega |\nabla w_n|^p dx \quad \text{as } k\to\infty. \tag{17}$$

Hence, in view of (13) and assumption (H2), we deduce

$$\int_\Omega |\nabla w_n^{\varepsilon_k}|^p\,dx \to \int_\Omega |\nabla w_n|^p dx \quad \text{as } k\to\infty. \tag{18}$$

Thus we finally find that (7) from the uniform convexity of $W_0^{1,p}(\Omega)$ since the argument above does not depend on the choice of $\{\varepsilon_k\}$.

(ii) The verification for (ii) is similar to that for (i), but it needs more delicate arguments. First of all we note that

$$|u_n|^{q-2}u_n \to |u|^{q-2}u \quad \text{strongly in } L^{\frac{q}{q-1}}(\Omega) \text{ as } n \to \infty. \tag{19}$$

Multiplying (3) by w_n, we have

$$\begin{aligned}\int_\Omega |w_n|^q dx + \int_\Omega a|\nabla w_n|^p dx &= \int_\Omega (1+b)|u_n|^{q-2}u_n w_n dx \\ &\leq (1+\|b\|_{L^\infty})\|u\|_{L^q(\Omega)}^{q-1}\|w_n\|_{L^q(\Omega)}.\end{aligned} \tag{20}$$

Then we can derive the a priori bounds for $\|w_n\|_{L^q(\Omega)}$ and $\|\nabla w_n\|_{L^p(\Omega)}$ since $a(x) \geq a_0$. Hence, by Rellich's compactness theorem and the demiclosedness of the operator: $w \mapsto |w|^{q-2}w$ from $L^1_{loc}(\Omega)$ into $L^{\frac{q}{q-1}}(\Omega)$, we can extract a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that

$$D_i w_{n_k} \rightharpoonup D_i w \qquad \text{weakly in} \qquad L^p(\Omega) \tag{21}$$

$$\begin{aligned} w_{n_k} \rightharpoonup w \qquad &\text{weakly in} \qquad L^q(\Omega) \\ &\text{and strongly in} \quad L^{q-\theta}_{loc}(\Omega) \text{ for all } \theta \in (0, q-1] \\ &\text{as} \quad k \to \infty, \end{aligned} \tag{22}$$

$$|w_{n_k}|^{q-2}w_{n_k} \rightharpoonup |w|^{q-2}w \quad \text{weakly in } L^{\frac{q}{q-1}}(\Omega) \text{ as } k \to \infty. \tag{23}$$

Since w_n is a solution of $(\mathrm{E})_n$, the definition of subdifferential operator gives

$$\begin{aligned} &\frac{1}{p}\int_\Omega a|\nabla v|^p dx - \frac{1}{p}\int_\Omega a|\nabla w_n|^p dx \\ &\geq \int_\Omega (-|w_n|^{q-2}w_n + (1+b)|u_n|^{q-2}u_n)(v-w_n)dx \\ &= \int_\Omega |w_n|^q dx - \int_\Omega |w_n|^{q-2}w_n v dx + \int_\Omega (1+b)|u_n|^{q-2}u_n(v-w_n)dx \end{aligned} \tag{24}$$

$$\text{for all } v \in C_0^\infty(\Omega).$$

Then by letting $n = n_k \to \infty$ in (24) and recalling (19), (21), (22) and (23), we obtain

$$\begin{aligned} &\frac{1}{p}\int_\Omega a|\nabla v|^p dx - \frac{1}{p}\int_\Omega a|\nabla w|^p dx \\ &\geq \int_\Omega (-|w|^{q-2}w + (1+b)|u|^{q-2}u)(v-w)dx \text{ for all } v \in C_0^\infty(\Omega). \end{aligned} \tag{25}$$

Now putting $v = w + tz$, with $z \in C_0^\infty(\Omega)$, and letting $t \to +0$, $t \to -0$ in (25), we can show that w satisfies

$$Aw + |w|^{q-2}w = (1+b)|u|^{q-2}u \quad \text{in } \mathcal{D}',$$

where $Aw := -\mathrm{div}\,(a|\nabla w|^{p-2}\nabla w)$. Whence follows

$$(|w|^{q-2}w + Aw) - (|u|^{q-2}u + Au) = 0 \quad \text{in } \mathcal{D}'.$$

Multiplying this by $w-u$, and using the monotonicity of A and strict monotonicity of the operator $u \mapsto |u|^{q-2}u$, we deduce $w = u$.

Therefore, in view of (19), (20), (21), (22) and the relation

$$\int_\Omega a|\nabla u|^p\, dx = \int_\Omega b|u|^q\, dx, \tag{26}$$

we get

$$\begin{aligned}\|u\|^q_{L^q(\Omega)} + \|b^{1/q}u\|^q_{L^q(\Omega)} &= \|u\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla u\|^p_{L^p(\Omega)} \\ &\le \liminf_{k\to\infty}(\|w_{n_k}\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)}) \\ &= \lim_{k\to\infty}(\|w_{n_k}\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)}) \\ &\le \|u\|^q_{L^q(\Omega)} + \|b^{1/q}u\|^q_{L^q(\Omega)},\end{aligned}$$

which yields

$$\lim_{k\to\infty}(\|w_{n_k}\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)}) = \|u\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla u\|^p_{L^p(\Omega)}.$$

Moreover we observe

$$\begin{aligned}\|u\|^q_{L^q(\Omega)} &\le \liminf_{k\to\infty}\|w_{n_k}\|^q_{L^q(\Omega)} \le \limsup_{k\to\infty}\|w_{n_k}\|^q_{L^q(\Omega)} \\ &= \limsup_{k\to\infty}(\|w_{n_k}\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)} - \|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)}) \\ &\le \limsup_{k\to\infty}(\|w_{n_k}\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)}) - \liminf_{k\to\infty}\|a^{1/p}\nabla w_{n_k}\|^p_{L^p(\Omega)} \\ &\le \|u\|^q_{L^q(\Omega)} + \|a^{1/p}\nabla u\|^p_{L^p(\Omega)} - \|a^{1/p}\nabla u\|^p_{L^p(\Omega)} = \|u\|^q_{L^q(\Omega)}.\end{aligned}$$

Consequently we find that $\lim_{k\to\infty}\|w_{n_k}\|_{L^q(\Omega)} = \|u\|_{L^q(\Omega)}$ and $\lim_{k\to\infty}\|\nabla w_{n_k}\|_{L^p(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$ since $a(x) \ge a_0$. Thus the uniform convexity of $L^p(\Omega)$ and $L^q(\Omega)$ assures the strong convergence. Since the above argument does not depend on the choice of subsequences, (ii) holds. □

Now we are ready to introduce "Pohozaev-type inequality" which is formulated in terms of solutions w_n^ε of approximate equations $(\mathrm{E})_n^\varepsilon$. After the manner of Pohozaev [8], we calculate $\displaystyle\lim_{\varepsilon\to 0}\sum_{i=1}^N\int_\Omega x_i D_i w_n^\varepsilon\,(\mathrm{E})_n^\varepsilon dx$. Then, we have:

Lemma 2.4. *The solution w_n of $(\mathrm{E})_n$ satisfies the following inequality.*

$$\begin{aligned}&\frac{N}{q}\int_\Omega |w_n|^q\,dx + \frac{N-p}{p}\int_\Omega a|\nabla w_n|^p\,dx + \frac{1}{p}\int_\Omega (x\cdot\nabla a)|\nabla w_n|^p\,dx \\ &+ \int_\Omega (1+b)|u_n|^{q-2}u_n\, x\cdot\nabla w_n\,dx + \mathcal{R}_n \le 0,\end{aligned} \tag{27}$$

where $\displaystyle\mathcal{R}_n = \limsup_{\varepsilon\to 0}\frac{\bar p - 1}{p}\int_{\partial\Omega} a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}(x\cdot\vec{n})\,dS$ *and* $\bar p = \min(p,2)$.

Proof. We are going to calculate $\displaystyle\sum_{i=1}^N\int_\Omega x_i D_i w_n^\varepsilon (\mathrm{E})_n^\varepsilon\,dx$. By applying the integration by parts several times and using the fact that $\nabla w_n^\varepsilon = \pm|\nabla w_n^\varepsilon|\cdot\vec{n}$ on $\partial\Omega$, we can derive the following identity.

$$\begin{aligned}&\frac{N}{q}\int_\Omega |w_n^\varepsilon|^q\,dx - \int_\Omega a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla w_n^\varepsilon|^2\,dx + \frac{N}{p}\int_\Omega a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}\,dx\\&+\frac{1}{p}\int_\Omega (x\cdot\nabla a)(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p/2}\,dx + \int_\Omega v_n^\varepsilon x\cdot\nabla w_n^\varepsilon\,dx\\&+\frac{p-1}{p}\int_{\partial\Omega} a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{p/2}(x\cdot\vec{n})\,dS\\&=\varepsilon\int_{\partial\Omega} a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(x\cdot\vec{n})\,dS.\end{aligned}\tag{28}$$

Since w_n^ε converges to w_n strongly in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ by (i) of Lemma 2.3, we can easily show that

$$a^{1/p}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{1}{2}} \to a^{1/p}|\nabla w_n| \quad \text{strongly in } L^p(\Omega),\tag{29}$$

$$(x\cdot\nabla a)^{1/p}(|\nabla w_n^\varepsilon|^2+\varepsilon)^{1/2} \to (x\cdot\nabla a)^{1/p}|\nabla w_n| \quad \text{strongly in } L^p(\Omega).\tag{30}$$

Moreover we observe

$$\begin{aligned}&\left|\int_\Omega a\{(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p-2)/2}|\nabla w_n^\varepsilon|^2 - (|\nabla w_n^\varepsilon|+\varepsilon)^{p/2}\}\,dx\right|\\&\le \int_\Omega a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p-2)/2}\varepsilon\,dx\\&\le \begin{cases}\left(\int_\Omega (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p/2}dx\right)\cdot \varepsilon a_1|\Omega|^{2/p} & (2<p<\infty),\\ \varepsilon^{p/2}a_1|\Omega| & (1<p\le 2).\end{cases}\end{aligned}\tag{31}$$

Hence (29) and (31) assure

$$\int_\Omega a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{(p-2)/2}|\nabla w_n^\varepsilon|^2\,dx \to \int_\Omega a|\nabla w_n|^p\,dx \quad \text{as } \varepsilon\to 0.\tag{32}$$

On the other hand, noting that $(x\cdot\vec{n})\ge 0$ on $\partial\Omega$, we get

$$\begin{aligned}&\varepsilon\int_{\partial\Omega} a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(x\cdot\vec{n})\,dS\\&\le \begin{cases}\int_{\partial\Omega}\varepsilon^{\frac{p}{2}}a(x\cdot\vec{n}(x))\,dS & \text{if } 1<p\le 2,\\ \frac{p-2}{p}\int_{\partial\Omega} a(|\nabla w_n^\varepsilon|^2+\varepsilon)^{\frac{p}{2}}(x\cdot\vec{n}(x))\,dS & \\ +\frac{2}{p}\int_{\partial\Omega}\varepsilon^{\frac{p}{2}}a(x\cdot\vec{n}(x))\,dS & \text{if } 2<p.\end{cases}\end{aligned}\tag{33}$$

Now, by letting $\varepsilon\to 0$ in (28), and making use of (7), (29), (30), and (33), we can deduce (27). □

We here proceed to the proof of Theorem 2.1.

Proof of the Theorem 2.1. By virtue of the fact that $u_n(x)\to u(x)$, $|u_n(x)|\le|u(x)|$ for a.e. $x\in\Omega$, and $x_i|u|^{q-2}u\in L^{\frac{p}{p-1}}(\Omega)$, we note

$$(1+b)x_i|u_n|^{q-2}u_n \to (1+b)x_i|u|^{q-2}u \quad \text{strongly in } L^{\frac{p}{p-1}}(\Omega) \text{ as } n\to\infty.$$

Hence we find

$$\int_\Omega (1+b)|u_n|^{q-2}u_n\, x\cdot\nabla w_n\, dx \to \int_\Omega (1+b)|u|^{q-2}u\, x\cdot\nabla u\, dx$$
$$= -\frac{1}{q}\int_\Omega |u|^q(x\cdot\nabla b)\, dx - \frac{N}{q}\int_\Omega (1+b)|u|^q\, dx \qquad \text{as} \quad n\to\infty.$$

Since w_n converges to u strongly in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ by (ii) of Lemma 2.3, we can repeat the same verification as for (29) and (30) to get

$$a|\nabla w_n| \to a|\nabla u| \quad \text{strongly in } L^p(\Omega),$$
$$(x\cdot\nabla a)|\nabla w_n| \to (x\cdot\nabla a)|\nabla u| \quad \text{strongly in } L^p(\Omega).$$

Consequently, by letting $n\to+\infty$ in (27), we obtain

$$-\frac{1}{q}\int_\Omega |u|^q(x\cdot\nabla b)\, dx - \frac{N}{q}\int_\Omega b|u|^q\, dx + \frac{N-p}{p}\int_\Omega a|\nabla u|^p\, dx$$
$$+\frac{1}{p}\int_\Omega |\nabla u|^p(x\cdot\nabla a)\, dx + \mathcal{R} \le 0. \tag{34}$$

Then to complete the proof, it suffices to use the relation (26). ☐

3. **Nonexistence results.** By virtue of the Pohozaev-type inequality we can get the following nonexistence theorem.

Theorem 3.1. *In addition to the assumption of Theorem 2.1, we assume*

$$x\cdot\nabla a \ge 0.$$

Then the following nonexistence results hold.

(i) *If* $\dfrac{(N-p)b(x)}{p} - \dfrac{x\cdot\nabla b}{q} > 0$ *a.e.* Ω, *there exists no nontrivial solution of* (E) *belonging to* $\mathcal{P}$.

(ii) *If* $\dfrac{(N-p)b(x)}{p} - \dfrac{x\cdot\nabla b}{q} = 0$ *a.e.* Ω, *there exists no positive solution of* (E) *belonging to* $\mathcal{P}$.

Remark 2. As for the case b is constant, (i) says that if q is super critical ($q > p^*$), (E) has no nontrivial solution and (ii) says that if q is critical ($q = p^*$), (E) has no positive solution (cf. [7], [5]).

Proof. Every solution u in $\mathcal{P}$ enjoys the Pohozaev-type inequality (1). Since $\mathcal{R}\ge 0$ and $x\cdot\nabla a\ge 0$, (1) with $\dfrac{(N-p)b(x)}{p} - \dfrac{x\cdot\nabla b}{q} > 0$ implies that $u = 0$ a.e. Ω. Thus the first assertion is verified.

To prove (ii), we further need delicate arguments. First of all, (1) with $\dfrac{(N-p)b(x)}{p} - \dfrac{x\cdot\nabla b}{q} = 0$ implies that $\mathcal{R} = 0$. Then for any $\eta > 0$, there exists $R_0 > 0$, $N_0\in\mathbb{N}$ and $\varepsilon_0 > 0$ such that

$$\int_{\partial\Omega} (|\nabla w_n^\varepsilon|^2+\varepsilon)^{p/2}(x\cdot\vec{n})\, dS < \eta \quad \text{for all } R\ge R_0,\ n\ge N_0,\ \varepsilon\in(0,\varepsilon_0). \tag{35}$$

Since

$$N|\Omega_0| = \int_\Omega \operatorname{div} x\, dx = \int_{\partial\Omega} x\cdot\vec{n}(x)\, dS,$$

there exist a positive number ρ and a relatively open subset $\Gamma_0 \subset \partial\Omega$ such that $x \cdot \vec{n} \geq \rho > 0$ on $\overline{\Gamma_0}$. Then it follows from (35) that

$$\int_{\Gamma_0} |\nabla w_n^\varepsilon|^p dS < \frac{\eta}{\rho} \quad \text{for all } n \geq N_0,\ \varepsilon \in (0, \varepsilon_0). \tag{36}$$

On the other hand, we can take a relatively open portion $I_0 \subset \Gamma_0$ which is parameterized as $I_0 = \{x_\lambda\}_{\lambda\in\Lambda}$ with $\Lambda = (0,1)^{N-1}$. Furthermore there exist $\{y_\lambda\}_{\lambda\in\Lambda}$ and a (sufficiently small) positive number ℓ such that $\{x_\lambda\} = \partial B_{2\ell}(y_\lambda) \cap \partial\Omega$, where $B_{2\ell}(y_\lambda)$ is a ball centered at y_λ with radius 2ℓ. Let $\Omega^\lambda = B_{2\ell}(y_\lambda) \setminus \overline{B_\ell(y_\lambda)}$ and introduce functions z_λ on Ω^λ by

$$z_\lambda(x) = \alpha(3\ell - r)^\delta - \alpha\ell^\delta,\ r = |x - y_\lambda|,\ (\alpha, \delta: \text{ positive parameters}).$$

Then, it is easy to see that there exist $\varepsilon_1 > 0$ and (sufficiently large) δ such that

$$\begin{cases} A_\varepsilon z_\lambda + z_\lambda^{q-1} \leq 0, & \text{in } \Omega^\lambda, \\ z_\lambda|_{\partial B_{2\ell}(y_\lambda)} = 0, & z_\lambda|_{\partial B_\ell(y_\lambda)} = \alpha(2^\delta - 1)\ell^\delta \end{cases} \tag{37}$$

for all $\varepsilon \in (0, \varepsilon_1)$. On the other hand, the comparison theorem (see Proposition 3) and the Harnack principle assure that (see proof of Theorem 2.1 of [5])

$$w_n(x) \uparrow u(x) \text{ for a.e. } x \in \Omega, \tag{38}$$

$$u(x) \geq w_n(x) > 0 \text{ for a.e. } x \in \Omega. \tag{39}$$

Then it follows from (38) and (39) that,

$$\inf_{n\geq N_0, x\in K_1} w_n(x) = \kappa_0 > 0,$$

where $K_1 = \bigcup_{\lambda\in\Lambda} \partial B_\ell(y_\lambda)$. Moreover, by virtue of Theorem in [6, p.251], we can deduce the equi-continuity of $\{w_n^\varepsilon\}_\varepsilon$, therefore we can extract a sequence ε_k which tends to 0 as $k \to \infty$ such that

$$w_n^{\varepsilon_k} \to w_n \text{ uniformly in } K_1. \tag{40}$$

Hence there exists k_0 such that

$$w_n^{\varepsilon_k}(x) \geq \frac{\kappa_0}{2} \quad \text{for all } x \in K_1 \text{ and } k \geq k_0.$$

Therefore, for a sufficiently small α independent of $\lambda \in \Lambda$, we get

$$w_n^{\varepsilon_k}(x) \geq z_\lambda(x) \quad \text{for all} \quad x \in \partial\Omega^\lambda, \quad n \geq N_0 \text{ and } k \geq k_0.$$

Since we can take $v_n^{\varepsilon_k}(x) \geq 0$ for all $x \in \Omega$, $w_n^{\varepsilon_k}$ satisfy $A_{\varepsilon_k} w_n^{\varepsilon_k} + w_n^{\varepsilon_k} \geq 0$ in Ω. Thus by virtue of a comparison theorem for A_ε (Proposition 3), we can deduce

$$z_\lambda(x) \leq w_n^{\varepsilon_k}(x) \text{ for all } x \in \Omega^\lambda,\ n \geq N_0 \text{ and } k \geq k_1,$$

where $\varepsilon_{k_1} < \varepsilon_1$ and $k_1 \geq k_0$. Then

$$z_\lambda(x_\lambda + t\vec{n}(x_\lambda)) - z_\lambda(x_\lambda) \leq w_n^{\varepsilon_k}(x_\lambda + t\vec{n}(x_\lambda)) - w_n^{\varepsilon_k}(x_\lambda)$$

holds for $t < 0$. Dividing both sides by $t < 0$ and letting $t \to -0$, we deduce

$$\frac{\partial w_n^{\varepsilon_k}}{\partial n}(x_\lambda) \leq \frac{\partial z_\lambda}{\partial n}(x_\lambda) = -\alpha\delta\ell^{\delta-1} < 0 \quad \text{for all } \lambda \in \Lambda, n \geq N_0 \text{ and } k \geq k_1.$$

Since $\dfrac{\partial w_n^{\varepsilon_k}}{\partial n}(x_\lambda) = -|\nabla w_n^{\varepsilon_k}(x_\lambda)|$, (36) gives

$$(\alpha\delta\ell^{\delta-1})^p |I_0| \leq \int_{I_0} |\nabla w_n^{\varepsilon_k}(x)|^p dS \leq \int_{\Gamma_0} |\nabla w_n^{\varepsilon_k}(x)|^p dS \leq \frac{\eta}{\rho},$$

which leads to a contradiction since α, δ, ℓ and ρ are independent of η. □

Proposition 3. *Let $u_1, u_2 \in C^2(\bar{\Omega})$ satisfies the following relations.*

$$\begin{cases} |u_1|^{q-2}u_1 + A_\varepsilon u_1 \le |u_2|^{q-2}u_2 + A_\varepsilon u_2 & \text{in } \Omega, \\ u_1 \le u_2 & \text{on } \partial\Omega, \end{cases} \tag{41}$$

where $A_\varepsilon u = -\mathrm{div}\{a(|\nabla u|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u\}$. Then we have $u_1 \le u_2$ in Ω.

Proof. Multiply (41) by $w(x) = [u_1 - u_2]^+(x) = \max(u_1(x) - u_2(x), 0)$. Since $w|_{\partial\Omega} = 0$, we have

$$\begin{aligned} 0 &\ge \int_\Omega (A_\varepsilon u_1 - A_\varepsilon u_2)[u_1-u_2]^+(x)dx + \int_\Omega (|u_1|^{q-2}u_1 - |u_2|^{q-2}u_2)[u_1-u_2]^+(x)dx \\ &= \int_{u_1\ge u_2} a\{(|\nabla u_1|^2+\varepsilon)^{(p-2)/2}\nabla u_1 - (|\nabla u_2|^2+\varepsilon)^{(p-2)/2}\nabla u_2\}\cdot\nabla(u_1-u_2)dx \\ &\quad + \int_{u_1\ge u_2} (|u_1|^{q-2}u_1 - |u_2|^{q-2}u_2)(u_1-u_2)dx = I_1 + I_2. \end{aligned}$$

Here it is easy to obtain

$$\begin{aligned} &\{(|\nabla u_1|^2+\varepsilon)^{(p-2)/2}\nabla u_1 - (|\nabla u_2|^2+\varepsilon)^{(p-2)/2}\nabla u_2\}\cdot(\nabla u_1 - \nabla u_2) \\ &\quad \ge \begin{cases} (p-1)\{(|\nabla u_1| + |\nabla u_2|)^2+\varepsilon\}^{(p-2)/2}|\nabla u_1 - \nabla u_2|^2 & \text{for } 1 < p \le 2, \\ \dfrac{1}{p-1}\left(\dfrac{1}{2}\right)^{p-1}|\nabla u_1 - \nabla u_2|^p & \text{for } 2 \le p. \end{cases} \end{aligned}$$

Since $a(x) \ge a_0 > 0$, it easily follows that $I_1 = I_2 = 0$, which implies $u_1 = u_2$ on $\{x \in \Omega : u_1 \ge u_2\}$, i.e., $u_1 \le u_2$ in Ω. □

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